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A Combinatorial Approach to the Tanny Sequence

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The Tanny sequence $T(i)$ is a sequence defined recursively as $T(i) = T(i-1-T(i-1)) + T(i-2-T(i-2))$, $T(0) = T(1) = T(2) = 1$. In the first part of this paper we give combinatorial proofs of all the results regarding $T(i)$, that Tanny proved in his paper "A well-behaved cousin of the Hofstadter sequence", Discrete Mathematics, 105(1992), pp. 227-239, using algebraic means. In most cases our proofs turn out to be simpler and shorter. Moreover, they give a "visual" appeal to the theory developed by Tanny. We also generalize most of Tanny's results. In the second part of the paper we present many new results regarding $T(i)$ and prove them combinatorially. Given two integers n and k, it is interesting to know if $T(n) = k$ or not. In this paper we characterize such numbers.

Keywords: Infinite complete binary tree, VLR-ordering, meta-Fibonacci sequence.

1 Introduction

Let $G = (V, E)$ be a simple, finite, undirected graph. For $S \subseteq V$, $G[S]$ denotes the subgraph of G induced by the vertices in S. Let T be a rooted tree. The *depth* of T is the number of nodes present in the longest path starting from the root and ending at a leaf. Let $T = (V, E)$ be a tree rooted at a vertex r. For a vertex $x \in V$, all the vertices from r to x are called *ancestors* of x. If x is an ancestor of y then y is called a *descendant* of x. Note that x is a descendant of x itself.

A fibonacci sequence is a sequence of numbers defined by the recurrence relation $a(n) = a(n-1) + a(n-2)$ with $a(0) = 1$ and $a(1) = 1$. In [11], Hofstadter defined the sequence $Q(n)$ by

$$Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2)), \quad n > 2$$

with $Q(1) = Q(2) = 1$. He remarked on the apparent parallel between $Q(n)$ and the usual fibonacci recursion as follows: "Each new value is a sum of two previous values-but not of the immediately two values". Guy [6] reports that Malm called $Q(n)$ a 'meta-Fibonacci sequence', motivated by the above observation.

In [18], Tanny defined a sequence recursively as $T(n) = T(n-1-T(n-1)) + T(n-2-T(n-2))$, $T(0) = T(1) = T(2) = 1$. This sequence is known as Tanny sequence which is a close relative of the

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Hofstadter sequence. He showed that in sharp contrast to the 'chaotic' behavior of $Q(n)$, $T(n)$ behaves in a completely predictable fashion. Jackson and Ruskey [13] were the first to give a combinatorial way to interpret some special meta-Fibonacci sequences.

A combinatorial interpretation of Tanny sequence similar to that given in [13] is developed in [2]. They used this interpretation to relate the Tanny sequence with the so called discrete connected isoperimetric problem on infinite complete binary trees. In this paper we will use the combinatorial interpretation developed in [2]. Next we describe the relevant results from [2] in detail.

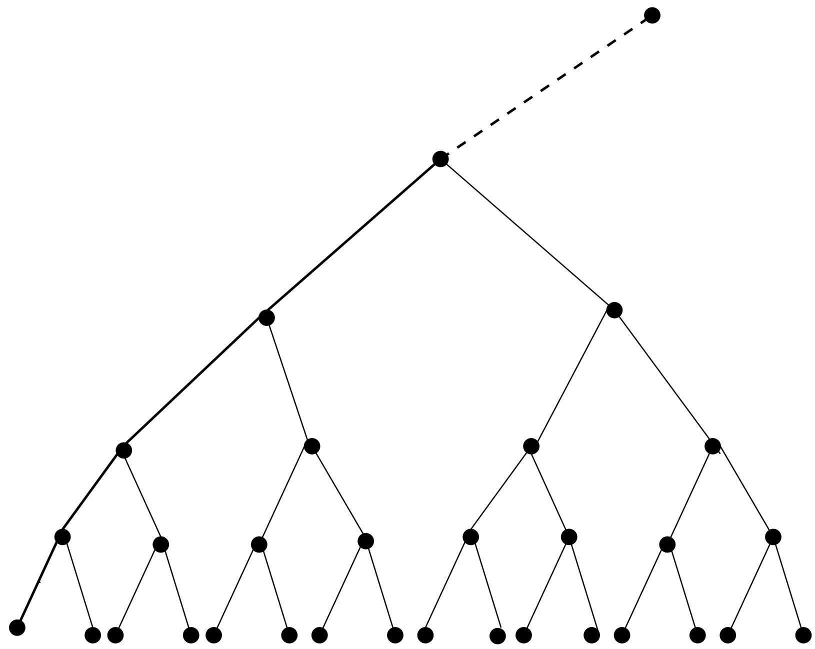

Fig. 1: The infinite complete binary tree T_2. The path shown in bold edges is the back bone of T_2.

Definition 1.1 *Let T be a tree rooted at r and has k children, say $c_1, c_2, \ldots, c_k$. Let $T_1, T_2, \ldots, T_k$ be the subtrees of T rooted at $c_1, c_2, \ldots, c_k$, respectively. The VLR-traversal of T $($also known as pre-order traversal$)$ is defined as follows:*

1. *Visit the root.*
2. *Visit T_1 in VLR-order.*

$\vdots$

$k+1$. *Visit T_k in VLR-order.*

The ordering of the vertices of T which is associated with the VLR-traversal is known as *VLR-ordering*.

Let T_2 be an infinite complete binary tree. See Figure 1 for illustration. We assume the leaf of T_2 to be at depth one. The *depth* of a vertex v, denoted as $dep(v)$, is the number of vertices present in the shortest path starting from v and ending at a leaf. Let T' be a subtree of T_2. A leaf of T' is called a *true leaf* if it is also a leaf of T_2. The *infinite path* starting from the left most leaf of T_2 to the root of T_2 is called the *back bone* of T_2. (It is shown in bold edges in Figure 1.)

Definition 1.2 *For a given integer i, $i > 0$, let d be the integer such that $2^{d-1} \leq i \leq 2^d - 1$ and T_2^d be the complete binary tree of depth d. Clearly there exists a subtree of T_2 isomorphic to T_2^d such that the leaves of this subtree are at depth one and its root situated on the backbone of T_2. A VLR-tree of i nodes, denoted as $VLR(i)$, is a subtree of T_2 induced by the set of first i vertices in the VLR-ordering of such a subtree of T_2 isomorphic to T_2^d.*

Note that, for a given i, $VLR(i)$ is uniquely defined. The combinatorial interpretation of $T(n)$ developed in [2] is summarized in the following theorem.

Theorem 1.1 ([2]) *For a positive integer n, the number of true leaves of $VLR(n)$ equals $T(n)$, the Tanny value at n.*

Definition 1.3 *For $S \subseteq V$, the edge boundary $\delta(S, G)$ is the set of edges of G with exactly one end point in S. In other words,*

$$\delta(S, G) = \{(u, v) \in E : u \in S \text{ and } v \in V - S\}$$

An edge which is present in $\delta(S, G)$ is called an *out going* edge of S.

Definition 1.4 *Let i be an integer where $1 \leq i \leq |V|$. For each i define the edge isoperimetric value $b_e(i, G)$ of G at i as follows*

$$b_e(i, G) = \min_{S \subseteq V;\ |S|=i} |\delta(S, G)|$$

The *discrete isoperimetric problem* on G is to find the value of $b_e(i, G)$, for each i, $1 \leq i \leq |V(G)|$. Discrete isoperimetric inequalities form a very useful and important subject in graph theory and combinatorics. See [1, 4, 16, 17] for more details.

Note that the subgraph $G[S]$ induced by the set of vertices of S which gives the minimum edge boundary need not be connected in general. Next we define a special kind of isoperimetric value by putting some restriction on the set S.

Definition 1.5 *Let i be an integer where $1 \leq i \leq |V|$. For each i define the connected edge isoperimetric value $b_c(i, G)$ of G at i as follows:*

$$b_c(i, G) = \min_{S \subseteq V; |S|=i;\ G[S] \text{ is connected}} |\delta(S, G)|$$

The *discrete connected isoperimetric problem* on G is to find the value of $b_c(i, G)$, for each i, $1 \leq i \leq |V(G)|$.

In this paper we deal with infinite complete binary trees only. So, for convenience we denote $b_c(i, T_2)$ as $b_c(i)$. Theorem 1.1 is used in [2] to relate the Tanny value to connected discrete isoperimetric problem on T_2.

Theorem 1.2 ([2]) *For T_2, $b_c(i) = i + 2 - 2T(i)$, for each i, $i \geq 1$, where $T(i)$ is the Tanny value at i.*

The following theorem is from [2] which relates the connected isoperimetric problem on T_2 with the boundary of the VLR tree.

Theorem 1.3 ([2]) *Let T_2 be an infinite complete binary tree. Then for a positive integer n, $b_c(n) = |\delta(VLR(n), T_2)|$.*

Remark 1. Since T_2 is an infinite complete binary tree, $b_c(n) \geq 1$, for every positive integer n. Also, it is easy to see from Theorem 1.3 that $1 \leq b_c(n) \leq \lceil \log n \rceil$.

Consider the tree $VLR(i)$. Let $2^{d-1} \leq i \leq 2^d - 1$. So, the depth of $VLR(i)$ is d. Let the root of the tree $VLR(i)$ be r and r_l and r_r be the left child and the right child of r, respectively. By the construction of $VLR(i)$ and by the choice of d, it is easy to see that the descendants of r_l in $VLR(i)$ form a complete binary tree of depth $d-1$. When $i > 2^{d-1}$, $i - 2^{d-1}$ vertices of $VLR(i)$ are present in the subtree (of $VLR(i)$) rooted at r_r.

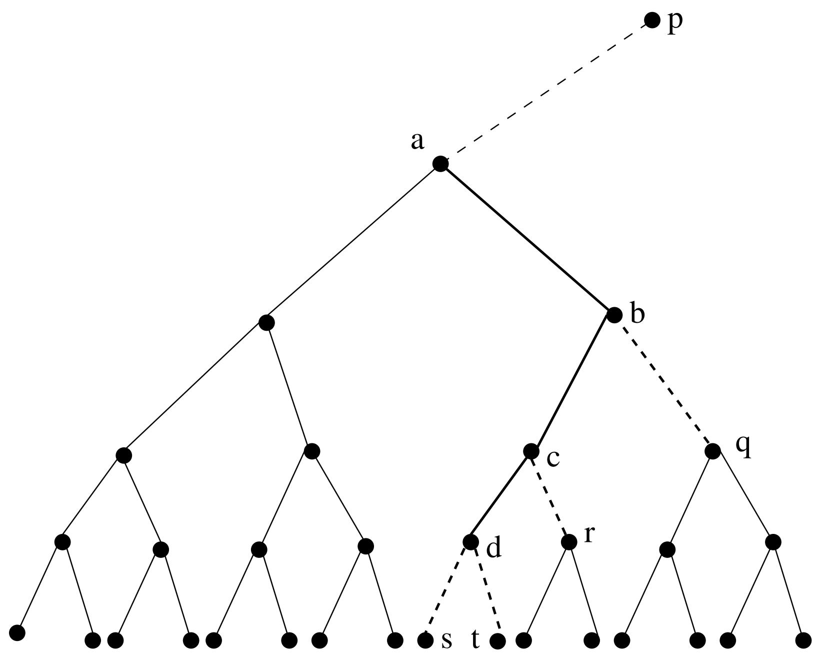

Fig. 2: Illustration of Primary path. The path a-b-c-d is the primary path of $VLR(19)$ and all the vertices of the primary path are active vertices.

Define a path $v_1, v_2, \ldots, v_k$ in $VLR(i)$ starting from the root as follows: $v_1 = r$ and v_j, $2 \leq j \leq k$, is the right child of v_{j-1} if the right child of v_{j-1} is present in $VLR(i)$, otherwise if only the left child of v_{j-1} is present in $VLR(i)$ then v_j is the left child of v_{j-1}. If both the children of v_{j-1} are not present in $VLR(i)$ then the path ends at v_{j-1}, that is $v_{j-1} = v_k$. This path is called the *primary path* of $VLR(i)$. Note that when $i = 2^{d-1}$, the primary path of $VLR(i)$ consists of only one vertex, the root r of $VLR(i)$. In Figure 2, the path $a - b - c - d$ is the primary path of $VLR(19)$.

By the definition of the *primary path* it is easy to see that if e is an *outgoing edge* of $VLR(i)$ then it is incident on some vertex of the primary path. If u is a vertex in the primary path such that an outgoing edge of $VLR(i)$ is incident on u then either the next node after u in the path is the left child of u or u is the last vertex of the primary path and it is not a true leaf. A vertex of the primary path on which an outgoing edge is incident is called an *active* vertex. It is easy to see that the first vertex of the primary path, that is the root of the tree $VLR(i)$ is always an active vertex. The end vertex of the primary path is called the *terminal* vertex. Note that, the terminal vertex need not be an active vertex. The active vertex on the primary path which is situated at the least depth is called the *tip* of the primary path. In a primary path, the terminal vertex and the tip vertex need not be same, in particular if the terminal vertex is a true leaf, then it cannot be the tip. In $VLR(19)$, the terminal vertex and the tip of the primary path are same. See Figure 2 for illustration.

Observation 1. Let P be the primary path of $VLR(i)$, $2^{d-1} < i < 2^d - 1$, and v_{tip}, v_t be the tip and the terminal vertex of P, respectively. If $v_{tip} = v_t$, then in $VLR(i+1)$, the primary path P' is the path $P \cup v_l$, where v_l is the left child of v_{tip} in T_2. If v_{tip} and v_t are not same, then the primary path P' of $VLR(i+1)$ is the path $P \cup v_r$, where v_r is the right child of v_{tip} in T_2.

Observation 2. Let P be the primary path of $VLR(i)$. If the tip of P is at depth two, then $VLR(i+1)$ contains one more true leaf than $VLR(i)$, otherwise number of true leaves in $VLR(i)$ and $VLR(i+1)$ are same.

Observation 3. Let P be the primary path of $VLR(i)$ for some i and v be the terminal vertex of P. If $|V(P)| \geq 2$ and v is not a true leaf then v is an active vertex and moreover two outgoing edges of $VLR(i)$ are incident on v.

Observation 4. Let $P = v_1, v_2, \ldots, v_p, \ldots, v_q$ be the primary path of $VLR(i)$, for some positive integer i such that v_q is a true leaf and v_p is the tip of P. Let $P' = v_1, v_2, \ldots, v_p$ and $S = \{v_1, v_p\} \cup \{v_i \in V(P') - \{v_1, v_p\} | v_i$ is not an active vertex of $P'\}$. Note that, since the tip of the primary path can never be a true leaf, we have $dep(v_p) \geq 2$. Now, clearly $VLR(i)$ contains exactly $N = \sum_{v_h \in S} 2^{dep(v_h)-2}$ number of true leaves. Let t be the smallest integer such that $VLR(t)$ has N number of true leaves, where N is a given positive integer. For $v_h \in S$, let $left(v_h)$ denote the complete subtree rooted at the left child of v_h. Since $V(P')$ does not contain any true leaf, the set of true leaves of $VLR(t)$ is exactly the union of true leaves of the complete binary trees $\{left(v_h) \ : \ v_h \in S\}$. A careful look at $VLR(t)$ will reveal that the vertices of $VLR(t)$ is exactly union of all the vertices in the complete sub-trees $\{left(v_h) : v_h \in S\}$ and $V(P')$. That is, $V(VLR(t)) = \bigcup_{v_h \in S} V(left(v_h)) \cup V(P')$.

Observation 5. Let N be a positive integer and $N = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_p}$, $0 \leq k_1 < k_2 < \cdots < k_p$. Let t be the smallest positive integer such that $VLR(t)$ contains N number of true leaves and $P = v_1, v_2, \ldots, v_r$ be the primary path of $VLR(t)$. Since t is the smallest such number, the terminal vertex of P is the last vertex in the VLR ordering of $VLR(i)$ and v_r is a true leaf in $VLR(t)$. Let v_{tip} be the tip of P. By Observation 4, it is clear that the depth of v_{tip} is $k_1 + 2$. Now, it is easy to see that for $j \in \{t, t+1, \ldots, t+k_1\}$, $VLR(j)$ has also N number of true leaves. Moreover, $VLR(t+k_1+1)$ has $N+1$ true leaves. That is, $t+k_1+1$ is the smallest integer such that $VLR(t+k_1+1)$ contains $N+1$ number of true leaves.

2 Graph theoretic proof of properties of $T(n)$

In this section we reprove the results of [18] combinatorially using Theorem 1.1 and the properties of $VLR(i)$. In most of the cases we have also generalized the results.

Let $T(n)$ be the Tanny sequence which is defined by the recurrence relation $T(n) = T(n-1-T(n-1)) + T(n-2-T(n-2)), T(0) = T(1) = T(2) = 1$.

The following Proposition of [18] follows from Observation 2.

Proposition 2.1 ([18]) *For $T(n)$ defined as above,*

(a) $T(n+1) = T(n)$ *or* $T(n+1) = T(n) + 1$.

(b) *For* $n \geq 2$, *if* $T(n)$ *is odd,* $T(n+1) = T(n) + 1$.

We can generalize Proposition 2.1(b) as follows:

Proposition 2.2 *If* $T(n) = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_p}$, $0 \leq k_1 < k_2 < \cdots < k_p$, *then* $T(n + k_1 + 1) > T(n)$.

The following theorem follows from Observation 2.

Theorem 2.3 *Let P be the primary path of $VLR(n)$. Then, $T(n+1) = T(n) + 1$ if and only if the tip of P is at depth 2.*

Let $\rho(N) = \{n : T(n) = N\}$. By Proposition 2.1, the elements of $\rho(N)$ consist of a sequence of consecutive integers. Let $|\rho(N)|$ denotes the cardinality of the set $\rho(N)$.

Proposition 2.4 ([18]) *If $N > 1$ is odd, then $|\rho(N)| = 1$. If N is even, then $|\rho(N)| \geq 2$.*

Proof: By the construction of the VLR tree it is easy to see that for any positive integer N, we can construct a VLR tree such that it contains exactly N number of true leaves. Hence for any $N \geq 1$, $\rho(N) \neq \emptyset$, that is the Tanny sequence hits every positive integer.

If N is odd, then $|\rho(N)| = 1$ by Proposition 2.1. Next suppose that N is even. Let $N = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_p}$, $0 \leq k_1 < k_2 < \cdots < k_p$. Since N is even, $k_1 \geq 1$. By Observation 5, $VLR(t+1)$ also contains N number of leaves. Hence $|\rho(N)| \geq 2$. □

It is easy to find the elements of $\rho(N)$ when N is of certain special form. For example, if $N = 2^x$ for some $x \geq 1$, it is easy to find out the elements of $\rho(N)$.

Proposition 2.5 ([18]) *For every non-negative integer x, $\rho(2^x)$=$\{2^{x+1} - 1$, 2^{x+1}, …,$2^{x+1} + x\}$ and $|\rho(2^x)| = x + 2$.*

Proof: Let $N = 2^x$, for some positive integer x and $n = 2^{x+1} - 1$. It is easy to see that n is the smallest positive integer such that the number of true leaves in $VLR(n)$ is N. Also the tree $VLR(n + 1 = 2^{x+1})$ has N number of true leaves. The depth of the tree $VLR(2^{x+1})$ is $x + 2$. Let $t = 2^{x+1} + (x + 1)$. By the property of the VLR tree, it is easy to see that t is the smallest integer such that $VLR(t)$ contains $N + 1$ number of true leaves. So, $t - 1$ is the largest integer j such that the tree $VLR(j)$ contains N number of true leaves. Hence, $\rho(2^x) = \{2^{x+1} - 1, 2^{x+1}, \ldots, 2^{x+1} + x\}$ and $|\rho(2^x)| = x + 2$. □

A formula for $|\rho(N)|$ for the remaining values of N is given by the following theorem which follows immediately from Observation 5.

Theorem 2.6 *Let N be a positive integer and $N = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_p}$ such that $p \geq 2$ and $0 \leq k_1 < k_2 < \cdots < k_p$. Then, $|\rho(N)| = k_1 + 1$.*

The following proposition present in [18] is an immediate consequence of Theorem 2.6

Proposition 2.7 ([18]) *Let α be any even integer, $2 \leq \alpha \leq 2^x - 2$, where $x \geq 2$. Then $|\rho(2^x + \alpha)| = |\rho(2^{x-1} + \frac{\alpha}{2})| + 1$.*

Proof: Let $2^x + \alpha = 2^x + 2^{k_p} + 2^{k_{p-1}} + \ldots + 2^{k_1}$, $0 \leq k_1 < k_2 < \cdots < k_p$. By Theorem 2.6, $|\rho(2^x + \alpha)| = k_1 + 1$. Now, $2^{x-1} + \frac{\alpha}{2} = \frac{2^x+\alpha}{2} = 2^{x-1} + 2^{k_p-1} + 2^{k_{p-2}} + \ldots + 2^{k_1-1}$, $0 \leq k_1 - 1 < k_2 - 1 < \cdots < k_p - 1$. By Theorem 2.6, $|\rho(2^{x-1} + \frac{\alpha}{2})| = k_1$. Hence, $|\rho(2^x + \alpha)| = |\rho(2^{x-1} + \frac{\alpha}{2})| + 1$. □

In fact we can easily generalize Proposition 2.7 as following in view of Theorem 2.6.

Theorem 2.8 *Let N_1 and N_2 be two positive integers such that $N_1 = 2^{\alpha_1} + 2^{\alpha_2} + \ldots + 2^{\alpha_{k_1}}$ and $N_2 = 2^{\beta_1} + 2^{\beta_2} + \ldots + 2^{\beta_{k_2}}$ such that $k_1, k_2 \geq 2$ and $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{k_1}$ and $0 \leq \beta_1 < \beta_2 < \cdots < \beta_{k_2}$. Then, $|\rho(N_1)| = |\rho(N_2)|$ if and only if $\alpha_1 = \beta_1$.*

Proof: Suppose $|\rho(N_1)| = |\rho(N_2)|$. By Theorem 2.6, $|\rho(N_1)| = \alpha_1 + 1$ and $|\rho(N_2)| = \beta_1 + 1$. That is, $\alpha_1 = \beta_1$.

Next suppose $\alpha_1 = \beta_1 = k$. By Theorem 2.6, $|\rho(N_1)| = \alpha_1 + 1 = k + 1$ and $|\rho(N_2)| = \beta_1 + 1 = k + 1$. That is, $|\rho(N_1)| = |\rho(N_2)|$. Hence the theorem. □

Proposition 2.9 ([18]) *Let N be a positive integer and $N = 2^{\alpha_1} + 2^{\alpha_2} + \ldots + 2^{\alpha_p}$ such that $p \geq 2$ and $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p$. The greatest element in $\rho(N)$ is given by $2N + \alpha_p - p + 1$.*

Proof: Let t be the smallest integer such that $VLR(t)$ contains N number of true leaves. Let $P = v_1, v_2, \ldots, v_m, \ldots, v_q$ be the primary path of $VLR(t)$. Since t is smallest, v_q is a true leaf. Let v_m be the tip of P. Let $P' = v_1, v_2, \ldots, v_m$ and $S = \{v_1, v_m\} \cup \{v_i \in V(P') - \{v_1, v_m\} | v_i$ is not an active vertex of $P'\}$. By Observation 4, t is the number of vertices in $\bigcup_{v_h \in S} left(v_h) \cup V(P')$. By Observation 5, we have $\bigcup_{v_h \in S} left(v_h) = (2^{\alpha_1+1} - 1) + (2^{\alpha_2+1} - 1) + \ldots + (2^{\alpha_p+1} - 1) = 2N - p$. It is easy to see that $dep(v_1) = \alpha_p + 2$ and $dep(v_m) = \alpha_1 + 2$. So, $|V(P')| = \alpha_p - \alpha_1 + 1$. Hence $t = 2N + \alpha_p - p - \alpha_1 + 1$. By Theorem 2.6, $t + \alpha_1$ is the largest integer such that $VLR(t + \alpha_1)$ contains N number of leaves. Hence the greatest element in $\rho(N)$ is $t + \alpha_1 = 2N + \alpha_p - p + 1$. □

Proposition 2.10 ([18]) *Let $N = 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_p}$, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p$. If $p = 1$, then $T(n) = n$ for all $\alpha_1 + 2$ consecutive integers $n = 2^{\alpha_1+1} - 1$ to $n = 2^{\alpha_1+1} + \alpha_1$. If $p \geq 2$ then $T(n) = N$ for all $(\alpha_1 + 1)$ consecutive integers from $n = 2N + (\alpha_p - \alpha_1) - p + 1$ to $n = 2N + \alpha_p - p + 1$.*

Proof: Let $N = 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_p}$, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p$. If $p = 1$, then the result follows from Proposition 2.5. If $p \geq 2$, then the result follows from Theorem 2.6 and Proposition 2.9. □

Proposition 2.11 ([18]) $\frac{1}{2} - (\frac{1}{n}) \log(\frac{n+1}{2}) \leq \frac{T(n)}{n} \leq \frac{1}{2} + \frac{1}{2n}$, *and* $\frac{T(n)}{n} \longrightarrow \frac{1}{2}$ *as* $n \longrightarrow \infty$.

By Remark 1, we have $1 \leq b_c(i) \leq \lceil \log n \rceil$. This immediately allows us to derive a statement which is slightly better than Proposition 2.11.

Proposition 2.12 $\frac{1}{2} - \frac{\lceil \log n \rceil - 2}{2} \leq \frac{T(n)}{n} \leq \frac{1}{2} + \frac{1}{2n}$.

Proof: From Theorem 1.2, we have $T(n) = \frac{n+2}{2} - \frac{b_c(n)}{2}$. By Remark 1, $1 \leq b_c(n) \leq \lceil \log n \rceil$. So, $\frac{T(n)}{n} \leq \frac{1}{2} + \frac{1}{2n}$ and $\frac{T(n)}{n} \geq \frac{1}{2} - \frac{\lceil \log n \rceil - 2}{2}$. □

3 New results about $T(n)$

We know from Theorem 1.2 that $T(i) \leq \frac{i+1}{2}$ and in fact for any i, $\frac{i+1}{2} - T(i) = \frac{j}{2}$, for some j, $0 \leq j \leq \lceil \log i \rceil - 1$. Keeping this in mind, we ask the following related question: Given a j, $0 \leq j \leq d-1$, can we characterize the set $\{i : \frac{i+2}{2} - T(i) = \frac{j}{2}\}$. In view of Theorem 1.2, this is exactly the set $\{i : b_c(i) = j + 1\}$.

From Theorem 1.2, we have

$$T(i) = \frac{i+1}{2} - \frac{b_c(i) - 1}{2} \tag{1}$$

Recall from Remark 1 that $b_c(i)$ takes the values between 1 and $\lceil \log i \rceil$. Since the depth of $VLR(i)$ is d, where $2^{i-1} \leq d \leq 2^d - 1$, $\lceil \log i \rceil = d$ and thus $1 \leq b_c(i) \leq d$. So, $0 \leq b_c(i) - 1 \leq d - 1$. From Equation 1 we have,

$$\frac{i+1}{2} - T(i) = \frac{j}{2} \tag{2}$$

where $0 \leq j = b_c(i) - 1 \leq d - 1$. For a given value of j, $0 \leq j \leq d - 1$, it is interesting to characterize the values of i for which $\frac{i+1}{2} - T(i) = \frac{j}{2}$. The following theorem gives a characterization of the values of i such that $\frac{i+1}{2} - T(i) = \frac{j}{2}$, for small values of j.

Theorem 3.1 (1) $T(i) = \frac{i+1}{2}$ *if and only if* $i = 2^d - 1$, *for some* $d \geq 1$.

(2) $T(i) = \frac{i}{2}$ *if and only if* $i = 2^d - 2^t$, *for some* $1 \leq t \leq d - 1$.

Proof: (1) From Equation 2 we have $T(i) = \frac{i+1}{2}$ if and only if $j = 0$. That is, $b_c(i) = 1$. But $b_c(i) = 1$ only when $i = 2^d - 1$, for some $d \geq 1$.

(2) From Equation 2 it is easy to see that $T(i) = \frac{i}{2}$ if and only if $j = 1$. That is, $b_c(i) = 2$. Note that the root of $VLR(i)$ is always an active vertex and therefore one out going edge is always incident on the root. Let P be the primary path of $VLR(i)$. If P consists of one vertex, i.e., the root of $VLR(i)$ only then $b_c(i) = 2$. In this case $i = 2^{d-1} - 1 + 1 = 2^{d-1}$. Other than this case, $b_c(i) = 2$ when P contains exactly one more active vertex (except the root) and exactly one out going edge is incident on this active vertex. This means that the terminal vertex of P is a true leaf by Observation 3. So, P contains d vertices in this case. Let t' be the depth of the active vertex (other than the root of $VLR(i)$) of P. Clearly $2 \leq t' \leq d - 1$.

Now, it is easy to see that $i = (2^d - 1) - (2^{t'-1} - 1) = 2^d - 2^{t'-1}$. Putting $t' - 1 = t$, we have $i = 2^d - 2^t$, $1 \leq t \leq d - 2$. Now, combining both the cases, we have $i = 2^d - 2^t$ for some t, $1 \leq t \leq d - 1$. □

For a given j let S_j represent the i's such that $\frac{i+1}{2} - T(i) = \frac{j}{2}$. That is, $S_j = \{i : \frac{i+1}{2} - T(i) = \frac{j}{2}\}$. The above result can be generalized as follows:

Lemma 3.2 *Let $S_j = \{i : \frac{i+1}{2} - T(i) = \frac{j}{2}\}$. For a given j, $i \in S_j$ if and only if there exist integers $d, t_1, t_2, \ldots, t_j$ such that $d - 2 \geq t_1 > t_2 > \ldots > t_{j-1} \geq t_j \geq 1$ and $i = 2^d - (2^{t_1} + 2^{t_2} + \ldots + 2^{t_j}) + (j - 1)$. Moreover, $|\{i \in S_j : 2^{d-1} \leq i \leq 2^d - 1\}| = \binom{d}{j+1}$.*

Proof: From Equation 2 it is easy to see that $T(i) = \frac{i+1-j}{2}$ if and only if $b_c(i) = j + 1$. Let P be the primary path of $VLR(i)$. We consider the following case:

The terminal vertex of P is a true leaf. In this case P contains d vertices and has exactly j active vertices other than the root. Clearly these j active vertices can be any of the vertices of P other than the root of $VLR(i)$ and the terminal vertex of P. So there are $\binom{d-2}{j}$, where $d = \lceil \log i \rceil$, different ways to choose these j active vertices on the primary path. Let $a_1, a_2, \ldots, a_j$ be these active vertices of P and $t'_1, t'_2, \ldots, t'_j$ be their depths, respectively. Clearly $d - 1 \geq t'_1 > t'_2 > \ldots > t'_{j-1} > t'_j \geq 2$. Now, it is easy to see that $i = (2^d - 1) - \{(2^{t'_1-1} - 1) + (2^{t'_2-1} - 1) + \ldots + (2^{t'_j-1} - 1)\} = 2^d - (2^{t'_1-1} + 2^{t'_2-1} + \ldots + 2^{t'_j-1}) + (j - 1)$. Taking $t'_i - 1 = t_i$, for $1 \leq i \leq j$, we have $i = 2^d - (2^{t_1} + 2^{t_2} + \ldots + 2^{t_j}) + (j - 1)$ where $d - 2 \geq t_1 > t_2 > \ldots > t_{j-1} > t_j \geq 1$.

The terminal vertex of P is not a true leaf. By Observation 3, the terminal vertex of P is an active vertex and two outgoing edges of $VLR(i)$ are incident on it. Since the root of $VLR(i)$ is always an active vertex, P will contain exactly $j - 2$ active vertices other than the root and the terminal vertex. Since the terminal vertex of P is not a true leaf, except the root and the terminal vertex, P can contain at most $d - 3$ vertices. These $j - 2$ active vertices can be chosen in $\binom{t-2}{j-2}$ many different ways provided t is the length P. Now, noting that t can vary from j to $d - 3$, we have $\sum_{t=j}^{d-1} \binom{t-2}{j-2}$ many distinct i's such that $VLR(i)$ has $j + 1$ out going edges and the primary path of $VLR(i)$ is not a true leaf. Let $a_1, a_2, \ldots, a_{j-2}$ be these active vertices of P and $t'_1, t'_2, \ldots, t'_{j-2}$ be their depths, respectively. Clearly $d - 1 \geq t'_1 > t'_2 > \ldots > t'_{j-2} \geq 3$. Let t'_{j-1} be the depth of the terminal vertex of P. Note that $t'_{j-1} \geq 2$. Now, it is easy to see that $i = (2^d - 1) - \{(2^{t'_1-1} - 1) + (2^{t'_2-1} - 1) + \ldots + (2^{t'_{j-2}-1} - 1) + 2(2^{t'_{j-1}-1} - 1)\}$ $= 2^d - (2^{t'_1-1} + 2^{t'_2-1} + \ldots + 2^{t'_{j-1}}) + (j - 1)$, where $d - 1 \geq t'_1 > t'_2 > \ldots > t'_{j-1} \geq 2$. Putting $t'_i - 1 = t_i$, for $1 \leq i \leq j$, we have $i = 2^d - (2^{t_1} + 2^{t_2} + \ldots + 2^{t_{j-1}}) + (j - 1)$ where $d - 2 \geq t_1 > t_2 > \ldots > t_{j-1} \geq 1$.

Combining both the above cases, we have $i = 2^d - (2^{t_1} + 2^{t_2} + \ldots + 2^{t_{j-1}} + 2^{t_j}) + (j - 1)$, where $d - 2 \geq t_1 > t_2 > \ldots > t_{j-1} \geq t_j \geq 1$.

From the above discussion, we have $|\{i \in S_j : 2^{d-1} - 1 \leq i \leq 2^d - 1\}| = \binom{d-2}{j} + \sum_{t=j}^{d-1} \binom{t-2}{j-2}$.

Claim 1. $\sum_{t=j}^{d-1} \binom{t-2}{j-2} = \binom{d-2}{j-1}$.

See [19] (Chapter 5) for proof of Claim 1. .

Now, $|\{i \in S_j : 2^{d-1} - 1 \leq i \leq 2^d - 1\}| = \binom{d-2}{j} + \binom{d-2}{j-1} = \binom{d-1}{j}$. Hence the theorem. $\square$

Though S_j is an infinite set, the distribution of the elements of S_j over the integers is nice. As shown in Lemma 3.2, The following theorem allows us to get the number of integers $i \in S_j$ that are strictly below 2^d.

Theorem 3.3 $|\{i \in S_j : i < 2^d\}| = \binom{d+1}{j+1}$.

Proof: It is easy to see that $|\{i \in S_j : i < 2^d\}| = \binom{d}{j} + \binom{d-1}{j} + \binom{d-2}{j} + \cdots + \binom{j}{j} = \sum_{j+1}^{d+1} \binom{t-1}{j}$. By Claim 1 of Lemma 3.2, we have, $\sum_{j+1}^{d+1} \binom{t-1}{j} = \binom{d+1}{j+1}$. Hence $|\{i \in S_j : i < 2^d\}| = \binom{d+1}{j+1}$. $\square$

Given two integers i and k it is interesting to know whether $T(i) = k$ or not. In [18], Tanny has already given a nice solution to this question (see Proposition 2.6 in [18]). In the following theorem we give a different way of looking at it. The answer to the question whether $T(i) = k$ is encoded in the bit pattern of the binary representation of $2^{\lceil \log i \rceil} - 2k$.

In the following theorem we characterize such numbers.

Theorem 3.4 *Let $x = 2^d - 2k$ where $d = \lceil \log i \rceil$ and let t be the number of 1's in the bit representation of x. Let t' be the greatest integer such that $2^{t'}$ divides x. Then, $T(i) = k$ if and only if $t' \geq i + 1 - 2k - t$.*

Proof:

Suppose

$$\begin{aligned} T(i) &= k & (3)\\ &= \frac{i+1}{2} - \frac{i+1-2k}{2} & (4)\\ &= \frac{i+1}{2} - \frac{j}{2} & (5) \end{aligned}$$

In equation 5, $j = i + 1 - 2k$. From Lemma 3.2, we know that for a given j, $i \in S_j$ if and only if there exists integers $d, t_1, t_2, \ldots, t_j$ such that $d - 2 \geq t_1 > t_2 > \cdots > t_{j-1} \geq t_j \geq 1$ and

$$\begin{aligned} 2^{t_1} + 2^{t_2} + \ldots + 2^{t_{j-1}} + 2^{t_j} &= 2^d - i + (j-1) & (6)\\ &= 2^d - i + (i + 1 - 2k - 1) & (7)\\ &= 2^d - 2k & (8) \end{aligned}$$

Let $x = 2^d - 2k$, where $d = \lceil \log i \rceil$. Suppose $(x)_2$ be the binary representation of x and t be the number of 1's present in $(x)_2$. Now, we have the following three cases:

Case 1. $t > j$.

Since $x = 2^d - 2k = 2^{t_1} + 2^{t_2} + \ldots + 2^{t_{j-1}} + 2^{t_j}$, where $d - 2 \geq t_1 > t_2 > \cdots > t_{j-1} \geq t_j \geq 1$, it is easy to see that the number of 1's in $(x)_2$ can be at most j. That is, $t \leq j$. So, in this case $T(i)$ cannot be equal to k.

Case 2. $t = j$.
If $t = j$, then it is easy to see that $T(i) = k$.

Case 3. $t < j$.
In this case $T(i)$ may or may not be equal to k. Let t' be the smallest positive integer such that $2^{t'}$ divides x. Note that $t' \geq 1$ (since $x(= 2^d - 2k)$ is even). That is, there is a 1 in $t'th$ position of $(x)_2$. Since t' is largest such number, all the bits after t' in $(x)_2$ are 0. So, if after $t'th$ position in $(x)_2$ there are enough bits, namely $j - t$ bits, then the rest $(j - t)$ number of 1's can be accommodated in $(x)_2$ after the $t'th$ bit. That is, we can find $t_1 > t_2 > \ldots > t_{j-1} \geq t_j$, which are nothing but the integers corresponding to the position of the j 1's in $(x)_2$ such that $x = 2^{t_1} + 2^{t_2} + \ldots + 2^{t_{j-1}} + 2^{t_j}$. So, $T(i) = k$ if and only if $t' \geq j - t$. But $j = i + 1 - 2k$. So, $T(i) = k$ if and only if $t' \geq i + 1 - 2k - t$. □

4 Conclusion

In [12], Isgur et. al. have given a combinatorial interpretation in terms of infinite binary trees to certain meta-Fibonacci sequences which can be seen as a generalization of Tanny sequence. As of now we do not know to which other meta-Fibonacci sequences our iso-perimetric analogue can be applied. It would be interesting to explore whether those meta-Fibonacci sequences also can be related to discrete iso-perimetric inequalities.

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