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Some Results on Stable Sets for k-Colorable P_6-Free Graphs and Generalizations

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This article deals with the Maximum Weight Stable Set (*MWS*) problem (and some other related NP-hard problems) and the class of P_6-free graphs. The complexity status of MWS is open for P_6-free graphs and is open even for P_5-free graphs (as a long standing open problem). Several results are known for MWS on subclasses of P_5-free: in particular, MWS can be solved for k-colorable P_5-free graphs in polynomial time for every k (depending on k) and more generally for (P_5,K_p)-free graphs (depending on p), which is a useful result since for every graph G one can easily compute a k-coloring of G, with k not necessarily minimum. This article studies the MWS problem for k-colorable P_6-free graphs and more generally for (P_6,K_p)-free graphs. Though we were not able to define a polynomial time algorithm for this problem for every k, this article introduces: (i) some structure properties of P_6-free graphs with respect to stable sets, (ii) two reductions for MWS on (P_6, K_p)-free graphs for every p, (iii) three polynomial time algorithms to solve MWS respectively for 3-colorable P_6-free, for 4-colorable P_6-free, and for (P_6,K_4)-free graphs (the latter allows one to state, together with other known results, that MWS can be solved for (P_6,F)-free graphs in polynomial time where F is any four vertex graph).

Keywords: Maximum Weight stable Set problem; P_6-free graphs; polynomial algorithms

1 Introduction

Let $G = (V, E)$ be a graph. A *stable set* (or an *independent set*) of G is a subset of pairwise nonadjacent vertices of G. a stable set of G is *maximal* if it is not properly contained in any other stable set of G. Let w be a weight function on V: the $weight$ of a stable set I of G is given by the sum of $w(v)$ for all $v \in I$.

The MWS problem is the following: Given a graph $G = (V, E)$ and a weight function w on V, determine a stable set of G of maximum weight. Let $\alpha_w(G)$ denote the maximum weight of any stable set of G. The MWS problem is called *MS* problem if all vertices v have the same weight $w(v) = 1$.

The MWS problem is NP-hard [25] and remains difficult for cubic or planar graphs respectively by [18], [17], for graphs not containing cycles below a certain length [38], in particular for triangle-free graphs [41], while it can be efficiently solved for various graph classes, such as e.g. perfect graphs [22], extensions of claw-free graphs [2, 3, 11, 12, 30, 33, 39, 44], and extensions of $2K_2$-free graphs [14, 15, 31].

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The class of P_5-free graphs is the unique minimal class, defined by forbidding a single connected subgraph, for which the computational complexity of M(W)S seems to be an open question $-$ see [1, 4].

Actually, this seems to be a long standing open question, though a subexponential time algorithm to solve MS for P_5-free graphs was recently introduced [42]. However several results are known for M(W)S on subclasses of P_5-free graph, such as e.g. MWS can be solved for (P_t,$K_{1,q}$)-free graphs for any fixed t and q in polynomial time (depending on t and on q) where $K_{1,q}$ is the graph formed by $q + 1$ vertices such that one vertex is adjacent to the other q vertices which form a stable set [32], and MWS can be solved for (P_5, F)-free graphs in polynomial where F is any five vertex graph different to a C_5 (by different respective references not reported here). Let us report the following results on k-colorable P_5-free graphs.

Graph $G = (V, E)$ is k-*partite* (equivalently called k-*colorable*) if V admits a partition, say $\{V_1, \ldots, V_k\}$ (equivalently called k-*coloring* of G), such that V_i is a stable set of G for $i = 1, \ldots, k$. A k-colorable graph will be also denoted as $(V_1 \cup \ldots \cup V_k, E)$. A 2-colorable graph is a bipartite graph. Let us observe that for every graph G one can easily find a k-coloring of G, with k not necessarily minimum.

A $clique$ is a set of pairwise adjacent vertices of a graph: a K_p is a clique of p elements, and a K_3 is also called $triangle$. Let us observe that every p-colorable graph is K_{p+1}-free (for any p), that is, K_{p+1}-free graphs are a generalization of p-colorable graphs (for any p).

- M(W)S can be solved for k-colorable P_5-free graphs for any fixed k in polynomial time (depending on k), and more generally for (P_5,K_p)-free graphs for any fixed p (depending on p) [20, 35].

- The above polynomial results were recently re-stated, by introducing a new structure property: every connected k-colorable P_5-free graph has a vertex whose non-neighbors induce a $(k - 1)$-colorable graph; this property allows one to state that the polynomial results hold also to other NP-hard problems related to MWS, and to improve the time bound of the previous algorithms for MWS on k-colorable P_5-free graphs [34].

The class of P_6-free graphs, which is a natural generalization of that of P_5-free graphs, was considered in several papers which introduce structure properties often applied to show that NP-hard problems can be solved for (subclasses of) such graphs in polynomial time. Let us report just some of them.

- [6] gives a characterization of graphs with no long paths with respect to the existence of the set of vertices whose distance from any other vertex is at most a fixed value: a corollary is that every connected P_6-free graph contains a vertex whose distance to any other vertex is at most 3. See also [13] for similar results. See also [5, 45] for extensions of the results in [6].

- [16] (see also [21]) describes a technique of decomposition of bipartite graphs and characterize the family of bipartite graphs, including bipartite P_6-free graphs, which are totally decomposable with respect to such a decomposition: in particular, the authors prove that bipartite P_6-free graphs have bounded-clique width and show that a large class of NP-hard problems, including MWS, can be solved for such graphs in $O(n + m)$ time.

- [32] shows that MWS can be solved for (P_t,$K_{1,q}$)-free graphs for any fixed t and q in polynomial time (depending on t and on q).

- [10] extends the above result on bounded clique-width of bipartite P_6-free graphs to (P_6,K_3)-free graphs and shows that a large class of NP-hard problems, including MWS, can be solved for such graphs in $O(n^2)$ time.

- [8] provides structure properties on (P_6, C_4)-free graphs and shows that MWS can be solved for such graphs in polynomial time (among other results).

- [24] shows that every connected P_6-free graph contains a dominating induced C_6 or a dominating (not necessarily induced) complete bipartite graph. See also [28, 27] for previous similar results.

- [36] shows that MWS can be solved for (P_6,diamond)-free graphs in polynomial time.

This article. This article studies the M(W)S problem for k-colorable P_6-free graphs and more generally for (P_6,K_p)-free graphs. Actually we were not able to define a polynomial time algorithm to solve this problem for every k (or for every p). However this article introduces: some structure properties of P_6-free graphs with respect to stable sets (Section 3 and Appendix); two reductions, one of which is just partial, for MWS in (P_6, K_p)-free graphs for every p (Section 4); three polynomial time algorithms to solve MWS respectively for 3-colorable P_6-free, for 4-colorable P_6-free, and for (P_6,K_4)-free graphs (Section 5); an open problem, i.e., the general case which we were not able to address (Section 6).

In what follows let us try to motivate the interest of this article by the following points.

- Since the complexity of the MWS problem is open for P_6-free graphs, the topic of this article may be of interest since for every graph G one can easily find a k-coloring of G (with k not necessarily minimum) for some natural k. In this context, let us recall that: to check whether a graph G is 3-colorable is NP-complete [19]; to check whether a P_6-free graph is 3-colorable is polynomial [43]; to check whether a P_6-free graph is k-colorable for $k > 3$ seems to be open [23].

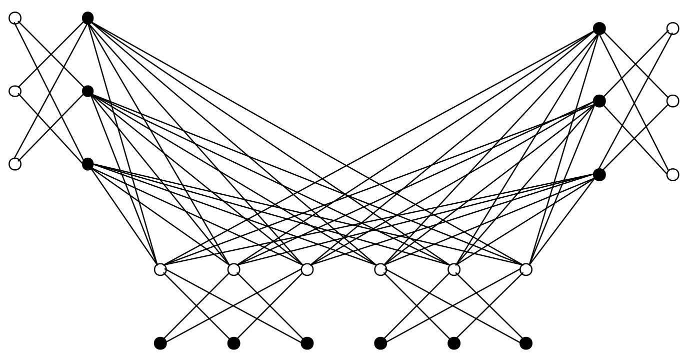
- The technique used to prove the polynomial results of Section 6 is based on three tools: a decomposition approach following the idea of [8], structure properties (both application of known ones [16] and introduction of new ones), and the anti-neighborhood (also called non-neighborhood) approach, i.e., the iterative search of a vertex v of G such that the problem can be solved in polynomial time for $G - v$.

  It follows that the polynomial results of Section 5 hold also for other NP-hard problems related to MWS. In particular they hold for: (i) the Minimum Weight Independent Dominating Set problem, i.e., Given a graph $G = (V, E)$ and a weight function w on V, determine a maximal stable set of G of minimum weight; (ii) the Maximum Induced Matching problem, i.e., Given a graph $G = (V, E)$ and a weight function w on E, determine a induced matching of G (that is, a matching of G such the vertices of the edges of the matching are pairwise nonadjacent) of maximum weight. Both these problems are NP-hard and their complexity for P_5-free graphs and for P_6-free graphs seems to be open, see respectively [7], [26].

- Concerning the techniques used in the mentioned analogous results on P_5-free graphs [20, 34, 35]: we were not able to apply that of [20, 35], and it was not possible to apply that of [34] (though we apply a similar technique looking for similar structure properties in Section 3), as briefly discussed below.

[20, 35] apply an augmenting graph technique, which is based on the the study of bipartite graphs. We were not able to deal with bipartite P_6-free graphs, which do not have the strong structure properties of bipartite P_5-free graphs. However, such a technique in itself does not provide results for the weighted case. Furthermore, it seems that all the known results obtained by such a technique for MS on subclasses of P_t-free graphs for $t > 5$ required the additional assumption of C_4-freeness (or more generally of banner-freeness, e.g. [29], or $K_{2,3}$-freeness, e.g. [37]).

[34] introduces the following structure property: every connected k-colorable P_5-free graph has a vertex whose non-neighbors induce a $(k-1)$-colorable graph, say property P. Then the anti-neighborhood approach applies. Then the polynomial results of [34] hold also for other NP-hard problems related to MWS (similarly to the point above). However property P does not hold for P_6-free graphs: a counter-example for $k = 2$ is a P_5 itself. Furthermore, even an iterated version of property P does not hold for P_6-free graphs, that is, a possible property such as: every connected k-colorable P_6-free graph has a vertex v such that each connected component H of the subgraph induced by the non-neighbors of v has a vertex whose non-neighbors in H induce a $(k-1)$-colorable graph. (Such a property would have solved the problem, by an iterated anti-neighborhood approach). A counter-example for $k = 2$ is the graph of the following figure. Also, the scheme of that counter-example allows one to construct repeatedly counter-examples showing that even a repeatedly iterated version of property P does not hold for P_6-free graphs.



- Though the complexity of the MWS problem is open even on P_5-free graphs, polynomial results for M(W)S on subclasses of P_6-free graphs may be of interest since: if MS should (be shown to) remain NP-hard for P_5-free graphs, then MS would remain NP-hard for P_6-free graphs too; if MS should (be shown to) be polynomial for P_5-free graphs, then by the mentioned result of [1] the class of P_6-free graphs would be one of the three minimal classes, defined by forbidding a single connected subgraph, for which the computational complexity of MS would be an open question.

- The polynomial results for MWS on 3-colorable P_6-free and on 4-colorable P_6-free graphs extend the respective mentioned analogous results on P_5-free graphs for the case $k = 3, 4$ [20, 34, 35]. Then let us observe that MS remains NP-hard for such graph classes without the assumption of P_6-freeness. That follows by the argument given in [41] to show that MS remains NP-hard for triangle-free graphs: given any graph $G = (V, E)$ one can construct a triangle-free graph $G' = (V', E')$ by a double subdivision of each edge of G, with $\alpha(G') = \alpha(G) + |E|$; now, since G' is 3-colorable too, the assertion follows.

- The polynomial result for MWS on (P_6,K_4)-free graphs extends the respective mentioned analogous results on P_5-free graphs for the case of (P_5,K_4)-free graphs [20, 34, 35] and mainly the mentioned result on (P_6,K_3)-free graphs [10] (with respect to the MWS problem). Then let us observe that MS remains NP-hard for such a graph class without the assumption of P_6-freeness, since it remains NP-hard for triangle-free graphs [41]. Then this polynomial result allows one to state that MWS can be efficiently solved for (P_6, F)-free graphs where F is any four vertex graph: that follows by the mentioned results for MWS on claw-free and $2K_2$-free graphs, by the fact that MWS on paw-free graphs can be reduced to MWS on triangle-free graphs (see [40]), and by the respective mentioned results for MWS on (P_6,C_4)-free and (P_6,diamond)-free graphs [8, 36].

- The preparatory results of Section 3 partly extended in the Appendix and the reductions of Section 4 might provide possible tools for new results on this topic.

2 Notation and Preliminaries

For any missing notation or reference, let us refer to [9].

Throughout this article let $G = (V, E)$ be a finite undirected graph without self-loops and multiple edges and let $|V| = n$, $|E| = m$. Let U, W be any two subsets of V. Let $N_U(W) = \{u \in U \setminus W \mid uv \in E\}$ be the set of *neighbors* of W in U. If $U = V$, then let us write $N(W)$ instead of $N_V(W)$. If $W = \{v\}$, then let us write $N_U(v)$ instead of $N_U(\{v\})$. Let us say that U has a *join* (a *co-join*, respectively) to W, if each vertex in U is adjacent (is nonadjacent) to each vertex in W. Let $v \in V$. Let us say that: v *contacts* U if v is adjacent to some vertex of U; v *dominates* U if v is adjacent to each vertex of U; v *is partial to* (or *distinguishes*) U if v contacts U but does not dominate U. Then let us say that U is a *module* of G if no vertex of $V \setminus U$ distinguishes U.

Let $G[U]$ denote the subgraph of G induced by the vertex subset U. For any graph F, G is *F-free* if G contains no induced subgraph isomorphic to F.

Graph G is *connected* if for any pair of vertices of G there exists a path in G joining them. A *component of* G is a maximal connected subgraph of G. A *component-set of* G is the vertex set of a component of G.

For any $U, W \subset V$ with $U \cap W = \emptyset$, let $\mathcal{C}(U, W)$ denote the family of component-sets of $G[U]$ which are distinguished by some vertex of W, and let $V(\mathcal{C}(U, W))$ denote the union of the elements of $\mathcal{C}(U, W)$ (then $V(\mathcal{C}(U, W)) \subseteq U$).

The following specific graphs are mentioned here. A P_k has vertices $v_1, v_2, \ldots, v_k$ and edges $v_j v_{j+1}$ for $1 \leq j < k$. A C_k has vertices $v_1, v_2, \ldots, v_k$ and edges $v_j v_{j+1}$ for $0 \leq j \leq k-1$ (index arithmetic modulo k).

Let us report from [16] (in a less deep form) a structure result on bipartite P_6-free graphs, together with its application to MWS.

A bipartite graph $G = (S_1 \cup S_2, E)$ is *complete* if $E = S_1 \times S_2$. Given a bipartite graph $G = (S_1 \cup S_2, F)$, the *bi-complemented* graph $\overline{G}^{bip}$ is the graph having the same vertex set $S_1 \cup S_2$ as G while its edge set is equal to $(S_1 \times S_2) \setminus F$.

Theorem 1 ([16]) *Let $G = (S_1 \cup S_2, F)$ be a connected bipartite P_6-free graph. Then one of the following cases (which can be verified in $O(n + m)$ time) occurs:*

(i) $\overline{G}^{bip}$ is disconnected;

(ii) there exist $S_1^* \subseteq S_1$ and $S_2^* \subseteq S_2$ such that $G[S_1^* \cup S_2^*]$ is complete bipartite, and $(S_1 \setminus S_1^*) \cup (S_2 \setminus S_2^*)$ is a stable set. □

Theorem 2 ([16]) *The MWS problem can be solved for bipartite P_6-free graphs in $O(n+m)$ time.* □

Let us report from [10] the following result.

Theorem 3 ([10]) *The MWS problem can be solved for (P_6,triangle)-free graphs in $O(n^2)$ time.* □

Obviously, the MWS problem on a graph G with vertex weight function w can be split to the same problem on subgraphs $G[V \setminus N(v)]$ and $G[V \setminus \{v\}]$ for any $v \in V$ in the following way:

$$\alpha_w(G) = \max\{\alpha_w(v) + \alpha_w(G[V \setminus N(v)]), \alpha_w(G[V \setminus \{v\}])\}$$

Thus, whenever one detects a vertex $v \in V$ such that MWS can be solved for $G[V \setminus N(v)]$ in polynomial time, the problem can be reduced to $G[V \setminus \{v\}]$. Such an approach, often called as antineighborhood (or non-neighborhood) approach, will concern directly or indirectly each section of this article.

3 Stable Sets in P_6-Free Graphs

Throughout this section let $G = (V, E)$ be a connected P_6-free graph and let:
B be any stable set of G, and $A = V \setminus B$;
$Z = \{Z_1, \ldots, Z_m\} = \mathcal{C}(A, B)$;
$B^* = \{b \in B : b$ contacts all the elements of $Z\}$.

Lemma 1 $B^* \neq \emptyset$.

Proof: The proof is by induction on the cardinality of Z.

The statement of Lemma 1 is obviously satisfied in case of $|Z| = 1$.

Then let us assume that the statement of Lemma 1 holds true for $|Z| \leq m-1$, and prove that it holds true for $|Z| = m$. Assume to the contrary that $B^* = \emptyset$, i.e., that there exists no vertex of B contacting all the elements of Z. By the inductive assumption, let $b \in B$ contact each element of $\{Z_2, \ldots, Z_m\}$. Then b does not contact Z_1 (by assumption of contradiction). Let $b_1 \in B$ be partial to Z_1. If b_1 contacts some element of $\{Z_2, \ldots, Z_m\}$, then b_1 contacts all the elements of $\{Z_2, \ldots, Z_m\}$: in fact, if $m = 2$, then the assertion follows trivially, while if $m \geq 3$, then the assertion follows since otherwise a P_6 arises involving two vertices of Z_1, b_1 and b. Then b_1 contacts no element of $\{Z_2, \ldots, Z_m\}$ (by assumption of contradiction). Notice that if a vertex v of $V \setminus (\{b_1\} \cup Z_1)$ contacts $\{b_1\} \cup Z_1$, then v can not dominate it: in fact, if $v \in A$ then v does not contact Z_1, while if $v \in B$ then v does not contact $\{b_1\}$ since B is a stable set. The same holds with $\{b\} \cup \{Z_2, \ldots, Z_m\}$ instead of $\{b_1\} \cup Z_1$. Then, since G is connected P_6-free, a shortest path from $\{b_1\} \cup Z_1$ to $\{b\} \cup \{Z_2, \ldots, Z_m\}$ contains exactly one vertex, i.e., there exists a vertex $v' \in V$ contacting both $\{b_1\} \cup Z_1$ and $\{b_2\} \cup \{Z_2, \ldots, Z_m\}$. If $v' \in A$, then two vertices of Z_1, b_1, v', b, and a vertex of Z_2 induce a P_6. If $v' \in B$, then v' contacts both Z_1 and at least one element of $\{Z_2, \ldots, Z_m\}$, say Z_2, while (by assumption of contradiction) v' does not contact at least one element of Z, say Z_m: then a P_6 is induced by a vertex of Z_m, b, a vertex of Z_2, v', a vertex of Z_1, and b_1, a contradiction. □

Then let us focus on the case in which $|Z| \geq 2$.

Lemma 2 *Assume that $|Z| \geq 2$. Let $b^* \in B^*$. Then one of the following statements holds:*

(i) each component-set of $G[V(Z) \setminus N(b^*)]$ is dominated by a vertex of $V(Z) \cap N(b^*)$;

(ii) b^* dominates all the elements of Z except one, say Z_1, and is the endpoint of an induced P_4 formed together with three vertices of Z_1.

Proof: If b^* dominates all the elements of Z, then $V(Z) \setminus N(b^*) = \emptyset$. If b^* does not dominate exactly one element of Z, say Z_1, then since b^* contacts Z_1 (being $b^* \in B$), either statement (i) or statement (ii) holds (to avoid a P_6, observing that b^* is adjacent to a vertex of an element say $Z_2 \neq Z_1$ of Z). If b^* does not dominate at least two elements of Z, then statement (i) holds: in fact, let K be a component of $G[V(Z) \setminus N(b^*)]$; since $b^* \in B^*$, there exists $a \in V(Z) \cap N(b^*)$ contacting K; in particular, a dominates K, otherwise a P_6 arises formed by two vertices of K, a, b^*, and two vertices of an element Z_i of Z not dominated by b^* (for which one has $Z_i \cap K = \emptyset$). □

Lemma 3 *Assume that $|Z| \geq 2$. Assume that no vertex of B^* enjoys (i) of Lemma 2. Let $b^* \in B^*$ enjoy (ii) of Lemma 2. Let $Y = \{b \in B : b$ distinguishes some element of $\mathcal{C}(Z_1 \setminus N(b^*), B)\}$. Let $\bar{y} \in Y$ such that $N_{Z_1 \setminus N(b^*)}(\bar{y})$ is maximal under inclusion in $\{N_{Z_1 \setminus N(b^*)}(y) : y \in Y\}$. Then:*

(i) $\bar{y}$ contacts all the elements of $\mathcal{C}(Z_1 \setminus N(b^*), B)$;

(ii) for all $Q \in \mathcal{C}(Z_1 \setminus N(b^*), B)$, each component-set of $G[Q \setminus N(\bar{y})]$ is dominated by some vertex of $Q \cap N(\bar{y})$.

Proof: Let us prove statement (i). According to (ii) of Lemma 2, b^* dominates all the elements of Z except one, namely Z_1, and is the endpoint of an induced P_4 formed together with three vertices of Z_1. Let $X = \{b \in B : b$ distinguishes some element of $Z \setminus \{Z_1\}\}$. Then for any $Z_i \in Z$ with $i \neq 1$ there exists $x_i \in X$ distinguishing Z_i. Then every vertex of X contacts Z_1 (otherwise a P_6 arises involving any vertex of X, an element of Z contacted by such a vertex, b^* and three vertices of Z_1). It follows that for any two vertices $x_i, x_j \in X$ with x_i distinguishing Z_i and x_j distinguishing Z_j, either x_i contacts Z_j or x_j contacts Z_i (otherwise a P_6 arises). This implies that there exists a vertex $x \in X$ contacting all the elements of $Z \setminus \{Z_1\}$. Then x contacts all the elements of Z. Since no vertex of B enjoys (i) of Lemma 2, x is the endpoint of an induced P_4 of G together with three vertices of an element of $Z \setminus \{Z_1\}$, say Z_2, and dominates Z_i for every $i \neq 2$. Then $x \notin Y$. Then: each vertex of Y contacts Z_2, otherwise any vertex of Y, a vertex of Z_1, x and three vertices of Z_2 induce a P_6. Now, by contradiction assume that $\bar{y}$ does not contact an element Q of $\mathcal{C}(Z_1 \setminus N(b^*), B)$. Let $y \in Y$ be partial to Q. Then, by definition of $\bar{y}$, there exists $z \in Z_1 \setminus N(b^*)$ adjacent to $\bar{y}$ and nonadjacent to y. Notice that z is nonadjacent to any vertex of Q, since z belongs to a different component-set of $G[Z_1 \setminus N(b^*)]$. Then, since both y and $\bar{y}$ contact Z_2, a P_6 arises, a contradiction.

Let us prove statement (ii). Let Q be any element of $\mathcal{C}(Z_1 \setminus N(b^*), B)$ not dominated by $\bar{y}$. By statement (i), $\bar{y}$ contacts Q. Then, each component-set K of $G[Q \setminus N(\bar{y})]$ is contacted by some vertex q of $Q \cap N(\bar{y})$, and in particular is dominated by q, otherwise a P_6 arises formed by two elements of K,q, $\bar{y}$, a vertex of Z_2, and b^*. □

4 Reductions for MWS on (P_6,K_p)-Free Graphs

In this section let us apply the properties of the previous section to introduce two reductions for MWS on (P_6,K_p)-free graphs, for $p \geq 3$, both relying to the following definition.

Definition 1 *A graph $G = (V, E)$ is* locally F-free*, where F is any fixed graph, if V admits a partition $\{A, B\}$ such that $G[A]$ is F-free and B is a stable set.*

The following general observations will be often used later.

Observation 1 *Let $G = (V, E)$ be a graph and W be a subset of V. Assume that MWS can be solved for each induced subgraph of $G[W]$ in polynomial time. Then while showing that MWS can be solved for G in polynomial time, $W \setminus V(\mathcal{C}(W, V \setminus W))$ can be treated as a stable set.*

Proof: In fact one can contract each component-set K of $G[W]$ which is a module of G into a singleton k with $w(k) = \alpha_w(G[K])$. That can be done in polynomial time by assumption. □

Observation 2 *Let $G = (V, E)$ be a graph, and $U \subseteq V$ with $|U| = k$. If one can solve MWS for each subgraph of $G[V \setminus U]$ in time T, then one can solve MWS for G in time $2^k(T + n^2)$.*

Proof: Let $I(U)$ be the family of stable sets of $G[U]$. Then to solve MWS for G one can solve MWS for $|I(U)|$ subgraphs of G, i.e., for $G[V \setminus U]$ and for $G[I \cup (V \setminus (N(I) \cup U))]$ for every $I \in I(U)$. Since $|I(U)| \leq 2^k$, and since $V \setminus (N(I) \cup U)$ can be computed in $O(n^2)$ for every $I \in I(U)$, the assertion follows. □

4.1 From (P_6,K_{p+1})-Free to (P_6, Locally K_p)-Free Graphs

The next lemma shows that the difficulty of solving MWS for (P_6, K_{p+1})-free graphs can be reduced to that of solving MWS for (P_6, locally K_p)-free graphs.

Lemma 4 *If MWS can be solved for (P_6, locally K_p)-free graphs in $O(n^t)$ time for some natural t, then MWS can be solved for (P_6, K_{p+1})-free graphs in $O(n^{t+p-1})$ time.*

Proof: Let $G = (V, E)$ be a (P_6, K_{p+1})-free graph. Assume without loss of generality that G is connected. To prove the lemma, let us show that for any $v \in V$ one can solve MWS for $G[V \setminus N(v)]$ in $O(n^{t+p-2})$ time.

Let $v \in V$. Let H be any component-set of $G[V \setminus N(v)]$. Since G is connected, there exists $u \in N(v)$ contacting H. Let us write: $A = H \cap N(u)$ and $B = H \setminus N(u)$. Then $\{A, B\}$ is a partition of $G[H]$. Since G is K_{p+1}-free, $G[A]$ is K_p-free.

Now to solve MWS for $G[H]$ one can proceed as follows: (i) compute a maximum weight stable set of $G[B]$, (ii) compute a maximum weight stable set of $G[H]$ containing at least one vertex of A, and finally (iii) choose a best solution, i.e., one of maximum weight.

To compute a maximum weight stable set of $G[B]$ one can proceed as follows.

Claim 1. MWS can be solved for $G[B]$ in $O(n^t)$ time.

Proof: To prove the claim, by the assumption of the lemma, it is sufficient to show that $G[B]$ can be treated as a locally K_p-free graph. Let Q be a component-set of $G[B]$. Since $G[H]$ is connected, there exists $a \in A$ such that $Q \cap N(a) \neq \emptyset$. Let $Q_1 = Q \cap N(a)$ and $Q_2 = Q \setminus N(a)$. Then $\{Q_1, Q_2\}$ is a partition of Q. Since G is K_{p+1}-free, $G[Q_1]$ is K_p-free. Let K be any component-set of $G[Q_2]$. Then, to avoid a P_6 involving v, u, a, each vertex of Q_1 either dominates or does not contact K. Then K is a module of $G[Q]$: in particular, since G is K_{p+1}-free, $G[K]$ is K_p-free. Then one can contract K into a singleton k with $w(k) = \alpha_w(K)$: since $G[K]$ is K_p-free that can be done in polynomial time by assumption. Then one can treat Q_2 as a stable set, and the assertion follows. □

To compute a maximum weight stable set of $G[H]$ containing at least one vertex of A one can proceed as follows.

For any $A' \cup B'$ with $A' \subseteq A$ and $B' \subseteq B$, let us define the *brown* decomposition of $A' \cup B'$ by the following two steps:

Step 1. First define a binary relation '$\geq$' on A', such that for any $x, y \in A'$ one has $x \geq y$ if $N_{B'}(x) \supseteq N_{B'}(y)$. Clearly, $(A', \geq)$ is a partially ordered set. Then define a topological total order '$>$' on A' with respect to $(A', \geq)$: let us write $A' = \{a_1, \dots, a_t\}$, with $a_1 > a_2 > \dots > a_t$, so that if $a_i \geq a_j$ then $a_i > a_j$ for any pair of indices i, j.

Step 2. Decompose $A' \cup B'$ into the following subsets: $W_1' = \{a_1\} \cup [A' \setminus N(a_1)] \cup [B' \setminus N(a_1)]$, $W_2' = \{a_2\} \cup [(A' \setminus \{a_1\}) \setminus N(a_2)] \cup [B' \setminus N(a_2)]$, ..., $W_t' = \{a_t\} \cup [(A' \setminus \{a_1, \dots, a_{t-1}\}) \setminus N(a_t)] \cup [B' \setminus N(a_t)]$. Let us say that sets W_k' for $k = 1, \dots, t$ are the *children* of $A' \cup B'$.

Let us observe that any maximum weight stable set of $G[A' \cup B']$ is contained in (exactly) one of the graphs $G[W_1'], \dots, G[W_t']$; in particular MWS can be efficiently solved for $G[A' \cup B']$ as soon as it can be so for graphs $G[W_1'], \dots, G[W_t']$, i.e., one has $\alpha_w G[A' \cup B'] = \max_{k=1,\dots,t}\{\alpha_w G[W_k']\}$.

A sufficient condition to efficiently solve MWS for graph $G[W_k']$, for some $k = 1, \dots, t$, is the following: If no vertex of $\{a_k\} \cup (A' \setminus \{a_1, \dots, a_{k-1}\}) \setminus N(a_k)$ distinguishes any component-set of $B' \setminus N(a_k)$, then by Observation 1 and Claim 1, $B' \setminus N(a_k)$ can be treated as a stable set: then $G[W_k']$ can be treated as a locally K_p-free graph and MWS can be solved for $G[W_k']$ in $O(n^t)$ time by assumption. Let us say that $G[W_k']$ is a *good subgraph* of $G[H]$ if this fact holds.

Then let us consider the following recursive Procedure Brown $(A \cup B)$:

(1) Apply the brown decomposition to $A \cup B$.

(2) For every child W_k of $A \cup B$, with $k = 1, \dots, |A|$, do: if $G[W_k]$ is not a good subgraph of G, then apply Procedure Brown (W_k).

This procedure may be represented by a tree $T(H)$ whose root is $A \cup B$, the children of $A \cup B$ are $W_1, W_2, \dots, W_t$, which are in turn the roots of subtrees representing the (possible) brown decomposition of $W_1, W_2, \dots, W_t$. Each leaf of $T(H)$ corresponds to a good subgraph of $G[H]$.

Claim 2. Tree $T(H)$ contains $O(n^{p-2})$ nodes.

Proof: To prove the claim, let us show that the depth of $T(H)$ is less or equal p.

Let $W'_k = \{a_k\} \cup [(A' \setminus \{a_1, \ldots, a_{k-1}\}) \setminus N(a_k)] \cup [B' \setminus N(a_k)]$ be a node of $T(H)$ having a leaf of $T(H)$ as a child. Then $G[W'_k]$ is not good, i.e., there exists a vertex $x \in (A' \setminus \{a_1, \ldots, a_{k-1}\}) \setminus N(a_k)$ distinguishing a component-set Q of $G[B' \setminus N(a_k)]$. Let $A' \cup B'$ be the father of W'_k. As $x \in (A' \setminus \{a_1, \ldots, a_{k-1}\}) \setminus N(a_k)$, with respect to the topological total order '>' one has $a_k > x$ (otherwise $x \in \{a_1, \ldots, a_{k-1}\}$). Thus, since x is adjacent to a vertex of $B' \setminus N(a_k)$, there exists a vertex $b_1 \in B'$ adjacent to a_k and nonadjacent to x. Then to avoid that $b_1, a_k, u, x,$ and two vertices of Q induce a P_6, b_1 contacts Q. To avoid that $v, u, a_k, b_1,$ and two vertices of Q induce a P_6, one has that: b_1 dominates Q and in particular $Q \cup \{b_1\}$ contains a K_3 (since Q is nontrivial); while a_k distinguishes $Q \cup \{b_1\}$.

Notice that in $T(H)$ if a node $A'' \cup B''$ is an ancestor of $A' \cup B'$, then $A'' \supset A'$ and $B'' \supset B'$. Then the argument of the previous paragraph can be iterated for the father of $A' \cup B'$, and so on until to reach the root of $T(H)$. At each iteration q, a new vertex b_q is detected such that b_q dominates $Q \cup \{b_1, \ldots, b_{q-1}\}$, that is, a K_{q+2} arises. Since G is K_{p+1}-free, at most $p-2$ iterations are possible, and the claim follows. □

By Claim 2, since by the above MWS can be solved for subgraphs of $G[H]$ corresponding to the leaves of $T(H)$ (i.e., the good subgraphs of $G[H]$) in $O(n^t)$ time, since by the above MWS can be solved for subgraphs of $G[H]$ corresponding to the internal nodes of $T(H)$ as soon as it can be solved for their children, and since the root of $T(H)$ corresponds to $G[H]$, one has that a maximum weight stable set of $G[H]$ containing at least one vertex of A can be computed in $O(n^{t+p-2})$.

This completes the proof of the lemma.

□

4.2 From (P_6, Locally K_p)-Free to (P_6, K_p)-Free Graphs (with one exception)

The next lemma shows that the difficulty of solving MWS for (P_6, locally K_p)-free graphs, with related partition $\{A, B\}$, can be reduced to that of solving MWS for (P_6, K_p)-free graphs, except for the case in which $|\mathcal{C}(A, B)| = 1$.

Lemma 5 *Let $G = (V, E)$ be a connected (P_6, locally K_p)-free graph, with related partition $\{A, B\}$. If MWS can be solved for (P_6,K_p)-free graphs in $O(n^t)$ time for some natural t, then:*

(i) if $|\mathcal{C}(A, B)| = 0$, then MWS can be solved for G in $O(n^t)$ time;

(ii) if $|\mathcal{C}(A, B)| \geq 2$, then there exists $v \in V$ (i.e., $v \in B$ and is easily detectable) such that MWS can be solved for $G[V \setminus \{v\}]$ in $O(n^{t+1})$ time.

Proof: Let us recall that, by definition of locally K_p-free graph, $G[A]$ is K_p-free and B is a stable set.

Let us prove statement (i). If $|\mathcal{C}(A, B)| = 0$, then by Observation 1 and since MWS can be solved for (P_6,K_p)-free graphs in $O(n^t)$ time, A can be treated as a stable set in $O(n^t)$ time. Then G can be treated as a bipartite graph and the statement follows.

Let us prove statement (ii). For brevity, let us write $Z = \mathcal{C}(A, B)$. By Observation 1 and by assumption, $A \setminus V(Z)$ can be treated as a stable set in $O(n^t)$ time.

By Lemma 1, there exists at least one vertex of B contacting all the elements of Z. Let $B^* = \{b \in B : b$ contacts all the elements of $Z\}$.

If there is $b \in B^*$ enjoying (i) of Lemma 2, then since $G[A]$ is K_p-free each component of $G[V(Z) \setminus N(b)]$ is K_{p-1}-free: then $G[V \setminus N(b)]$ is K_p-free, and MWS can be solved for $G[V \setminus N(b)]$ in $O(n^t)$ time by assumption.

Then assume that no vertex of B^* enjoys (i) of Lemma 2. Let $b^* \in B^*$ enjoy (ii) of Lemma 2.

Then b^* dominates all the elements of Z apart from one, say Z_1, and is partial to Z_1. Let $Y = \{b \in B : b$ distinguishes some element of $\mathcal{C}(Z_1 \setminus N(b^*), B)\}$. Let '$\geq$' be a binary relation on Y, such that for any $u, v \in Y$ one has $u \geq v$ if $N_{Z_1 \setminus N(b^*)}(u) \supseteq N_{Z_1 \setminus N(b^*)}(v)$. Clearly, $(Y, \geq)$ is a partially ordered set. Then one can define a topological total order '$>$' on Y with respect to $(Y, \geq)$: let us write $Y = \{y_1, \ldots, y_h\}$, with $y_1 > y_2 > \ldots > y_t$, so that if $y_i \geq y_j$ then $y_i > y_j$ for any pair of indices i, j.

One can solve MWS in $G[V \setminus N(b^*)]$ by solving MWS in: $G[\{y_1\} \cup (V \setminus N(b^*) \setminus N(y_1))]$, in $G[\{y_i\} \cup (V \setminus N(b^*) \setminus \{y_1, \ldots, y_{i-1}\} \setminus N(y_i))]$ for $i = 2, \ldots, h$, and in $G[V \setminus N(b^*) \setminus Y]$. Concerning MWS for $G[\{y_1\} \cup (V \setminus N(b^*) \setminus N(y_1))]$: by Observation 1 and since MWS can be solved for (P_6,K_p)-free graphs in $O(n^t)$ time, $Z_1 \setminus N(b^*) \setminus V(\mathcal{C}(Z_1 \setminus N(b^*), B)$ can be treated as a stable set; then, by Lemma 3, each component of $G[Z_1 \setminus N(b^*) \setminus N(y_1)]$ can be treated as a K_{p-1}-free graph; then $G[\{y_i\} \cup (V \setminus N(b^*) \setminus N(y_1))]$ can be treated as a K_p-free graph, and MWS can be solved for $G[V \setminus N(b^*) \setminus N(y_1)]$ in $O(n^t)$ time by assumption. Similarly, by the mentioned total order, MWS can be solved for each of the remaining mentioned graphs in $O(n^t)$ time. Then MWS can be solved for $G[V \setminus N(b^*)]$ in $O(n^{t+1})$ time. □

Let us point out the following easy sub-result of Lemma 5, referred to k-colorable graphs, which will be useful later.

Lemma 6 *Let $G = (V, E)$ be a connected k-colorable P_6-free graph $G = (V, E)$, where $\{A, B\}$ is a partition of V with $G[A]$ $(k-1)$-colorable and B stable set. If MWS can be solved for $(k-1)$-colorable P_6-free graphs, in $O(n^t)$ time for some natural t, then:*

(i) if $|\mathcal{C}(A, B)| = 0$, then MWS can be solved for G in $O(n^t)$ time;

(ii) if $|\mathcal{C}(A, B)| \geq 2$, then there exists $v \in V$ (i.e., $v \in B$ and is easily detectable) such that MWS can be solved for $G[V \setminus \{v\}]$ in $O(n^{t+1})$ time. □

5 Some Polynomially Solvable Cases for MWS

In this section let us prove that MWS can be solved for P_6-free graphs being respectively (i) 3-colorable, (ii) 4-colorable, and (iii) K_4-free. The case (i) is somehow basic and is applied in the other two cases. The cases (i)-(ii) require Lemma 6, while the case (iii) requires Lemma 4.

5.1 MWS for 3-Colorable P_6-Free Graphs

To solve MWS for 3-colorable P_6-free graphs, let us follow the idea of Brandstädt and Hoàng in [8], where the authors combine two decomposition approaches, namely by homogeneous sets and by clique separators, in order to obtain a binary tree which gives a refinement of the decompositions obtained separately. Here we combine two "local" decomposition approaches, namely by Theorem 1 of Fouquet, Giakoumakis and Vanherpe on bipartite P_6-free graphs and by Lemma 6, in order to obtain a binary tree which gives a refinement of the decompositions obtained separately.

Let $G = (V, E) = (V_1 \cup V_2 \cup V_3, E)$ be a 3-colorable P_6-free graph. Let us write: $A = V_1 \cup V_2$; $B = V_3$; $Z = \mathcal{C}(A, B)$.

By Observation 1 and Theorem 2, $A \setminus V(Z)$ can be treated as a stable set: in particular, if $|Z| = 0$, then G can be treated as a bipartite graph.

The method will adopt the following four decompositions of a 3-colorable P_6-free graph G into at most four subgraphs G_1, G_2, G_3, G_4, in order that if the MWS problem can be solved for G_1, G_2, G_3, G_4 in polynomial time, then so can the problem for G.

Case 1. If G is disconnected, then G is decomposed into subgraphs $G_1 = G[W_1]$ and $G_2 = G[W_2]$, where $\{W_1, W_2\}$ is any partition of V such that W_1 has a co-join to W_2 in G. In particular $\alpha_w(G) = \alpha_w(G_1) + \alpha_w(G_2)$.

Case 2. If G is connected, $|Z| = 1$ (i.e., let $Z = \{T\}$ where $G[T] = (S_1 \cup S_2, F)$ is a connected bipartite graph), and $G[T]$ fulfills the condition (i) of Theorem 1 (i.e., let $\{L, R\}$ be any partition of T such that L has a co-join to R in $\overline{G[T]}^{bip}$), then G is decomposed into subgraphs $G_1 = G[W_1]$, $G_2 = G[W_2]$, $G_3 = G[W_3]$, $G_4 = G[W_4]$, where $W_1 = V \setminus V_1$, $W_2 = V \setminus S_2$, $W_3 = V \setminus L$, $W_4 = V \setminus R$. In particular, $\alpha_w(G) = \max_{i=1,\dots,4}\{\alpha_w(G_i)\}$.

In fact let S be a maximum (weight) stable set of G: if S contains no vertex from S_1, then $S \subseteq W_1$; if S contains no vertex from S_2, then $S \subseteq W_2$; if F contains both a vertex from S_1 and a vertex from S_2, then either $S \subseteq W_3$ or $S \subseteq W_4$, by the condition (i) of Theorem 1.

Case 3. If G is connected, $|Z| = 1$ (i.e., let $Z = \{T\}$ where $G[T] = (S_1 \cup S_2, F)$ is a connected bipartite graph), and $G[T]$ fulfills the condition (ii) of Theorem 1, then G is decomposed into subgraphs $G_1 = G[W_1]$ and $G_2 = G[W_2]$, where $W_1 = V \setminus S_1^*$ and $W_2 = V \setminus S_2^*$. In particular, $\alpha_w(G) = \max_{i=1,2}\{\alpha_w(G_i)\}$.

In fact let S be a maximum (weight) stable set of G: if S contains no vertex from S_1^*, then $S \subseteq W_1$; if S contains no vertex from S_2^*, then $S \subseteq W_2$; the case in which S contains both a vertex from S_1^* and a vertex from S_2^* can not occur, since S_1^* has a join to S_2^*, by the condition (ii) of Theorem 1.

Case 4. If G is connected, $|Z| \geq 2$, and $B \neq \emptyset$ (i.e., G admits a vertex v according to Lemma 6 (ii)), then G is decomposed into subgraphs $G_1 = G[W_1]$ and $G_2 = G[W_2]$, where $W_1 = V \setminus N(v)$ and $W_2 = V \setminus \{v\}$. Let us say that the graph G_1 obtained by such a decomposition is a *good subgraph* of G. In particular, $\alpha_w(G) = \max_{i=1,2}\{\alpha_w(G_i)\}$.

Then let us consider the following recursive Procedure Gray (G):

(1) Decompose G according to one of the above four decompositions, depending on the case, so to obtain graphs (at most) G_1, G_2, G_3, G_4.

(2) For every G_k, with $k = 1, \dots, 4$, do: if G_k is not bipartite and if G_k is not a good subgraph of G, then apply Procedure Gray (G_k).

This procedure may be represented by a tree $T(G)$ whose root is G, the children of G are (at most) G_1, G_2, G_3, G_4, which are in turn the roots of subtrees representing the (possible) decomposition of G_1, G_2, G_3, G_4. Each leaf of $T(G)$ corresponds either to a bipartite subgraph of G or to a good subgraph of G.

Lemma 7 *Tree $T(G)$ contains $O(n^3)$ nodes.*

Proof: Let us show that each internal node of $T(G)$ can be labeled with a distinct 3-tuple (x, y, z) where x, y, z are three vertices of G. One only needs to label internal nodes that correspond to graphs with at least three vertices.

Let G_H denote the induced subgraph of G that corresponds to an internal node H of $T(G)$.

If G is decomposed by Case 1 into graphs G_1, G_2, then label H with (x, y, z) where: x is any vertex in G_1, while y, z are two adjacent vertices of A in G_2. Let us say H is a *node of type* 1.

If G is decomposed by Case 2, then label H with (x, y, z) where: x is any vertex in G, y is any vertex of L, z is any vertex of R, such that y is adjacent to z. Let us say H is a *node of type* 2.

If G is decomposed by Case 3, then label H with (x, y, z) where: x is any vertex of G, y is any element of S_1^*, z is any element of S_2^* (then y is adjacent to z). Let us say H is a *node of type* 3.

If G is decomposed by Case 4, then label H with (x, y, z) where: $x = v$, while y, z are two adjacent vertices of A. Let us say H is a *node of type* 4.

Assume that there are two nodes H, K in $T(G)$ with the same 3-tuple (x, y, z); in particular one has $x, y, z \in G_H \cap G_K$.

Suppose first that K is a descendent of H. The above choice of the labels implies that, whether H is of type 1 or 2 or 3 or 4, there is at least one vertex in the label of G_H that does not belong to G_K, a contradiction.

Now, one may assume that K is not a descendent of H and H is not a descendent of K. Let J be the lowest common ancestor of H and K in $T(G)$. For simplicity, one may assume that $H(K)$ either is the left (right) child of J, or is a descendent of the left (right) child of J. If J is a node of type 1, then H and K have no vertex in common, and thus cannot have the same 3-tuple. If J is a node of type 2 or 3, then $G_H[A]$ and $G_K[A]$ have no edge (i.e., no pair of adjacent vertices) in common; then, since for each internal node Q of $T(G)$ the elements y, z of the 3-tuple (x, y, z) of Q are adjacent vertices of $G_Q[A]$, one has that H and K can not have the same 3-tuple (x, y, z). If J is a node of type 4, then either H or K is a yellow subgraph of G, i.e., a leaf of $T(G)$, a contradiction since H and K are internal nodes of $T(G)$. $\Box$

Then Lemma 7 implies:

Corollary 1 *If the MWS problem can be solved in polynomial time for every bipartite subgraph of G and for every good subgraph of G, then so can the problem for G.* $\Box$

By Theorem 2 MWS can be solved for bipartite subgraphs of G in $O(n + m)$ time. By Lemma 6 MWS can be solved for good subgraphs of G in $O(nm)$ time. Then Corollary 1 implies:

Theorem 4 *The MWS problem can be solved for 3-colorable P_6-free graphs in $O(n^4 m)$ time.* $\Box$

5.2 MWS for 4-Colorable P_6-Free Graphs

Let $G = (V, E) = (V_1 \cup V_2 \cup V_3 \cup V_4, E)$ be a 4-colorable P_6-free graph. Let us write: $A = V_1 \cup V_2$, $B = V_3$, $C = V_4$.

Lemma 8 *If $|\mathcal{C}(A, B)| = 0$ or $|\mathcal{C}(A, C)| = 0$, then MWS can be solved for G in $O(n^6 m)$ time.*

Proof: By symmetry, the proof is given only for the case in which $|\mathcal{C}(A, C)| = 0$. Without loss of generality, assume that G is connected. To prove the lemma let us show that either MWS can be solved for G in $O(n^4 m)$ time, or there exists a vertex $v \in V$ such that MWS can be solved for $G[V \setminus N(v)]$ in $O(n^5 m)$ time. Then one can (possibly) iteratively apply this argument to each component of $G[V \setminus \{v\}]$.

Assume that $|\mathcal{C}(A \cup B, C)| = 0$. Then by Lemma 5 (i) and Theorem 4, MWS can be solved for G in $O(n^4m)$ time.

Assume that $|\mathcal{C}(A \cup B, C)| \geq 2$. Then by Lemma 5 (ii) and Theorem 4, there exists $c \in C$ such that MWS can be solved for $G[V \setminus N(c)]$ in $O(n^5m)$ time.

Assume that $|\mathcal{C}(A \cup B, C)| = 1$. Let us write $\mathcal{C}(A \cup B, C) = \{Q\}$. By Observation 1 and Theorem 4, $(A \cup B) \setminus Q$ can be treated as a stable set in $O(n^4m)$ time.

If $|\mathcal{C}(A \cap Q, B \cap Q)| = 0$, then since $|\mathcal{C}(A, C)| = 0$ (and thus $|\mathcal{C}(A \cap Q, C)| = 0$), by Observation 1 and Theorem 2, $A \cap Q$ can be treated as a stable set; then G can be treated as a 3-colorable graph and MWS can be solved for G in $O(n^4m)$ time.

If $|\mathcal{C}(A \cap Q, B \cap Q)| = 1$, then let us write $\mathcal{C}(A \cap Q, B \cap Q) = \{T\}$. Since $|\mathcal{C}(A, C)| = 0$ (and thus $|\mathcal{C}(A \cap Q, C)| = 0$), by Observation 1 and Theorem 2, $(A \cap Q) \setminus T$ can be treated as a stable set. If no vertex of C contacts T, then G is 3-colorable and MWS can be solved for G in $O(n^4m)$ time. If some vertex of C, say c, contacts T, then c dominates T since $|\mathcal{C}(A, C)| = 0$; then $G[V \setminus N(c)]$ is 3-colorable and MWS can be solved for $G[V \setminus N(c)]$ in $O(n^4m)$ time.

If $|\mathcal{C}(A \cap Q, B \cap Q)| \geq 2$, then since $|\mathcal{C}(A, C)| = 0$ and $G[(A \cap Q) \cup (B \cap Q)]$ is connected being $G[Q]$ connected, one can apply an argument similar to that of Lemma 5 to show that there exists $b \in B$ such that MWS can be solved for $G[V \setminus N(b)]$ in $O(n^5m)$ time (i.e., by reducing the problem to instances of 3-colorable graphs). □

Let us say that G is C-*compact* if the following facts hold: (i) $|\mathcal{C}(A \cup B, C)| = 1$, say $\mathcal{C}(A \cup B, C) = \{Q\}$; (ii) $|\mathcal{C}(A \cap Q, B)| = |\mathcal{C}(A \cap Q, C)| = 1$; (iii) $\mathcal{C}(A \cap Q, B) = \mathcal{C}(A \cap Q, C)$.

Lemma 9 *If G is not C-compact, then there exists $v \in V$ (easily detectable) such that MWS can be solved for $G[V \setminus N(v)]$ in $O(n^7m)$ time.*

Proof: Without loss of generality, assume that G is connected.

Assume that $|\mathcal{C}(A \cup B, C)| = 0$. Then by Lemma 5 (i) and Theorem 4, MWS can be solved for G in $O(n^4m)$ time.

Assume that $|\mathcal{C}(A \cup B, C)| \geq 2$. Then by Lemma 5 (ii) and Theorem 4, there exists $v \in C$ such that MWS can be solved for $G[V \setminus N(v)]$ in $O(n^5m)$ time.

Assume that $|\mathcal{C}(A \cup B, C)| = 1$, say $\mathcal{C}(A \cup B, C) = \{Q\}$. By Observation 1 and Theorem 4, $(A \cup B) \setminus Q$ can be treated as a stable set: then from now on let us assume that $G[A \cup B]$ is formed by one nontrivial component, namely Q, and by isolated vertices.

Let $|\mathcal{C}(A \cap Q, B \cap Q)| = 0$. Then $|\mathcal{C}(A, B)| = 0$ and by Lemma 8 MWS can be solved for G in time $O(n^6m)$.

Let $|\mathcal{C}(A \cap Q, B \cap Q)| = 1$, say $\mathcal{C}(A \cap Q, B \cap Q) = \{T\}$. Since G is not C-compact, $\mathcal{C}(A \cap Q, C) \neq \{T\}$. Then one of the following two cases occurs: (i) there exists a vertex $c \in C$ (contacting and so) dominating T: then $G[V \setminus N(c)]$ enjoys the assumption of Lemma 8 (since $|\mathcal{C}((A \cap Q) \setminus N(c), (B \cap Q) \setminus N(c))| = 0$) and MWS can be solved for $G[V \setminus N(c)]$ in $O(n^6m)$ time; (ii) there exists no vertex of C contacting T: then (recalling that $A = V_1 \cup V_2$, and that $(A \cup B) \setminus Q$ is a stable set) one can re-define $A := A \setminus (V_2 \cap T) \setminus ((A \cup B) \setminus Q)$, $B := B \cup ((A \cup B) \setminus Q)$ and $C := C \cup (V_2 \cap T)$, so that $|\mathcal{C}(A \cap Q, B \cap Q)| = 0$ and then G enjoys the assumptions of Lemma 8 and MWS can be solved for G in time $O(n^6m)$.

Let $|\mathcal{C}(A \cap Q, B \cap Q)| \geq 2$. Let us write: $A' = A \cap Q$, $B' = B \cap Q$, $\mathcal{C}(A', B') = Z$.

By Lemma 1 (which can be applied since Q is connected), there exists at least one vertex of B' contacting all the elements of Z. Let $B^* = \{b \in B' : b \text{ contacts all the elements of } Z\}$.

If there is $b \in B^*$ enjoying (i) of Lemma 2, then since $G[A']$ is bipartite each component-set of $G[A' \setminus N(b)]$ is a stable set: then each component-set of $G[A' \setminus N(b)]$ is not distinguished by any vertex of B'. Then $G[V \setminus N(b)]$ enjoys the assumption of Lemma 8, and MWS can be solved for $G[V \setminus N(b)]$ in $O(n^6 m)$ time.

Then assume that no vertex of B^* enjoys (i) of Lemma 2. Let $b^* \in B^*$ enjoy (ii) of Lemma 2.

Then b^* dominates all the elements of Z apart from one, say Z_1, and is partial to Z_1. Let $Y = \{b \in B' :$ b distinguishes some element of $\mathcal{C}(Z_1 \setminus N(b^*), B)\}$. Let '$\geq$' be a binary relation on Y, such that for any $u, v \in Y$ one has $u \geq v$ if $N_{Z_1 \setminus N(b^*)}(u) \supseteq N_{Z_1 \setminus N(b^*)}(v)$. Clearly, $(Y, \geq)$ is a partially ordered set. Then one can define a topological total order '$>$' on Y with respect to $(Y, \geq)$: let us write $Y = \{y_1, \ldots, y_h\}$, with $y_1 > y_2 > \ldots > y_t$, so that if $y_i \geq y_j$ then $y_i > y_j$ for any pair of indices i, j.

One can solve MWS in $G[V \setminus N(b^*)]$ by solving MWS in: $G[\{y_1\} \cup (V \setminus N(b^*) \setminus N(y_1))]$, in $G[\{y_i\} \cup (V \setminus N(b^*) \setminus \{y_1, \ldots, y_{i-1}\} \setminus N(y_i))]$ for $i = 2, \ldots, h$, and in $G[V \setminus N(b^*) \setminus Y]$. Concerning MWS for $G[\{y_1\} \cup (V \setminus N(b^*) \setminus N(y_1))]$: by Lemma 3 and since $G[A']$ is bipartite, each component-set of $G[Z_1 \setminus N(b^*) \setminus N(y_1)]$ is either a stable set or is not distinguished by any vertex of B' (anyway): then each component-set of $G[A' \setminus N(b^*) \setminus N(y_1)]$ is not distinguished by any vertex of B'; then $G[\{y_1\} \cup (V \setminus N(b^*) \setminus N(y_1))]$ enjoys the assumption of Lemma 8, and MWS can be solved for $G[\{y_1\} \cup (V \setminus N(b^*) \setminus N(y_1))]$ in $O(n^6 m)$ time. Similarly, by the mentioned total order, MWS can be solved in each of the remaining mentioned graphs in $O(n^6 m)$ time. Then MWS can be solved for $G[V \setminus N(b^*)]$ in $O(n^7 m)$ time. □

At this point, MWS can be solved for G by a decomposition approach similar to that introduced for 3-colorable P_6-free graphs in the previous subsection. In details:

If G is disconnected, then decompose G according to Case 1.

If G is connected and is C-compact (i.e., let $\mathcal{C}(A \cap Q, B) = \mathcal{C}(A \cap Q, C) = \{T\}$, where $G[T] = (S_1 \cup S_2, F)$ is a connected bipartite graph), then decompose G according either to Case 2 (if $G[T]$ fulfills the condition (i) of Theorem 1), or to Case 3 (if $G[T]$ fulfills the condition (ii) of Theorem 1).

If G is connected and is not C-compact, then decompose G according to Case 4, by referring to Lemma 9 instead of Lemma 5.

Theorem 5 *The MWS problem can be solved for 4-colorable P_6-free graphs in $O(n^{10} m)$ time.* □

5.3 MWS for (P_6, K_4)-Free Graphs

Lemma 10 *Let $G = (V, E)$ be a connected (P_6, locally triangle)-free graph, with related partition $\{A, B\}$. If $|\mathcal{C}(A, B)| = 1$, then there exists a vertex $b \in B$ (easily detectable) such that MWS can be solved for $G[V \setminus N(b)]$ in $O(n^5 m)$ time.*

Proof: Let us recall that, by definition of locally triangle-free graph (i.e., locally K_3-free graph), $G[A]$ is triangle-free and B is a stable set.

Let $\mathcal{C}(A, B) = \{Q\}$. By Observation 1 and Theorem 3, $A \setminus Q$ can be treated as a stable set.

If $G[Q]$ is C_5-free, then $G[Q]$ is bipartite, i.e., G is 3-colorable and the lemma follows by Theorem 4.

Then assume that $G[Q]$ contains a C_5, say C, with vertices v_i and edges $v_i v_{i+1}$, $i \in \{0, \ldots, 4\}$ (index arithmetic modulo 5). Let $N(C)$ be the set of vertices from $Q \setminus C$ which are adjacent to some vertex of C. For any subset S of C, let M_S be the set formed by those elements of $N(C)$ which are adjacent to each element of S and are nonadjacent to each element of $C \setminus S$. In particular, let us write M_1 for $S = \{v_1\}$,

$M_{1,2}$ for $S = \{v_1, v_2\}$, and so on. Then let us denote as $Z(k)$ the set of the vertices of $V \setminus C$ with exactly k neighbors in C.

Since $G[Q]$ is triangle-free: $Z(5) \cup Z(4) \cup Z(3) = \emptyset$; each vertex of $Z(2)$ belongs to some of the sets $M_{i,i+2}$, $i \in \{0, \dots, 4\}$ (index arithmetic modulo 5); the sets M_i and $M_{i,i+2}$, $i \in \{0, \dots, 4\}$ (index arithmetic modulo 5), are stable sets. Since $G[Q]$ is P_6-free: M_i has a co-join to $M_{i-1} \cup M_{i+1}$, and has a join to $M_{i-2} \cup M_{i+2}$, for every $i \in \{0, \dots, 4\}$ (index arithmetic modulo 5). Since $G[Q]$ is (P_6,triangle)-free: $Z(0)$ has a co-join to $Z(1)$; if a vertex $z \in Z(2)$ contacts a component-set K of $Z(0)$, then z dominates K; (consequently, since $G[Q]$ is triangle-free) $Z(0)$ is a stable set.

Let us fix any vertex of C, say v_2, and let us prove that MWS can be solved for $G[V \setminus N(v_2)]$ in $O(n^5 m)$.

A partition of $Q \setminus N(v_2)$ is given by $\{\{v_2, v_4, v_5\}, M_1, M_3, M_4, M_5, M_{1,3}, M_{1,4}, M_{3,5}, Z(0)\}$. Then, by Observation 2, to our aim it is sufficient to prove that MWS can be efficiently solved for (each induced subgraph of) $G[U \cup B]$, where a partition of U is given by $\{M_1, M_3, M_4, M_5, M_{1,3}, M_{1,4}, M_{3,5}, Z(0)\}$.

Case 1. $M_{1,4} \cup M_{3,5} = \emptyset$

MWS can be solved in $G[U \cup B]$ by solving MWS in $G[(U \setminus M_1) \cup B]$ and in $G[(U \setminus N(a)) \cup B]$, for every $a \in M_1$.

That can be done in $O(n^5 m)$ time. First, let us consider $G[(U \setminus M_1) \cup B]$. Notice that $\{M_3 \cup M_{1,3}, M_4 \cup M_5 \cup Z(0)\}$ is a bipartition of $U \setminus M_1$. Then $G[(U \setminus M_1) \cup B]$ is 3-colorable, and the assertion follows by Theorem 4. Then, let us consider $G[U \setminus N(a)]$ for some $a \in M_1$. Notice that $(M_3 \cup M_4) \setminus N(a) = \emptyset$, otherwise a P_6 arises. Thus $\{(M_1 \cup M_{1,3}) \setminus N(a), (M_5 \cup Z(0)) \setminus N(a)\}$ is a bipartition of $U \setminus N(a)$. Then $G[(U \setminus N(a)) \cup B]$ is 3-colorable, and the assertion follows by Theorem 4.

Case 2. $M_{1,4} \cup M_{3,5} \neq \emptyset$

MWS can be solved in $G[U \cup B]$ by solving MWS in $G[(U \setminus (M_{1,4} \cup M_{3,5})) \cup B]$ and in $G[(U \setminus N(a)) \cup B]$, for every $a \in (M_{1,4} \cup M_{3,5})$.

That can be done in $O(n^5 m)$ time. Concerning $G[U \setminus (M_{1,4} \cup M_{3,5}) \cup B]$, one can refer to Case 1. Then let us consider $G[(U \setminus N(a)) \cup B]$, for some $a \in (M_{1,4} \cup M_{3,5})$. Assume without loss of generality, by symmetry, that $a \in M_{3,5}$. Since G is P_6-free: either $M_1 \setminus N(a) = \emptyset$ or $M_4 \setminus N(a) = \emptyset$; $M_5 \setminus N(a)$ has a co-join to $M_{1,4} \setminus N(a)$. At this point, one can verify $-$ by a not complicated check which is omitted here $-$ that $G[U \setminus N(a)]$ contains no C_5, i.e., $G[U \setminus N(a)]$ is bipartite. Then $G[(U \setminus N(a)) \cup B]$ is 3-colorable, and the assertion follows by Theorem 4.

□

Lemma 11 *The MWS problem can be solved for (P_6, locally triangle)-free graphs in $O(n^6 m)$ time.*

Proof: Let G be a (P_6, locally triangle)-free graph. To prove the lemma let us show that either MWS can be solved for G in $O(n^6 m)$ time, or there exists a vertex $v \in V$ such that MWS can be solved for $G[V \setminus N(v)]$ in $O(n^5 m)$ time. Then one can (possibly) iteratively apply this argument to each component of $G[V \setminus \{v\}]$.

By Observation 1 and Theorem 3, $A \setminus V(\mathcal{C}(A, B))$ can be treated as a stable set. If $|\mathcal{C}(A, B)| = 0$, then by Lemma 10 (i) MWS can be solved for G in $O(n^3)$ time. If $|\mathcal{C}(A, B)| = 1$, then by Lemma 10 (ii)

there exists $v \in V$ such that one can solve MWS in $O(n^5 m)$ time. If $|\mathcal{C}(A, B)| \geq 2$, then by Lemma 5 and Theorem 3 there exists $v \in V$ such that MWS can be solved for $G[V \setminus N(v)]$ in $O(n^3)$ time. □

By Lemmas 4 and 11 one obtains the following fact.

Theorem 6 *The MWS problem can be solved for (P_6, K_4)-free graphs in $O(n^9 m)$ time.* □

6 Conclusions

Let us formalize as possible open problem the natural extension of the polynomial results of this article.

Open Problem. What is the complexity of MWS for P_6-free graphs which are either k-colorable for $k \geq 5$ or K_p-free for $p \geq 5$?

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7 Appendix: Stable sets in P_6-free graphs II

This appendix is to introduce two lemmas which are not applied in the article. These lemmas actually are the report of an unsuccessfully attempt to extend the polynomial results of this article. However we hope that they could be useful to a reader for new results on this topic.

Let us adopt the notation of Section 3. Throughout this appendix also let:
$W = A \setminus V(Z)$, with $W \neq \emptyset$;
$b^* \in B^*$, such that $N_W(b^*)$ is maximal under inclusion in $\{N_W(b) : b \in B^*\}$;
$B_0 = \{b \in B : b$ is adjacent to some vertex of $W \setminus N(b^*)\}$.

Lemma 12 $B_0 \subset B^*$.

Proof: Let $b_0 \in B_0$ be adjacent to $w \in W \setminus N(b^*)$. Let us prove the following claim.

Claim 1. b_0 contacts at least one element of Z.

Proof: Assume that b_1 is nonadjacent to w. Since G is connected P_6-free and B is a stable set, there exists a vertex $c \in V$ contacting both $\{b_0, w\}$ and $\{b_1\} \cup Z_1$: in particular $c \in B$, otherwise a P_6 arises involving two vertices of Z_1. Then c is adjacent to w (thus $c \neq b^*$), and contacts Z_1. Note that there exists $z_1 \in Z_1$ such that z_1 is adjacent to both c and b_1, otherwise a P_6 arises involving b_0, w, c, two vertices of Z_1, and b_1. Similarly, there exists $z_1^* \in Z_1$ such that z_1^* is adjacent to both c and b^*. It follows that $c \in B^*$: in fact, if c does not contact an element Z_i of Z, $i \neq 1$, then b_0, w, c, z_1^*, b^* and a vertex of Z_i induce a P_6. Since c is adjacent to w, by definition of b^* there exists $w^* \in W$ such that w^* is adjacent to b^* and nonadjacent to c. Then w^* is adjacent to b_0, otherwise $w^*, b^*, z_1^*, c, w, b_0$

induce a P_6. Then w^* is adjacent to b_1, otherwise w^*, b_0, w, c, z_1, b_1 induce a P_6. But then w, b_0, w^*, b_1 and two vertices of Z_1 induce a P_6, a contradiction.

Assume that b_1 is adjacent to w. Note that there exists $z_1 \in Z_1$ such that z_1 is adjacent to both b_1 and b^*, otherwise a P_6 arises involving b_0, w, b_1, two vertices of Z_1, and b^*. It follows that $b_0 \in B^*$: in fact, if b_1 does not contact an element Z_i of Z, $i \neq 1$, then b_0, w, b_1, z_1, b^* and a vertex of Z_i induce a P_6. Since b_1 is adjacent to w, by definition of b^* there exists $w^* \in W$ such that w^* is adjacent to b^* and is nonadjacent to b_1. Then w^* is adjacent to b_0, otherwise $w^*, b^*, z_1, b_1, w, b_0$ induce a P_6. But then w^*, b_0, w, b_1 and two vertices of Z_1 induce a P_6, a contradiction. □

By Claim 1, b_0 contacts at least one element of Z, say Z_1. To show that b_0 contacts each element of Z, assume to the contrary there exists an element of Z, say Z_2, such that b_0 does not contact Z_2. Let $b_2 \in B$ be partial to Z_2.

Assume that b_2 is nonadjacent to w. Then $b_2 \neq b^*$ otherwise w, b_0, a vertex of Z_1, b^* and two vertices of Z_2 induce a P_6. Then b_2 contacts Z_1, otherwise w, b_0, a vertex of Z_1, b^*, a vertex of Z_2 and b_2 induce a P_6. But then w, b, a vertex of Z_1, b_2 and two vertices of Z_2 induce a P_6, a contradiction.

Assume that b_2 is adjacent to w. Then b_2 contacts each element $Z_i \in Z$ contacted by b_0 (otherwise two vertices of Z_2, b_2, w, b_0, and a vertex of Z_i induce a P_6), and contacts each element $Z_j \in Z$ not contacted by b_0 (otherwise b_0, w, b_2, a vertex of Z_2, b^* and a vertex of Z_j induce a P_6). That is, $b_2 \in B^*$. Since b_2 is adjacent to w, by definition of b^* there exists $w^* \in W$ such that w^* is adjacent to b^* and is nonadjacent to b_2. Then w^* is adjacent to b_0, otherwise w^*, b^*, a vertex of Z_2, b_2, w, b_0 induce a P_6. But then w^*, b_0, w, b_2 and two vertices of Z_2 induce a P_6, a contradiction. □

Lemma 13 *Assume that $|Z| \geq 2$. Let $Y = \{b \in B : b$ distinguishes some element of $\mathcal{C}(V(Z) \setminus N(b^*), B)\}$. Then:*

(i) no vertex in B_0 distinguishes any component-set of $G[V(Z) \setminus N(b^*)]$;

(ii) each $y \in Y$ contacts exactly one element of Z;

(iii) Y has a co-join to W;

(iv) B_0 has a co-join to $V(\mathcal{C}(V(Z) \setminus N(b^*), B))$.

Proof: Let us prove statement (i). Let $b_0 \in B_0$. Assume to the contrary that there exists a component Q of $G[V(Z) \setminus N(b^*)]$ such that b_0 distinguishes Q. Assume without loss of generality that $Q \subset Z_1$. Since $|Z| \geq 2$, there exists $Z_2 \in Z$. By Lemma 12, $b_0 \in B^*$. Since $b_0 \in B_0$, by definition of b^* there exists $w^* \in W$ such that w^* is adjacent to b^* and is nonadjacent to b_0. Then w^*, b^*, a vertex of Z_2, b_0 and two vertices of Q induce a P_6, a contradiction.

Let us prove statement (ii). Let $y \in Y$. Assume without loss of generality that y distinguishes a component-set Q of $G[V(Z) \setminus N(b^*)]$. Let us show that y does not contact any $Z_i \in Z$, for every $i \neq 1$. Assume to the contrary that y contacts $Z_i \in Z$ for some $i \neq 1$. Let $w \in W \setminus N(b^*)$, and let $b \in B$ be adjacent to w. By Lemma 12 $b \in B^*$. By statement (i) and by definition of Y, y is nonadjacent to w. By definition of b^*, let $w^* \in W \cap N(b^*)$ such that w^* is adjacent to b^* and nonadjacent to b. If y is adjacent to w^*, then: if b contacts Q, then w, b, a vertex of Q, y, w^*, b^* induce a P_6; if b does not contact Q, then w, b, a vertex of Z_i, y and two vertices of Q induce a P_6. If y is nonadjacent to w^*, then w^*, b^*, a vertex of Z_i, y and two vertices of Q induce a P_6 a contradiction.

Let us prove statement (iii). Let $y \in Y$. Let $w^* \in W \cap N(b^*)$, and $w \in W \setminus N(b^*)$. Assume without loss of generality that y distinguishes a component-set Q of $G[Z_1 \setminus N(b^*)]$. By statement (ii), y does not contact say Z_2. Then y in nonadjacent to w^*, otherwise a vertex of Z_2, b^*, w^*, y and two vertices of Q induce a P_6. Also, y is nonadjacent to w, otherwise by Lemma 12 one should have $y \in B^*$, which contradicts statement (ii).

Let us prove statement (iv). Let $b_0 \in B_0$. Assume to the contrary b_0 contacts (and thus dominates, by Lemma 12) a component-set Q of say $G[V(Z) \setminus N(b^*)]$ (without loss of generality), such that there exists $y \in B$ distinguishing Q (thus $y \in Y$). Then by statement (ii), y does not contact any $Z_i \in Z$, for every $i \neq 1$. By definition of b^*, there exists $w^* \in W$ such that w^* is adjacent to b^* and nonadjacent to b_0. By statement (iii), w^* is nonadjacent to y. Then w^*, b^*, a vertex of say Z_2, b_0, a vertex of Q and y induce a P_6, a contradiction. □