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Krausz dimension and its generalizations in special graph classes

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A *Krausz* (k, m)*-partition* of a graph G is a decomposition of G into cliques, such that any vertex belongs to at most k cliques and any two cliques have at most m vertices in common. The m*-Krausz* dimension $kdim_m(G)$ of the graph G is the minimum number k such that G has a Krausz (k, m)-partition. In particular, 1-Krausz dimension or simply Krausz dimension $kdim(G)$ is a well-known graph-theoretical parameter.

In this paper we prove that the problem "$kdim(G) \leq 3$" is polynomially solvable for chordal graphs, thus partially solving the open problem of P. Hlineny and J. Kratochvil. We solve another open problem of P. Hlineny and J. Kratochvil by proving that the problem of finding Krausz dimension is NP-hard for split graphs and complements of bipartite graphs. We show that the problem of finding m-Krausz dimension is NP-hard for every $m \geq 1$, but the problem "$kdim_m(G) \leq k$" is is fixed-parameter tractable when parameterized by k and m for $(\infty, 1)$-polar graphs. Moreover, the class of $(\infty, 1)$-polar graphs with $kdim_m(G) \leq k$ is characterized by a finite list of forbidden induced subgraphs for every $k, m \geq 1$.

Keywords: Krausz dimension, intersection graph, linear k-uniform hypergraph, chordal graph, split graph, polar graph

1 Introduction

In this paper we consider finite undirected graphs without loops and multiple edges. The vertex and the edge sets of a graph (hypergraph) G are denoted by $V(G)$ and $E(G)$ respectively. $N(v) = N_G(v)$ is the neighborhood of a vertex v in G and $deg(v)$ is the degree of v. Let $G(X)$ denote the subgraph of G induced by a set $X \subseteq V(G)$ and $ecc_G(v)$ is the eccentricity of a vertex $v \in V(G)$.

A *Krausz partition* of a graph G is a decomposition of G into cliques, which are called *clusters* of the partition, such that every edge of G belongs to exactly one cluster. If every vertex of G belongs to at most k clusters, then the partition is called *Krausz* k*-partition*. The *Krausz dimension* $kdim(G)$ of the graph G is a minimal k such that G has Krausz k-partition.

Krausz k-partitions are closely connected with the representation of a graph as an intersection graph of a hypergraph. The *intersection graph* $L(H)$ of a hypergraph $H = (V(H), E(H))$ is defined as follows:

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1. the vertices of $L(H)$ are in a bijective correspondence with the edges of H;
2. two vertices are adjacent in $L(H)$ if and only if the corresponding edges have a nonempty intersection.

Hypergraph H is called *linear* if any two of its edges have at most one common vertex; k-*uniform* if every edge contains k vertices.

The *multiplicity* of the pair of vertices u, v of the hypergraph H is the number $m(u,v) = |\{\mathcal{E} \in E(H) : u, v \in \mathcal{E}\}|$; the *multiplicity* $m(H)$ of the hypergraph H is the maximum multiplicity of the pairs of its vertices. So, linear hypergraphs are the hypergraphs with the multiplicity 1.

Denote by H^* the dual hypergraph of H and by $H_{[2]}$ the 2-section of H, i.e., the simple graph obtained from H by the transformation of each edge into a clique. It follows immediately from the definition that

$$L(H) = (H^*)_{[2]}. \tag{1}$$

First this relation was implicitly formulated by C. Berge in [3]. (1) implies that a graph G has Krausz k-partition if and only if it is an intersection graph of linear k-uniform hypergraph.

A graph G is called *polar* [7, 21] if there exists a partition of its vertex set

$$V(G) = A \cup B,\ A \cap B = \emptyset \tag{2}$$

such that all connected components of the graphs $\overline{G}(A)$ and $G(B)$ are complete graphs. The partition (A, B) is called *bipartition*. If, in addition, α and β are fixed integers and the orders of connected components of the graphs $\overline{G}(A)$ and $G(B)$ are at most α and β respectively, then the polar graph G with bipartition (2) is called (α, β)-*polar* [5]. Given a polar graph G with bipartition (2), if the order of connected components of the graph $\overline{G}(A)$ (the graph $G(B)$) is not restricted above, then the parameter α (respectively β) is supposed to be equal ∞. Thus an arbitrary polar graph is (∞, ∞)-polar, and $(1, 1)$-polar graphs are exactly well-known *split graphs*.

Denote by $KDIM(k)$ the problem of determining whether $kdim(G) \leq k$ and by $KDIM$ the problem of finding the Krausz dimension.

Graphs with Krausz dimension at most 2 are exactly line graphs. The class of line graphs has been studied for a long time. It is characterized by a finite list of forbidden induced subgraphs [1], efficient algorithms for solving the problem $KDIM(2)$ and constructing the corresponding Krausz 2-partition are also known [6, 12, 18, 19].

The situation changes radically if one takes $k = 3$ instead of $k = 2:$ the problem $KDIM(k)$ is NP-complete for every fixed $k \geq 3$ [9]. The case $k = 3$ was studied in the different papers [10, 15, 16, 17, 20], and several graph classes, where the problem $KDIM(3)$ is polynomially solvable or remains NP-complete, were found.

In [9] P. Hlineny and J. Kratochvil studied the computational complexity of the Krausz dimension in detail. Besides other results, the following findings were obtained in their paper:

R1) The problem $KDIM$ is polynomially solvable for graphs with bounded treewidth. In particular, it is polynomially solvable for chordal graphs with bounded clique size.

R2) For the whole class of chordal graphs, the problem $KDIM(k)$ is NP-complete for every $k \geq 6$.

R3) For every fixed k the problem $KDIM(k)$ is polynomially solvable in the class of split graphs. Note, that this result was independently obtained in [13].

Note, that the study of Krausz dimension of split graphs were motivated by results of L. Beineke and I. Broere [2], who studied the Krausz dimension of complete graph without one edge, which appeared to be non-trivial problem.

Based on their results, P. Hlineny and J. Kratochvil posed the following open problems:

P1) decide the complexity of $KDIM(k)$ restricted to chordal graphs for $k = 3, 4, 5$;

P2) decide the complexity of $KDIM$ for split graphs and complements of bipartite graphs.

In this paper we solve Problem P2 and partially solve Problem P1. In Section 2 we prove that the problem $KDIM(3)$ is polynomially solvable in the class of chordal graphs. In Section 3 we show that the problem $KDIM$ is NP-hard for both classes of split graphs and complements of bipartite graphs.

In Section 4 we consider a natural generalization of the Krausz dimension. A *Krausz (k, m)-partition* of a graph G is a decomposition of G into cliques, which are called *clusters* of the partition, such that any vertex belongs to at most k clusters, and any two clusters have at most m vertices in common. As shown above, the relation (1) implies the following statement:

Proposition 1 *A graph G has Krausz (k, m)-partition if and only if it is the intersection graph of a k-uniform hypergraph with the multiplicity at most m.*

The *m-Krausz* dimension $kdim_m(G)$ of the graph G is the minimum k such that G has a Krausz (k, m)-partition. The Krausz dimension in those terms is the 1-Krausz dimension.

Denote by $KDIM_m$ the problem of determining the m-Krausz dimension of graph, by $KDIM_m(k)$ the problem of determining whether $kdim_m(G) \leq k$ and by L_k^m the class of graphs with a Krausz (k, m)-partition. It was proved in [11] that the class L_3^m could not be characterized by a finite set of forbidden induced subgraphs for every $m \geq 2$, but the complexity of the problem $KDIM_m$ for an arbitrary m has not been established yet. We prove that the problem $KDIM_m$ is NP-hard for every $m \geq 1$.

The class L_k^m is hereditary, i.e., closed with respect to deleting the vertices. Therefore it can be characterized in terms of forbidden induced subgraphs. We prove that for every fixed integers m, k such finite characterization of the class exists when restricted to $(\infty, 1)$-polar graphs. In particular, it follows that the problem $KDIM_m(k)$ is fixed-parameter tractable when parameterized by k and m. It is the strong generalization of the result $R3$. Note also, that the class of $(\infty, 1)$-polar graphs could be recognized in polynomial time [5].

2 Krausz 3-partitions of chordal graphs

Let F be a family of cliques of graph G. The cliques from F are called *clusters of F*. Denote by $l_F(v)$ the number of clusters from F covering the vertex v and by $C(v_1, \ldots, v_r)$ the cluster containing vertices $v_1, \ldots, v_r$.

A maximal clique with at least $k^2 - k + 2$ vertices is called a *k-large* clique. For such cliques the following statement holds:

Lemma 2 ([9, 10, 17]) *Any k-large clique of a graph G belongs to every Krausz k-partition of G.*

Further in this section 3-large clique will be called simply *large clique*.

Let G be a graph with $kdim(G) \leq 3$ and Q be some its Krausz 3-partition. Any subset $F \subseteq Q$ is called *a fragment of the Krausz 3-partition Q* or simply *a fragment*.

Let F be some fragment of Krausz 3-partition Q and H be the subgraph of G obtained by deleting edges covered by F. Further we will use the notation $H = G - E(F)$. The set F could be empty. Fix some vertex $a \in V(H)$ and positive integer k. Denote by $B_k[a]$ the kth neighborhood of a in H, i.e. the set of vertices at distance at most k from a. In particular, $N(a) = B_1[a] \setminus \{a\}$ denote the set of neighbors of a. A family of cliques $F_k(a)$ in H is called (a,k)-*local fragment* or simply a *local fragment*, if

1. every edge with at least one end in $B_k[a]$ is covered by some cluster of $F_k(a)$,
2. every vertex $v \in B_k[a]$ belongs to at most $3 - l_F(v)$ clusters of $F_k(a)$,
3. every two clusters of $F_k(a)$ have at most one common vertex.

A clique C of H is called *special* if C is a cluster of every (a,k)-local fragment for some a and k. In particular, by Lemma 2 large cliques are special.

The following statement is evident.

Lemma 3 *Let G be a graph with a Krausz 3-partition Q, F be a fragment of Q and $H = G - E(F)$. Then the following propositions hold:*

1. *If $deg(v) \geq 19$ for some vertex $v \in V(G)$, then v is contained in some large clique.*
2. *If $l_F(v) = 2$, then $C = N_H(v) \cup \{v\}$ is a special clique.*
3. *If $v \in B_k[a]$ is adjacent to at least $4 - l_F(v)$ vertices of the cluster C of some local fragment $F_k(a)$, then $v \in C$.*
4. *For every $a \in V(H)$ and every k there exists at least one (a,k)-local fragment;*
5) *If the clique C is special, then $F \cup \{C\}$ is a fragment.*

Proof:

1. If v is contained in clusters $C_1, C_2, C_3 \in Q$, then $|C_1| + |C_2| + |C_3| = deg(v) + 3 \geq 22$, which implies, that at least one of the clusters C_1, C_2, C_3 has more than 7 vertices.
2. If $F_k(a)$ is a local fragment, then by the definition v belongs to at most one cluster $C \in F_k(a)$. Therefore all neighbors of v in H belong to C, i.e. $C = N_H(v) \cup \{v\}$.
3. If $v \in B_k[a]$ is adjacent to vertices $v_1, ..., v_{4-l_F(v)} \in C \in F_k(a)$, but $v \notin C$, then the edges $vv_1, \ldots, vv_{4-l_F(v)}$ should be covered by different clusters of $F_k(a)$. It contradicts (2).
4. The family of cliques $X = \{C \in Q \setminus F : C \cap B_k[a] \neq \emptyset\}$ is a local fragment.
5. Let X is the same as in the proof of 4). Since C is special, $C \in X$ and therefore $C \in Q \setminus F$.

□

Lemma 4 *Let G be a chordal graph and S be its cycle with length at least 4. Then for every two consecutive vertices of S at least one of them belongs to some chord.*

Proof: Let $S = (a_1, \ldots, a_k)$, $a_i a_{i+1} \in E(G)$, indices are taken modulo k. The statement of lemma is obvious, if $k = 4$. Let $k \geq 5$. Without lost of generality consider two adjacent vertices a_k, a_1 of S. If the statement of lemma is not true, then one can choose the chord $a_p a_q$, $1 < p < q < k$ such, that $(p-1) + (k-q)$ is minimal. But then $(a_1, \ldots, a_p, a_q, \ldots, a_k)$ is a chordless cycle. $\square$

Denote by $lc(H)$ the length of a longest induced cycle of the graph H.

Lemma 5 *Let G be a chordal graph with a Krausz 3-partition Q, F be a fragment of Q and $H = G - E(F)$. Let further there be no $(a, 2)$-local fragments, which contain special cliques in H. Then $lc(H) \leq 6$.*

Proof: Suppose the contrary, i.e., let $a_1, \ldots, a_k$ form the induced cycle $S \cong C_k$ in H, $k \geq 7$, $a_i a_{i+1} \in E(H)$, indices are taken modulo k.

Since each a_i has two nonadjacent neighbors in H, in every local fragment with center in a_i it is covered by at least 2 clusters. It implies $l_F(a_i) \leq 1$ for every $i = 1, \ldots, k$.

As G is a chordal graph, there exist chords of the cycle S covered by the fragment F. As $l_F(a_i) \leq 1$, for every vertex a_i chords incident to this vertex are covered by exactly one cluster of F. It implies that no pair of chords has the form $\{a_i a_j, a_i a_{j+1}\}$, since in this case the vertices a_i, a_j, a_{j+1} are covered by one cluster of F and thus the edge $a_j a_{j+1}$ should be covered by F.

Assume without loss of generality that a chord of S contains a_1. Let us show that all chords of S are covered by the cluster $C_{chord} \supseteq \{a_1, a_3, \ldots, a_{k-1}\}$. Indeed, suppose that some chords of S are covered by the cluster $C \supseteq \{a_{i_1}, \ldots, a_{i_r}\}$, $i_1 < i_2 < \ldots < i_r$, $i_1 = 1$, $C \neq C_{chord}$. Then there exists $1 \leq p < r$ such that $i_{p+1} - i_p \geq 3$. So, $(a_{i_p}, a_{i_p+1}, \ldots, a_{i_{p+1}-1}, a_{i_{p+1}})$ is a cycle of length at least 4, in which at least one of the vertices $a_{i_p}, a_{i_{p+1}}$, say a_{i_p}, belongs to some chord according to Lemma 4. That chord should be covered by a cluster $C' \in F$, $C' \neq C$. So, we have $l_F(a_{i_p}) \geq 2$, the contradiction.

In particular, this proposition implies the following:

1. k is even;
2. for any odd i and even j such that $a_i a_j$ is not the edge of S, the vertices a_i and a_j are nonadjacent in G. It follows from the fact, that otherwise $l_F(a_i) \geq 2$.

Since there are no special cliques in H there exist two different local fragments $F_2(a_1) \supseteq \{C(a_1, a_2), C(a_1, a_k)\}$ and $F'_2(a_1) \supseteq \{C'(a_1, a_2), C'(a_1, a_k)\}$, such that without loss of generality $C(a_1, a_2) \setminus C'(a_1, a_2) \neq \emptyset$.

Let $v \in C(a_1, a_2) \setminus C'(a_1, a_2)$. Then $v \in C'(a_1, a_k)$ and therefore $a_1 v, a_2 v, a_k v \in E(H)$.

The vertices $a_2, v, a_k, a_{k-1}, a_3$ form a cycle in G. It should have at least 2 chords. Since $a_2 a_{k-1}, a_3 a_k \notin E(G)$, there are edges $v a_3, v a_{k-1} \in E(G)$. The edges $v a_3, v a_{k-1}$ are not covered by F, because otherwise $v \in C_{chord}$ and thus $\{a_1, v\} \subseteq C_{chord} \cap C(a_1, a_2)$ contradicting the definition of H. Hence $v a_3, v a_{k-1} \in E(H)$.

$a_3 \in B_2[a_1]$, therefore by the definition of local fragment all edges of H adjacent to a_3 are covered by $F_2(a_1)$. It implies that $F_2(a_1) \ni C(a_3, a_2), C(a_3, a_4)$. Since $v \in C(a_1, a_2)$ we have $v \in C(a_3, a_4)$. So, $v a_4 \in E(H)$ (see Figure 1). Note that since $k \geq 7$, we have $a_4 a_{k-1} \notin E(H)$.

Recall that the vertex v in the local fragment $F'_2(a_1)$ is covered by the cluster $C'(v, a_1, a_k)$. All other edges of H incident to v should be covered by at most two clusters of $F'_2(a_1)$. But this is impossible since the vertices a_2, a_4, a_{k-1} are pairwise nonadjacent. This contradiction proves Lemma 5. $\square$

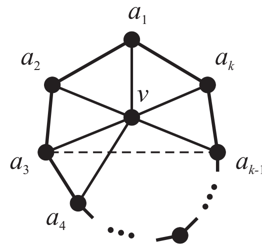

Figure 1:

The considerations above suggest the following algorithm which reduces the problem of recognition chordal graphs with Krausz dimension at most 3 to the same problem for graphs with bounded maximum degree and maximum induced cycle length.

Algorithm 1
Input: *chordal graph G.*
Output: *One of the following:*
1) graph H^* with $\Delta(H^*) \leq 18$ and $lc(H^*) \leq 6$ such that $kdim(G) \leq 3$ if and only if $kdim(H^*) \leq 3$;
2) the answer "$kdim(G) > 3$".
begin
 $F := \emptyset$; $H := G$; $isContinue := true$;
 while ($isContinue =$ **true**)
  if *there exists a vertex $v \in V(H)$ such that $l_F(v) = 2$*
   $C := N(v) \cup \{v\}$;
   if *C is a clique*
    $F := F \cup \{C\}$; **continue** *to the next iteration of the cycle;*
   else *the answer is "$kdim(G) > 3$";* **stop;**
  if *there exists a vertex $v \in V(H)$ with $deg(v) \geq 19$*
   if *v is contained in a clique C with $|C| \geq 8$*
    extend C to a maximal clique; $F := F \cup \{C\}$;
    continue *to the next iteration of the cycle;*
   else *the answer is "$kdim(G) > 3$";* **stop;**
  For every non-isolated vertex $v \in V(H)$ generate all
  (v, e)-local fragments, $e = min\{ecc_H(v), 2\}$;
  if *there exists a vertex $v \in V(H)$ such that there are*
   no (v, e)-local fragments
   the answer is "$kdim(G) > 3$"; **stop;**
  if *there exists a special clique C*
   $F := F \cup \{C\}$; **continue** *to the next iteration of the cycle;*
  $isContinue := false$
 endwhile;
 construct a graph H^* by adding a pendant edge vp_v to every vertex $v \in V(H)$ with $l_F(v) = 1$;
end.

Theorem 6 ([4]) *Let $lc(H^*) \leq s+2$, $\Delta(H^*) \leq \Delta$. Then $treewidth(H^*) \leq \Delta(\Delta-1)^{s-1}$.*

Theorem 7 *The problem $KDIM(3)$ is polynomially solvable for chordal graph.*

Proof: Let us first prove the correctness of Algorithm 1. Once the "while" loop of Algorithm 1 is completed, we obtain the graph H and the set of cliques F. By the construction F contains only special cliques. Therefore if the Krausz 3-partition of G exists then it has the form $F \cup R$, where R is a clique partition of H such that every vertex $v \in V(H)$ belongs to at most $3 - l_F(v)$ clusters of R and every edge of H belongs to exactly one cluster of R. By the construction of Algorithm 1 $l_F(v) \in \{0,1\}$ for every non-isolated vertex $v \in V(H)$. Therefore R is a clique partition of H with required properties if and only if $R \cup \{vp_v : l_F(v) = 1\}$ is a Krausz 3-partition of H^*.

Now we will show that the Algorithm 1 is polynomial. Indeed, the procedure of finding large clique which contains the fixed vertex $v \in V(H)$ has the complexity $O(m)$. We start to generate all possible (v,e)-local fragments for a vertex $v \in V(H)$ only when $deg(v) \leq 18$. This implies $|B_e[v]| \leq const$ and thus the complexity of this procedure is constant. The outer loop of Algorithm 1 is performed at most m times.

After performing the Algorithm 1, we obtain the graph H^* with bounded maximum degree and the length of the longest induced cycle. By Theorem 6 H^* has bounded treewidth. For such a graph the problem of determining its Krausz dimension is polynomially solvable [9]. $\square$

3 Krausz dimension of split graphs and complements of bipartite graphs

Theorem 8 *The problem $KDIM$ is NP-hard in the class of split graphs.*

Proof: We will reduce the problem $KDIM$ for general graphs to the problem $KDIM$ for split graphs.

Let G be a connected graph with n vertices and m edges and $k \leq n-2$ be an integer. Modify G by adding additional vertices and edges as follows:

1. For every $u, v \in V(G)$ such that $uv \notin E(G)$, $u \neq v$ add vertices x_{uv}, y_{uv} and edges uv, $x_{uv}u$, $x_{uv}v$, $y_{uv}u$, $y_{uv}v$, $x_{uv}y_{uv}$. Let $X = \{x_{uv} : uv \notin E(G), u \neq v\}$, $|X| = \binom{n}{2} - m$.
2. For every $u, v, w \in V(G)$ such that $uv \notin E(G)$ and every $w \in V(G) \setminus \{u, v\}$ add a vertex y_{uvw} and edges $x_{uv}w$, $x_{uv}y_{uvw}$, wy_{uvw};
3. For every two vertices $x_{uv}, x_{sw} \in X$ add a vertex z_{uvsw} and edges $x_{uv}z_{uvsw}$, $x_{sw}z_{uvsw}$, $x_{uv}x_{sw}$;
4. For every $v \in V(G)$ add $n-2-k$ vertices $r_{v,1}, ..., r_{v,n-2-k}$ and edges $vr_{v,1}, ..., vr_{v,n-2-k}$.

Let H be obtained graph (see an example of graph H for the graph $G = \overline{K_1 + P_3}$ on Figure 2). Note, that $A = V(G) \cup X$ and $B = V(H) \setminus A$ is a clique and a stable set in H, respectively. So, H is a split graph. We will show, that G has Krausz k-partition if and only if H has Krausz k'-partition, where $k' = \binom{n}{2} + n - m - 2$.

Suppose that Q is Krausz k'-partition of H. For every $u, v \in V(G)$, $uv \notin E(G)$ consider a set

$$N_{uv} = N(x_{uv}) \cap B = \{y_{uv}\} \cup \{y_{uvw} : w \in V(G) \setminus \{u, v\}\} \cup \{z_{uvsw} : x_{sw} \in X \setminus \{x_{uv}\}\} \quad (3)$$

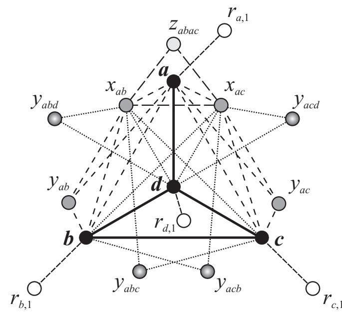

Figure 2:

Let $Q_{uv} \subseteq Q$ be the set of clusters containing x_{uv}. N_{uv} is a stable set, and $|N_{uv}| = 1 + (n-2) + (\binom{n}{2} - m - 1) = k'$. Therefore every cluster of Q_{uv} contains exactly one vertex from N_{uv}, i.e.

$$Q_{uv} = \{C(y_{uv}, x_{uv})\} \cup \{C(y_{uvw}, x_{uv}) : w \in V(G) \backslash \{u, v\}\} \cup \{C(z_{uvsw}, x_{uv}) : x_{sw} \in X \backslash \{x_{uv}\}\} \quad (4)$$

Every vertex of H, which is adjacent to x_{uv}, belongs to one of those clusters. For every vertex from $N(x_{uv}) \setminus N_{uv} = A \setminus \{x_{uv}\}$ we have the following.

1. The only vertex from N_{uv}, which is adjacent to u and v, is y_{uv}. Moreover, $N(y_{uv}) = \{u, v, x_{uv}\}$. Therefore $C(y_{uv}, x_{uv}) = \{u, v, x_{uv}, y_{uv}\}$.
2. The only vertex from N_{uv}, which is adjacent to $w \in V(G) \backslash \{u, v\}$, is y_{uvw}. Moreover, $N(y_{uvw}) = \{w, x_{uv}\}$. Therefore $C(y_{uvw}, x_{uv}) = \{w, x_{uv}, y_{uvw}\}$.
3. The only vertex from N_{uv}, which is adjacent to $x_{sw} \in X \backslash \{x_{uv}\}$, is z_{uvsw}. Moreover, $N(z_{uvsw}) = \{x_{uv}, x_{sw}\}$. Therefore $C(z_{uvsw}, x_{uv}) = \{x_{uv}, x_{sw}, z_{uvsw}\}$.

For every $v \in V(G)$ Krausz k'-partition Q contains clusters $\{v, r_{v,i}\}$, $i = 1, ..., n-2-k$. So, Q contains a fragment

$$Q' = \{Q_{uv} : x_{uv} \in X\} \cup \{\{v, r_{v,i}\} : v \in V(G), i = 1, ..., n-2-k\}. \quad (5)$$

For every $v \in V(G)$ let $Q'(v)$ be the set of clusters of Q' which contain v. We have

$$\begin{aligned} Q'(v) = &\{\{v, r_{v,i}\} : i = 1, ..., n-2-k\} \cup \{\{u, v, x_{uv}, y_{uv}\} : u \in V(G), uv \notin E(G)\} \cup \\ &\cup \{\{x_{sw}, y_{swv}, v\} : s, w \in V(G) \setminus \{v\}, sw \notin E(G)\}\} \end{aligned} \quad (6)$$

So,

$$|Q'(v)| = n - 2 - k + (\binom{n}{2} - m) = k' - k \quad (7)$$

The graph obtained from H by deleting edges covered by Q' and isolated vertices is exactly the graph G. Consider the fragment $Q'' = Q \setminus Q'$. By (7) Q'' is Krausz k-covering of G.

Conversely, it is easy to see, that if Q'' is Krausz k-covering of G, then $Q' \cup Q''$ is Krausz k'-covering of H. □

Corollary 9 *The problem $KDIM$ is NP-hard in the class of complements of bipartite graphs.*

Proof: Let G be a split graph on n vertices, $V(G) = A \cup B$ be the partition of its vertex set into clique A and stable set B and k be the natural number. Build the graph H as follows:

1. For every $a \in A$ add the new vertex x_a and the edge ax_a;
2. If $n < (k+1)^2 - (k+1) + 2$, add new vertices $y_1, \ldots, y_{(k+1)^2-(k+1)+2-n}$; let
$$B' = B \cup \{x_a : a \in A\} \cup \{y_1, \ldots, y_{(k+1)^2-(k+1)+2-n}\};$$
3. Add all possible edges between vertices of B'.

H is a complement of bipartite graph. Taking into account Lemma 2, it is easy to see that Q is a Krausz k-partition of G if and only if $Q \cup \{B'\} \cup \{\{a, x_a\} : a \in A\}$ is a Krausz $(k+1)$-partition of H. □

4 m-Krausz dimension of graphs

We will start with proving the NP-hardness of the problem $KDIM_m$.

Theorem 10 *The problem $KDIM_m$ is NP-hard for every fixed $m \geq 1$.*

Proof: Let us reduce to the problem $KDIM_m$ the following special case of the 3-dimensional matching problem, which we will call Problem A:

Given: Non-intersecting sets X, Y, Z , such that $|X| = |Y| = |Z| = q$; $M \subseteq X \times Y \times Z$, such that the following condition holds:

$$(*) \qquad \text{if } (a, b, w), (a, x, c), (y, b, c) \in M, \text{ then } (a, b, c) \in M.$$

The question: Does M contain a subset $M' \subseteq M$ such that $|M'| = q$ and every two elements of M' does not have common coordinates? The subset M' is called *3-dimensional matching*.

It is known that Problem A is NP-complete [8]. Let X, Y, Z, M, $|X| = |Y| = |Z| = q$, be the input of the problem A. Let us reduce problem A to the problem of determining if $kdim_m(G) \leq 2q-1$. Construct the graph G as follows:

$$V(G) = X \cup Y \cup Z \cup \{v, v_1, \ldots, v_{q-1}\}; \tag{8}$$
$$E(G) = \bigcup_{(a,b,c)\in M} \{ab, bc, ac\} \cup \{v_i v : i = 1, \ldots, q-1\} \cup \{vd : d \in X \cup Y \cup Z\} \tag{9}$$

(see Figure 3). Let us show that M contains the 3-dimensional matching M' if and only if there exists a Krausz $(2q-1, m)$-partition of G.

Suppose that $M' = \{(a_i, b_i, c_i) : i = 1, \ldots, q\}$ is the 3-dimensional matching. Let $Q_1 = \{\{v, a_i, b_i, c_i\} : i = 1, \ldots, q\}$, $Q_2 = \{\{v, v_i\} : i = 1, \ldots, q-1\}$, $H = G - E(Q_1 \cup Q_2)$. All edges adjacent to v are covered by $2q-1$ cliques of $Q_1 \cup Q_2$ and every vertex $u \in V(G) \setminus \{v\}$ is adjacent to at most $2q-2$ vertices in H. Therefore if $Q_3 = \{\{z, t\} : zt \in E(H)\}$, then $Q = Q_1 \cup Q_2 \cup Q_3$ is Krausz $(2q-1, m)$-partition of G.

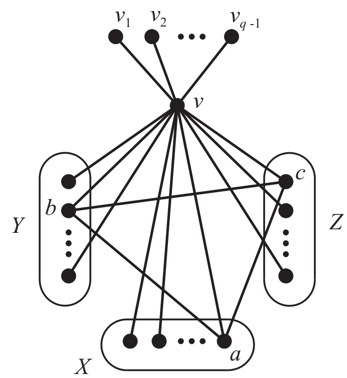

Figure 3:

Let now Q be Krausz $(2q-1, m)$-partition of G. Denote by $Q(v)$ the set of clusters of Q, which contain the vertex v. Since the vertices v_i, $i = 1, \ldots, q-1$, have degree 1, there exist $q-1$ clusters from $Q(v)$ of the form $\{v, v_i\}$, $i = 1, \ldots, q-1$. Let $C_1, \ldots, C_p$ be the remaining clusters from $Q(v)$, $p \leq q$. Then $(C_1 \cup \ldots \cup C_p) \setminus \{v\} = X \cup Y \cup Z$. Since X, Y, Z are stable sets of G, we have $|C_i| \leq 4$, $i = 1, \ldots, p$. As $|X \cup Y \cup Z| = 3q$, we have $p = q$, $|C_i| = 4$, $C_i \cap C_j = \{v\}$, $i, j = 1, \ldots, p$, $i \neq j$.

Let $C_i = \{a_i, b_i, c_i, v \,:\, a_i \in X,\ b_i \in Y,\ c_i \in Z\}$, $i = 1, \ldots, q$. The property (*) implies that $M' = \{(a_i, b_i, c_i) \,:\, i = 1, \ldots, q\} \subseteq M$ and, by the consideration above, M' is the 3-dimensional matching. $\square$

Now we turn to the complexity of the recognition problem $KDIM_m(k)$ in the class of $(\infty, 1)$-polar graphs.

A maximal clique with at least $f(k, m) = m(k^2 - k + 1) + 1$ vertices is called a (k, m)-*large* clique.

In [14] the following two statements were proved. Since they were published only in Russian in a journal that is of difficult access for general readers, we repeat their proofs here.

Theorem 11 *Any (k, m)-large clique C of a graph G belongs to every Krausz (k, m)-partition of G.*

Proof: Let A be a Krausz (k, m)-partition of graph G, $A_1, A_2, \ldots, A_t$ be those clusters of A which have common vertices with C. Assume that $C \notin A$. Then the family $B = (B_1, B_2, \ldots, B_t)$, where $B_i = A_i \cap C$, is a Krausz (k, m)-partition of the graph $G(C)$, and by maximality of C $B_i \neq C$ for every $i = 1, 2, \ldots, t$.

Let us show that $|B_i| \leq mk$ for any $i = 1, 2, \ldots, t$. Consider a cluster of B, say B_1, and a vertex $u \in C \setminus B_1$. No edge of the form ux, where $x \in B_1$, is contained in B_1. Moreover, by the definition of Krausz (k, m)-partition each cluster of B different from B_1 contains at most m such edges. Taking into account that the vertex u belongs to at most k clusters of B, we obtain the inequality $|B_1| \leq mk$.

Now we will prove that if $B_i \setminus B_j \neq \emptyset$ for some clusters $B_j \in B$, then $|B_j \setminus B_i| \leq m(k-1)$. Consider a vertex $u \in B_i \setminus B_j$. Any edge of the form ux, where $x \in B_j \setminus B_i$ (if such one exists) is contained neither in B_i, nor in B_j. Besides, no cluster of B contains more than m of such edges by definition of Krausz (k, m)-partition. Taking into account that u belongs to at most $k-1$ clusters of B different from B_i, we obtain the inequality $|B_j \setminus B_i| \leq m(k-1)$.

Consider an arbitrary vertex v of the clique C. Let, without loss of generality, it belong to the clusters $B_1, B_2, \ldots, B_s$ of B, $s \leq t$. We show that $|B_1 \cup B_2 \cup \ldots \cup B_s| \leq mk + (s-1)m(k-1)$. The following equality is obvious

$$|B_1 \cup B_2 \cup \ldots \cup B_s| = |B_1| + |B_2 \setminus B_1| + |B_3 \setminus (B_1 \cup B_2)| + \ldots + |B_s \setminus (B_1 \cup B_2 \ldots \cup B_{s-1})|. \quad (10)$$

If $B_1 \setminus B_2 \neq \emptyset$, $(B_1 \cup B_2) \setminus B_3 \neq \emptyset, \ldots, (B_1 \cup B_2 \cup \ldots \cup B_{s-1}) \setminus B_s \neq \emptyset$, then by the statement proved above, each term in the right part of the equality (10), starting from the second, does not exceed $m(k-1)$. Hence we have $|B_1 \cup B_2 \cup \ldots \cup B_s| \leq mk + (s-1)m(k-1)$. Let, on the contrary, $i \in \{2, \ldots, s\}$ be the maximal number such that $(B_1 \cup \ldots \cup B_{i-1}) \setminus B_i = \emptyset$. Then $B_1 \subseteq B_i$, $B_2 \subseteq B_i, \ldots, B_{i-1} \subseteq B_i$, and the sum of the first i terms in the right part of (10) is equal to $|B_1 \cup B_2 \cup \ldots \cup B_i| = |B_i| \leq mk$. Each of the other terms does not exceed $m(k-1)$ by the maximality of i. Hence

$$|B_1 \cup B_2 \cup \ldots \cup B_s| \leq mk + (s-i)m(k-1) < mk + (s-1)m(k-1).$$

So, in any case we obtain that the inequality $|B_1 \cup B_2 \cup \ldots \cup B_s| \leq mk + (s-1)m(k-1)$ holds. Taking into account that $C = B_1 \cup B_2 \cup \ldots \cup B_s$ and $s \leq k$, we have

$$|C| \leq mk + (k-1)m(k-1) = m(k^2 - k + 1) < f(k, m).$$

The obtained contradiction proves the lemma. □

Theorem 12 *For every fixed $k, m \geq 1$, there exists a finite set $\mathcal{F}_0$ of forbidden induced subgraphs such that a split graph G belongs to the class L_k^m if and only if no induced subgraph of G is isomorphic to an element of $\mathcal{F}_0$.*

Proof: Denote by R_p the graph obtained from the complete graph $H \cong K_{f(k,m)}$ by adding a new vertex and connecting it with exactly p vertices of H. Put $\mathcal{F}_0 = \{R_p : km + 1 \leq p \leq f(k,m) - 1\} \cup \{K_{1,k+1}\}$. Using Theorem 11 one can immediately verify that no graph from $\mathcal{F}_0$ belongs to L_k^m.

Let, without loss of generality, G be a connected graph, and $V(G) = C \cup S$ be a bipartition of $V(G)$ into clique C and stable set S such that C is a maximal clique. Let also no induced subgraph of G be isomorphic to an element of $\mathcal{F}_0$. Put $S = \{v_1, \ldots, v_s\}$. Consider two cases:

1) $|C| > (km-1)k + 1$.

In this case we have

$$|C| \geq (km-1)k + 2 = mk^2 - (k-1) + 1 \geq mk^2 - m(k-1) + 1 = f(k, m).$$

Then, since no induced subgraph of G is isomorphic to a graph R_p, $km + 1 \leq p \leq f(k,m) - 1$, we have $\deg(v_i) \leq km$ for any $i = 1, 2, \ldots, s$. Since G contains no induced $K_{1,k+1}$, we have $|N(u) \cap S| \leq k$ for any vertex u from C. Moreover, we prove that for any vertex u from C the inequality $|N(u) \cap S| \leq k-1$ holds. Assume this is not true. Let, without loss of generality, some vertex u from C be adjacent to the vertices $v_1, \ldots, v_k$ from S, $k \leq s$. Since $deg(v_i) \leq km$, $i = 1, 2, \ldots, k$, and $u \in \bigcap\limits_{i=1}^{k} N(v_i)$, then $|\bigcup\limits_{i=1}^{k} N(v_i)| \leq \sum\limits_{i=1}^{k}(deg(v_i) - 1) + 1 \leq (km-1)k + 1 < \varphi(G)$. Hence, there exists a vertex u'

from C, which is not adjacent to any vertex from $v_1, \ldots, v_k$. But then $G(u, u', v_1, \ldots, v_k) \cong K_{1,k+1}$, a contradiction.

Now we can construct a Krausz (k, m)-partition of G. Since $deg(v_i) \leq km$ for any $i = 1, 2, \ldots, s$, then there exists a partition $N(v_i) = C_{i_1} \cup \ldots \cup C_{i_{s_i}}$, where $C_{i_j} \cap C_{i_l} = \emptyset$, $j, l \in \{1, \ldots, s_i\}$, $j \neq l$, $|C_{i_j}| \leq m$, $s_i \leq k$. Obviously, the list of cliques $\{C_{i_j} \cup \{v_i\} : i = \overline{1, s}, j = \overline{1, s_i}\}$ together with the clique C is a Krausz (k, m)-partition of graph G.

2) $|C| \leq (km - 1)k + 1$.

Since G contains no induced $K_{1,k+1}$, we have $|N(u) \cap S| \leq k$ for any vertex u from C. Therefore, as G is connected,

$$|G| = |C| + |S| \leq |C| + \sum_{u \in C} |N(u) \cap S| \leq ((km-1)k+1) + ((km-1)k+1)k = ((km-1)k+1)(k+1),$$

i. e., the order of graph G is bounded above by a value, depending on k and m. Add to the list $\mathcal{F}_0$ all such split graphs H, that $H \notin L_k^m$ and $|H| \leq ((km - 1)k + 1)(k + 1)$.

Obviously, the constructed in the cases 1) and 2) finite list $\mathcal{F}_0$ is a required list of forbidden induced subgraphs. $\square$

Since $K_{1,k+1} \notin L_k^m$, the heredity of L_k^m immediately implies

Lemma 13 *A bipartite graph G belongs to the class L_k^m if and only if no induced subgraph of G is isomorphic to $K_{1,k+1}$.*

Theorem 14 *For every fixed $k, m \geq 1$, there exists a finite set $\mathcal{F}$ of forbidden induced subgraphs such that an $(\infty, 1)$-polar graph G belongs to the class L_k^m if and only if no induced subgraph of G is isomorphic to an element of $\mathcal{F}$.*

Proof: Without loss of generality, we can suppose that $(\infty, 1)$-polar graph G is connected. Let G have bipartition (A, B); A_i, $i = 1, 2, \ldots, t$, be the vertex sets of connected components of $\overline{G}(A)$; $\mathcal{F}_0$ be the set of split graphs from Theorem 12. Denote by $\mathcal{F}_1$ the set of $(\infty, 1)$-polar graphs which do not belong to the class L_k^m and have order at most $(k + 1)k(f(k, m) - 1)$.

Put $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \{K_{1,k+1}, K_{f(k,m)+1} - e\}$, where $K_{f(k,m)+1} - e$ is the graph obtained from the complete graph $K_{f(k,m)+1}$ after deleting an edge. The set $\mathcal{F}$ is finite, since $\mathcal{F}_0$ and $\mathcal{F}_1$ are finite. According to Theorem 11, there is no Krausz (k, m)-partition for $K_{f(k,m)+1} - e$. Therefore, $K_{f(k,m)+1} - e \notin L_k^m$. Thus, $\mathcal{F} \cap L_k^m = \emptyset$. The necessity of the statement follows from the heredity of the class L_k^m.

Now let G contain no induced subgraph isomorphic to an element from $\mathcal{F}$. If $G(A)$ is complete, then G is a split graph and by Theorem 12 $G \in L_k^m$. If $G(A)$ is empty, then G is a bipartite graph and by Lemma 13 $G \in L_k^m$.

Now suppose that $G(A)$ is neither complete nor empty graph. Then $2 \leq t \leq |A| - 1$. Since $K_{1,k+1} \in \mathcal{F}$, then $|A_i| \leq k$ for any $i = 1, 2, \ldots, t$. Now we will prove that since $K_{f(k,m)+1} - e \in \mathcal{F}$, then $t \leq f(k, m) - 1$. Let, to the contrary, $t \geq f(k, m)$. As $G(A)$ is not a complete graph, there exists an index $i_0 \in \{1, 2, \ldots, t\}$ such that $|A_{i_0}| \geq 2$. Consider the set $S = \{a_1, a_2, \ldots, a_{i_0-1}, a'_{i_0}, a''_{i_0}, a_{i_0+1}, \ldots, a_t\}$, where $a_i \in A_i$ for any $i \in \{1, 2, \ldots, t\} \setminus \{i_0\}$ and $a'_{i_0}, a''_{i_0} \in A_{i_0}$. Then $G(S)$ contains $K_{f(k,m)+1} - e$ as an induced subgraph, a contradiction. Therefore

$$|A| \leq \sum_{i=1}^{t} |A_i| \leq k(f(k, m) - 1).$$

Since $|N(a) \cap B| \leq k$ for any vertex $a \in A$ and G is connected, we have

$$|G| \leq |A| + |B| \leq |A| + \sum_{a \in A} |N(a) \cap B| \leq k(f(k,m)-1) + k^2(f(k,m)-1) = (k+1)k(f(k,m)-1).$$

It follows from the inclusion $\mathcal{F}_1 \subseteq \mathcal{F}$ that $G \in L_k^m$. □

Corollary 15 *The problem $KDIM_m(k)$ is fixed parameter tractable in the class of $(\infty,1)$-polar graphs, when parameterized by $k, m \geq 1$.*

Proof: The proofs of Theorems 12, 14 and Lemma 13 imply the following algorithm of recognition whether a given $(\infty,1)$-polar graph belong to the class L_k^m.

Algorithm 2
Input: *connected $(\infty,1)$-polar graph G with bipartition (A,B), $A = A_1 \cup A_2 \cup \ldots \cup A_t$ where A_i, $i = 1,2,\ldots,t$, be the vertex sets of connected components of $\overline{G}(A)$, integers $k, m \geq 1$.*
Output: *1 if $kdim_m(G) \leq k$, and 0 otherwise.*

1. **begin**
2. **if** $|N(a) \cap B| \leq k$ *for every* $a \in A$
3. **if** $t = 1$, *i.e. graph G is bipartite,*
4. **if** $deg(b) \leq k$ *for every* $b \in B$
5. **return** *1;*
6. **else return** *0;*
7. **if** $t = |A|$, *i.e. graph G is split,*
8. **if** $|A| \geq (km-1)k + 2$
9. **if** *($|N(a) \cap B| \leq k-1$ for every $a \in A$) AND ($deg(b) \leq km$ for every $b \in B$)*
10. **return** *1;*
11. **else return** *0;*
12. **if** $2 \leq t \leq |A| - 1$
13. **if** *($|A_i| \geq k+1$ for some $i = 1,2,\ldots,t$) AND ($t \geq f(k,m)$)*
14. **return** *0;*
15. **if** $kdim_m(G) \leq k$
16. **return** *1;*
17. **else return** *0;*
18. **return** *0;*
19. **end**.

The complexity of the algorithm in lines 1–14 is $O(n^2)$ at worst, where $n = |V(G)|$. Since the order of the graph G in line 15 is bounded above by value $g(k,m) = \max\{((km-1)k+1)(k+1), (k+1)k(f(k,m)-1)\}$ the problem "$kdim_m(G) \leq k$" in this line could be solved in time $O(h(k,m))$, where h is some function.

So, the total complexity of Algorithm 2 is $O(n^2 + h(k,m))$. □

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