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The Asymmetric Leader Election Algorithm with swedish stopping: A probabilistic analysis

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Dedicated to the memory of Philippe Flajolet.

We study a leader election protocol that we call the Swedish leader election protocol. This name comes from a protocol presented by L. Bondesson, T. Nilsson, and G. Wikstrand (2007). The goal is to select one among $n > 0$ players, by proceeding through a number of rounds. If there is only one player remaining, the protocol stops and the player is declared the leader. Otherwise, all remaining players flip a biased coin; with probability q the player survives to the next round, with probability $p = 1 - q$ the player loses (is killed) and plays no further … unless all players lose in a given round (null round), so all of them play again. In the classical leader election protocol, any number of *null* rounds may take place, and with probability 1 some player will ultimately be elected. In the Swedish leader election protocol there is a maximum number τ of consecutive null rounds, and if the threshold is attained the protocol fails without declaring a leader.

In this paper, several parameters are asymptotically analyzed, among them: Success Probability, Number of rounds $\mathcal{R}_n$, Number of null rounds $\mathcal{T}_n$.

This paper is a companion paper to Louchard, Martinez and Prodinger (2011) where De-Poissonization was used, together with the Mellin transform. While this works fine as far as it goes, there are limitations, in particular of a computational nature. The approach chosen here is similar to earlier efforts of the same authors - Louchard and Prodinger (2004,2005,2009). Identifying some underlying distributions as Gumbel (type) distributions, one can start with approximations at a very early stage and compute (at least in principle) all moments asymptotically. This is in contrast to the companion work, where only expected values were considered. In an appendix, it is shown that, wherever results are given in both papers, they coincide, although they are presented in different ways.

Keywords: asymmetric leader election, probabilistic analysis, urn models, limiting distributions

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1 Introduction

We present a probabilistic analysis, based on an urn model, of a leader election protocol that we call the Swedish leader election protocol. This name comes from a protocol presented by L. Bondesson, T. Nilsson, and G. Wikstrand in [1]. The goal is to select one among $n > 0$ players, by proceeding through a number of rounds. If there is only one player remaining, the protocol stops and the player is declared the leader. Otherwise, all remaining players flip a biased coin; with probability q the player survives to the next round, with probability $p = 1 - q$ the player loses (is killed) and plays no further ... unless all players lose in a given round (null round), so all of them play again. We say that a round is *null* if every active player tosses tails (all killed). In the classical leader election protocol, any number of *null* rounds may take place, and with probability 1 some player will ultimately be elected. In the Swedish leader election protocol there is a maximum number τ of consecutive null rounds, and if the threshold is attained the protocol fails without declaring a leader.

In this paper, several parameters are asymptotically analyzed, starting with n players, n large:[(i)] success probability S_n, number of rounds $\mathcal{R}_n$, number of null rounds $\mathcal{T}_n$, number $\mathcal{L}_n$, in case of failure, of players that were active at the last non-null round, the so-called *left-overs*, and the total number $\mathcal{C}_n$ of coins flipped. As suggested by one referee, other RVs of interest could be considered, for instance the number of useless coin flips (i.e. coin flips that lead to null rounds or rounds where every player goes forward to the next round). These RVs can indeed be analyzed with the same techniques, but due to length constraints, we refrain to do this here. Such considerations would make an excellent project for research students.

This work is a companion paper to [7] where De-Poissonization was used, together with the Mellin transform. Here, we approximate the relevant quantities at an early stage; the loss in accuracy results in a gain when it comes to the complexity of the necessary computations. Indeed, we can deal here with all moments, whereas in [7] only expected values could be computed.

2 Urn model

In this section we show how to proceed from the leader election algorithm to an urn model. Then, within this model, we explain how to derive the asymptotic distributions of our random variables (RVs). Finally we obtain (via Laplace transforms) all asymptotic distributions and moments we need.

By the definition of the protocol each player does have a life duration, being independent of all other players, given by a geometrically distributed random variable X, with probability mass function $\mathbb{P}(X = j) = pq^{j-1}$, where p denotes the killing probability, and q the probability of survival. Proceeding as in [10] the leader election protocol can be considered as an urn model in the following way: we have urns numbered $1, 2, \ldots$, and n balls are thrown into the urns. The probability of each out of the n balls falling into an urn numbered j is being given by pq^{j-1}. The balls contained in the urn numbered j represent the candidates who are killed at level j of the leader election protocol. The advantage of the viewpoint is that the parameters of interest can be interpreted in terms of statistics on the urn model. In order to do so, we will introduce several additional parameters. Let I denote the number of balls contained in the maximal (highest numbered) non-empty urn. If it contains only a single ball, then the leader election protocol has succeeded, and a single player has survived. If it contains more than one ball, say $i \geq 2$, this implies that a null round has taken place, and the process has to be restarted with i balls (all remaining i players have been killed and play again in the next round). A restarted process does not necessarily lead to a successful

[(i)] We use the same notations as in [7], to ease further comparisons, with the exception of I_n, replaced here by $\mathcal{T}_n$

election, additional null rounds may happen. Note that no null rounds have occurred if the maximal non-empty urn contains only a single ball. We will use another random variable J denoting either the position of the maximal non-empty urn, if it contains $i > 1$ balls, or the position of the last non-empty urn *before* the maximal non-empty urn if the latter contains $i = 1$ balls. Moreover, in the case $i = 1$ we denote with K the number of balls in the urn j.

Instead of directly analyzing the parameters of interest (compare with [7]), we can analyze suitable statistics on the urn model, avoiding more cumbersome calculations. For example, for the analysis of the success probability S_n we will study the joint distribution $\mathbb{P}(J = j, I = i)$ of I and J, distinguishing between the cases $i = 1$ and $i \geq 2$.

2.1 Asymptotic analysis

We state, for later use, the following properties, for large n, which are crucial for our approach - early stage approximation of the relevant quantities defined on the urn model.

- ASYMPTOTIC INDEPENDENCE. We have asymptotic independence of urns, for all events related to urn j containing $\mathcal{O}(1)$ balls. This means, for instance that

$$\begin{aligned}&\mathbb{P}[\text{urn } j_1 \text{ contains } i_1 \text{ balls ; urn } j_2 \text{ contains } i_2 \text{ balls ; } \dots \text{; urn } j_k \text{ contains } i_k \text{ balls,}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{where } i_1, i_2, \dots, i_k = \mathcal{O}(1)]\\ &\sim \mathbb{P}[\text{urn } j_1 \text{ contains } i_1 \text{ balls }] \times \mathbb{P}[\text{urn } j_2 \text{ contains } i_2 \text{ balls }] \times \cdots \times \mathbb{P}[\text{urn } j_k \text{ contains } i_k \text{ balls }].\end{aligned}$$

  This is proved, by Poissonization-De-Poissonization, in [9], [11] and [4] (in this paper for $p = 1/2$, but the proof is easily adapted). The error term is $\mathcal{O}(n^{-C})$ where C is a positive constant.

- ASYMPTOTIC DISTRIBUTIONS. We obtain asymptotic distributions of the interesting RVs as follows. The number of balls in each urn is asymptotically Poisson-distributed with parameter npq^{j-1} in urn j containing $\mathcal{O}(1)$ balls (this is the classical asymptotic for the Binomial distribution). This means that the asymptotic number ℓ of balls in urn j is given by

$$\exp\left(-npq^{j-1}\right) \frac{\left(npq^{j-1}\right)^{\ell}}{\ell!},$$

  and with $Q := 1/q, L := \ln Q, \eta = j - \log np/q$, this is equivalent to $\mathcal{P}\left(e^{-L\eta}, \ell\right)$, where $\mathcal{P}(\lambda, u) := e^{-\lambda}\lambda^u/u!$. The asymptotic distributions are related to Gumbel distribution functions (given by $\exp\left(-e^{-x}\right)$) or convergent series of such. The error term is $\mathcal{O}(n^{-1})$.

- EXTENDED SUMMATIONS. Some summations now go to ∞. This is justified, for example, in [9].

- UNIFORM INTEGRABILITY. We have uniform integrability for the moments of our RVs. To show that the limiting moments are equivalent to the moments of the limiting distributions, we need a suitable rate of convergence. This is related to a uniform integrability condition (see Loève [6, Section 11.4]). For the kind of limiting distributions we consider here, the rate of convergence is analyzed in detail in [8] and [11]. The error term is $\mathcal{O}(n^{-C})$.

- MELLIN TRANSFORM. Asymptotic expressions for the moments are obtained by Mellin transforms. The error term is $\mathcal{O}(n^{-C})$. We proceed as follows (see [8] for detailed proofs): from the asymptotic properties of the urns, we have obtained the asymptotic distributions of our RV of interest. Next we compute the Laplace transform $\phi(\alpha)$ of these distributions, from which we can derive the dominant part of probabilities and moments as well as the (tiny) periodic part in the form of a Fourier series. This connection will be detailed in the next sections. Note that we will also need the first values of probabilities and moments, obtained via some recurrences.

- SLOW INCREASE PROPERTY. $\Gamma(s)$ decreases exponentially in the direction $i\infty$:

$$|\Gamma(\sigma+it)| \sim \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}.$$

  Also, we this property is true for all other functions we encounter. So inverting the Mellin transforms is easily justified.

If we compare the approach in this paper with other ones that appeared previously, then we can notice the following. Traditionally, one would stay with exact enumerations as long as possible, and only at a late stage move to asymptotics. Doing this, one would, in terms of asymptotics, carry many unimportant contributions around, which makes the computations quite heavy, especially when it comes to higher moments. Here, however, approximations are carried out as early as possible, and this allows for streamlined (and often automatic) computations of asymptotic distributions and higher moments.

The present paper falls into the paradigm of *combinatorics of geometrically distributed words*; many papers have been written on that since the mid-nineties. We cite only the first one in this series: [12]

3 Notation and Plan of the paper

3.1 Notation

We will use several abbreviations for probabilities and moments in order to derive more compact expressions. Throughout this work we denote with n the number of initial players; we assume that n is large, we derive asymptotic expansions with respect to $n \to \infty$. Moreover, let $n^* = n \cdot \frac{p}{q}$. Since the Poisson distribution is crucial for our approach - early stage asymptotic analysis - we use the shorthand notation $\mathcal{P}(\lambda, u) := e^{-\lambda}\lambda^u/u!$. For the asymptotic analysis we use $\eta := j - \log n^*$ or $\eta := \kappa - \log n^*$. The fractional part of x is denoted by $\{x\}$.

In the context of combinatorics of geometrically distributed words the following notation is frequently used: $Q := 1/q$, $\log := \log_Q$, and $L := \ln Q$. Moreover, $M := \log p$, and $\chi_l := \frac{2l\pi\mathbf{i}}{L}$, where $\mathbf{i}$ denotes the imaginary unit. The expected values of the parameters of interest number of rounds $\mathcal{R}_n$, number of null rounds $\mathcal{T}_n$, number $\mathcal{L}_n$, in case of failure, of players that were active at the last non-null round, the so-called *left-overs*, and the total number $\mathcal{C}_n$ of coins flipped, will be denoted with

$$R_n = \mathbb{E}(\mathcal{R}_n), \quad T_n = \mathbb{E}(\mathcal{T}_n), \quad L_n = \mathbb{E}(\mathcal{L}_n), \quad C_n := \mathbb{E}(\mathcal{C}_n).$$

Moreover, the second moments will be denoted with

$$R_n^{(2)} = \mathbb{E}(\mathcal{R}_n^2), \quad T_n^{(2)} = \mathbb{E}(\mathcal{T}_n^2), \quad L_n^{(2)} = \mathbb{E}(\mathcal{L}_n^2), \quad C_n^{(2)} := \mathbb{E}(\mathcal{C}_n^2).$$

Note that the "′" notation will always be used, in the sequel, in relation with the failure case, and the tilde will denote that the players were obtained in a null round; for example $\tilde{T}_i'$ denotes the average number of null rounds, starting with i players, with failure at the end, given that the i players were obtained in a null round, not preceded by another null round.

In our analysis we will encounter certain sums: $\Sigma_1(i,\tau) := \sum_{v=0}^{\tau-1}(p^i)^v$, $\Sigma_2(i,\tau) := \sum_{v=0}^{\tau-1}(p^i)^v(v+1)$, $\Sigma_3(i,\tau) := \sum_{v=0}^{\tau-1}(p^i)^v(v+1)^2$, $\Sigma_4(i,\tau) := \sum_{v=0}^{\tau-1}(p^i)^v v$ and $\Sigma_5(i,\tau) := \sum_{v=0}^{\tau-1}(p^i)^v v^2$. Closed form expressions can readily be obtained and are given in the Appendix. We will first analyze *Model 1*: we fail if we have τ consecutive null records. The *Model 2*: we fail if we have τ null records, consecutive or not, will be briefly considered in Appendix G.

3.2 Plan of the paper

The paper is organized as follows: Section 4 is devoted to the success probability, Section 5 to 8 give the asymptotic distribution and first two moments of the RVs of interest (all moments can be derived by the same technique): we compute the dominant and periodic part, both in the success and failure case. Appendices A summarizes some definitions and identities. In [7], was presented an analytic treatment of the Swedish leader election protocol. In particular, the success probability and the dominant part of the mean of the following RV was computed: total number of rounds, total number of null rounds, number of left-overs. We prove, in Appendices B to E, the equivalence with the results given in this paper. Some matrix expressions are presented in Appendix F and Appendix G briefly presents the second model where we fail if we have τ null records, consecutive or not. Appendix H gives the proof of Theorem 8.2.

4 Success probability

In order to analyze the success probability S_n we proceed in three steps. First, we deduce basic recurrence relations for the success probability based on the leader election algorithm. Second, we reformulate the problem in terms of the urn model. Third, we carry out the asymptotic analysis: we compute the Laplace transform of the asymptotic distributions, and finally derive the asymptotic expressions for them. It is beneficial to distinguish between starting with i players (used in the recurrences and the asymptotics) and n which denotes the initial number of players. Moreover, for stating the recurrence relations we use the notation $P(i) = S_i$, and $\tilde{P}(i)$ for the probability that, starting with i players, we succeed given that the i players were obtained in a null round, not preceded by another null round. We will first show how to compute the basic probabilities, which will be used in the sequel. Next, as explained in Section 2, we proceed to some asymptotic distributions.

Since we can have up to $\tau-1$ or $\tau-2$ null rounds (all killed), followed by ℓ survivors, we readily obtain

$$P(1) = 1, \quad \tilde{P}(1) = 1,$$
$$P(i) = \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P(\ell) = \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P(\ell),\ i\geq 2,$$
$$\tilde{P}(i) = \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P(\ell) = \Sigma_1(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P(\ell) = \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)}P(i),$$

$i \geq 2$. Note that $\ell = i$ in the right-hand side leads to $P(i)$.

4.1 Urn model correspondance

Instead of working with the basic recurrence relations, compare with [7], we use the fact that the leader election algorithm succeeds if in the corresponding urn model the maximal nonempty urn $I = 1$ has a single entry, or if it succeeds after a restart (null round) ($I > 1$). We consider the joint distribution of I and J. Recall that we have asymptotic independence of urns, for all events related to urn j containing $\mathcal{O}(1)$ balls. Also the number of balls in each such urn is now Poisson-distributed with parameter npq^{j-1} in urn j. The asymptotic number ℓ of balls in urn j is given by

$$\exp\left(-npq^{j-1}\right)\frac{\left(npq^{j-1}\right)^{\ell}}{\ell!} = \mathcal{P}\left(e^{-L\eta},\ell\right),$$

with $\eta = j - \log n^*$. The asymptotic probability that urns $j+1, j+2, \dots$ are empty is given by

$$\prod_{\ell=j+1}^{\infty} e^{-npq^{\ell-1}} = \mathcal{P}\Big(\frac{q}{p}e^{-L\eta},0\Big).$$

Similar simple calculations lead to

$$\mathbb{P}(J=j,I=i) \sim \mathcal{P}\Big(\frac{q}{p}e^{-L\eta},0\Big)\mathcal{P}\left(e^{-L\eta},i\right),\ i \geq 2,$$
$$\mathbb{P}(J=j,I=1) \sim \mathcal{P}\Big(\frac{q}{p}e^{-L\eta},1\Big)\left[1-\mathcal{P}\left(e^{-L\eta},0\right)\right].$$

Note that the case $i \geq 2$ does not necessarily lead to a success: urn J corresponds to the first null round, hence the multiplication by the success probability $\tilde{P}(i)$. On the other hand, the case $i = 1$ does lead to a success: urn J corresponds to a round with one single player alive, which is immediately declared as the leader (there are no null rounds before). We have from [10], (here and in the sequel $\sim$ always denotes $\sim_{n\to\infty}$), with $\eta := j - \log n^*$,

$$\mathbb{P}(J=j,I=i) \sim \exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{e^{-L\eta i}}{i!},\ i \geq 2, \tag{1}$$
$$\mathbb{P}(J=j,I\geq 2,S) \sim f_1(\eta),$$
$$f_1(\eta) := \sum_{i=2}^{\infty}\exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{e^{-L\eta i}}{i!}\tilde{P}(i),$$
$$\mathbb{P}(J=j,I=1) \sim f_2(\eta),$$
$$f_2(\eta) := \exp\Big(-\frac{q}{p}e^{-L\eta}\Big)\frac{q}{p}e^{-L\eta}\left(1-\exp\left(-e^{-L\eta}\right)\right). \tag{2}$$

4.2 Asymptotic analysis

We now compute the Laplace transform. This gives

$$\phi_1(\alpha) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_1(\eta)d\eta = \sum_{i=2}^{\infty}\frac{(1/p)^{-i+\tilde{\alpha}}}{Li!}\Gamma(i-\tilde{\alpha})\tilde{P}(i).$$

From [8] the corresponding dominant part of S_n is given by

$$\phi_1(0) = \sum_{i=2}^{\infty}\frac{p^i}{Li}\frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)}P(i).$$

The corresponding periodic part is given by

$$\omega_{1,1} = \sum_{l\neq 0} \varphi_{1,1}(\chi_l) e^{-2l\pi \mathbf{i}\{\log n^*\}},$$

with

$$\varphi_{1,1}(\chi_l) = \phi_1(\alpha)\big|_{\alpha=-L\chi_l}.$$

We obtain

$$\varphi_{1,1}(\chi_l) = \sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!} \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)} P(i).$$

Also,

$$\phi_2(\alpha) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_2(\eta) d\eta = \frac{1}{L}\left[\left(\frac{q}{p}\right)^{\tilde{\alpha}} - q\left(\frac{1}{p}\right)^{\tilde{\alpha}}\right]\Gamma(1-\tilde{\alpha}).$$

Hence

$$\phi_2(0) = \frac{p}{L} = \text{Probability that the maximal non-empty urn contains one ball,} \tag{3}$$

$$\omega_{2,1} = \sum_{l\neq 0} \varphi_{2,1}(\chi_l) e^{-2l\pi \mathbf{i}\{\log n^*\}},$$

$$\varphi_{2,1}(\chi_l) = \frac{1}{L}\left[\left(\frac{q}{p}\right)^{-\chi_l} - q\left(\frac{1}{p}\right)^{-\chi_l}\right]\Gamma(1+\chi_l) = \frac{p^{1+\chi_l}}{L}\Gamma(1+\chi_l).$$

And finally, with notations provided in the Appendix A, we have the following result.

Theorem 4.1 *The success probability S_n of the asymmetric leader election algorithm with swedish stopping satisfies the asymptotic expansion*

$$\begin{aligned} S_n &\sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)} P(i) + \frac{p}{L} + \sum_{l\neq 0} \varphi_{1,1}(\chi_l) e^{-2l\pi \mathbf{i}\{\log n^*\}} + \sum_{l\neq 0} \varphi_{2,1}(\chi_l) e^{-2l\pi \mathbf{i}\{\log n^*\}} \\ &=: V_1 + \frac{p}{L} + \sum_{l\neq 0} \varphi_{1,1}(\chi_l) e^{-2l\pi \mathbf{i}\{\log n^*\}} + \sum_{l\neq 0} \varphi_{2,1}(\chi_l) e^{-2l\pi \mathbf{i}\{\log n^*\}}. \end{aligned}$$

It might look paradoxical at first sight, that the asymptotic formula involves the quantities $P(i)$ on the right site. However, that happens often, and convergence is quite good, so that with only a few terms (obtained directly from the recursion) a good approximation of the numerical values can be obtained. This phenomenon will appear in all our subsequent analyses.

Of course, the failure probability $F_n = 1 - S_n$. Note also that as $\tau \to \infty$, the dominant part gives

$$\sum_{i=2}^{\infty} \frac{p^i}{Li} + \frac{p}{L} = 1$$

as expected. For further use, we denote

$$\Pi_1 := \frac{p}{L} \quad \text{(one ball in the maximal non-empty urn),}$$

$$\Pi_2(i) := \frac{p^i}{Li},$$
$$Pd(S) = V_1 + \frac{p}{L} \quad \text{(dominant part of } S_n\text{)}.$$

5 Asymptotic distribution and moments of $\mathcal{R}_n - \log n^*$

5.1 Asymptotic distribution and moments of $\mathcal{R}_n - \log n^*$ success case

We denote with $P_R(i,k)$ the probability that, starting with i players, we succeed after k rounds, and with $\tilde{P}_R(i,k)$ the probability that, starting with i players, we succeed after k rounds, given that the i players were obtained in a null round. Furthermore, let R_i denote the average number of rounds, starting with i players, with success at the end, and $\tilde{R}_i$ the average number of rounds, starting with i players, with success at the end, given that the i players were obtained in a null round, not preceded by another null round. We will first compute some asymptotic distributions, then the recurrences for the moments and finally the asymptotics for distributions and moments. In case of success, the moments of $\mathcal{R}_n - \log n^*$ are computed as in [10], and expressed with some $\tilde{R}_i$, $\tilde{R}_i^{(2)}$, instead of $x_{i,S}$, $x_{i,S}^{(2)}$ used in [10]. They are computed as described in the sequel. Here and in the sequel, we give the first two moments. All moments could be computed, only with more (algebraic and Maple) efforts.

We use $f_1(\eta)$ as given by (4) of [10], with $\tilde{P}_R(i,k)$ instead of $P(i,k)$ and $\eta := \kappa - \log n^*$, i.e.,

$$f_1(\eta) = \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \exp\Big(-\frac{1}{p}e^{Lk}e^{-L\eta}\Big)\frac{e^{-L\eta i}e^{Lki}}{i!}\tilde{P}_R(i,k).$$

Above we deal here with the case $I \geq 2$. The probability that $\mathcal{R} = \kappa$ is given by the probability that, at some time $\kappa - k$, all i players are dead multiplied by the probability that, starting with i players, we succeed after k rounds.

The quantity $f_2(\eta)$ is given by (2). Indeed, in the case $I = 1$, $\mathcal{R}$ is identical to J.

The first few exact values of the probabilities will be needed in the asymptotic formulæ. We can have up to $\tau - 1$ or $\tau - 2$ null rounds (all killed), followed by ℓ survivors. This takes $s + 1$ rounds already. This leads to the following recurrences.

$$P_R(1,0) = 1, \quad \tilde{P}_R(1,0) = 1,$$
$$P_R(1,\geq 1) = 0, \quad \tilde{P}_R(1,\geq 1) = 0,$$
$$P_R(i,k) = \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_R(\ell,k-1-s),\ i \geq 2,$$
$$\tilde{P}_R(i,k) = \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_R(\ell,k-1-s),\ i \geq 2.$$

Concerning the first two moments we obtain the following results.

$$R_i = \sum_k P_R(i,k)k = \sum_k \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_R(\ell,k-1-s)[k-1-s+s+1],$$

$$
\begin{aligned}
&= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_\ell + \sum_k \sum_{s=0}^{\tau-1} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_R(\ell, k-1-s) \\
&= \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_\ell + \Sigma_2(i,\tau) P(i)/\Sigma_1(i,\tau),\ R_1 = 0, \\
\tilde{R}_i &= \sum_k \tilde{P}_R(i,k) k = \sum_k \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_R(\ell, k-1-s)[k-1-s+s+1], \\
&= \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_\ell + \sum_k \sum_{s=0}^{\tau-2} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_R(\ell, k-1-s) \\
&= \Sigma_1(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_\ell + \Sigma_2(i,\tau-1) P(i)/\Sigma_1(i,\tau),\ \tilde{R}_1 = 0, \\
R_i^{(2)} &= \sum_k P_R(i,k) k^2 = \sum_k \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_R(\ell, k-1-s)[k-1-s+s+1]^2, \\
&= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_\ell^{(2)} + \sum_k \sum_{s=0}^{\tau-1} (p^i)^s (s+1)^2 \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_R(\ell, k-1-s) \\
&+ 2 \sum_{s=0}^{\tau-1} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_\ell \\
&= \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_\ell^{(2)} + \Sigma_3(i,\tau) P(i)/\Sigma_1(i,\tau) + 2\Sigma_2(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_\ell,\ R_1^{(2)} = 0, \\
\tilde{R}_i^{(2)} &= \sum_k \tilde{P}_R(i,k) k^2 = \sum_k \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_R(\ell, k-1-s)[k-1-s+s+1]^2, \\
&= \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_\ell^{(2)} + \sum_k \sum_{s=0}^{\tau-2} (p^i)^s (s+1)^2 \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_R(\ell, k-1-s) \\
&+ 2 \sum_{s=0}^{\tau-2} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_\ell \\
&= \Sigma_1(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_\ell^{(2)} + \Sigma_3(i,\tau-1) P(i)/\Sigma_1(i,\tau) \\
&+ 2\Sigma_2(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_\ell,\ \tilde{R}_1^{(2)} = 0.
\end{aligned}
$$

Note that, in previous expressions, we could replace

$$\sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_\ell \qquad \text{by} \qquad [R_i - \Sigma_2(i,\tau)P(i)/\Sigma_1(i,\tau)]/\Sigma_1(i,\tau).$$

As the details of the Laplace transforms are given in [10], we will not repeat them here. Let us however summarize the general connection between $\phi(\alpha)$ and the moments of the corresponding RV. From [8], we have

$$\begin{aligned}
\mathbb{E}(\mathcal{R}_n - \log n^*)^k &\sim \tilde{m}_k + w_k,\\
\tilde{m}_k &= \phi^{(k)}(0),\\
w_k &= \sum_{l\neq 0} \Upsilon_k^*(\chi_l) e^{-2l\pi i\{\log n^*\}},\\
\Upsilon_k^*(s) &= \phi^{(k)}(\alpha)\Big|_{\alpha=-Ls},
\end{aligned}$$

To obtain the moments of $\mathcal{R}_n - \log n^*$, we plug, *mutatis mutandis*, $\tilde{R}_i$, $\tilde{R}_i^{(2)}$ into the moments given in [10]. Note that to each value $I = i \geq 2$ corresponds $\tilde{P}(i)$ as explained in Section 4. Also $A_0(\chi_l)$ is no more null here and $\tilde{P}(1) = 1$ by convention. This leads, with the quantities defined in the Appendix A, to the next result.

Theorem 5.1 *The asymptotic distribution and moments of the number of rounds $\mathcal{R}_n - \log n^*$ in the asymmetric leader election algorithm with swedish stopping, the success case, satisfies*

$$\begin{aligned}
&\mathbb{P}(\mathcal{R}_n = \kappa) \sim f_1(\eta) + f_2(\eta),\\
&R_n - \log n^* = \mathbb{E}(\mathcal{R}_n - \log n^*) \sim U_1 - MV_1 - \frac{V_2}{L} + \frac{p\gamma}{L^2} - \frac{1+pM}{L}\\
&\qquad + \sum_{l\neq 0}\left[B_1(\chi_l) - MA_0(\chi_l) - \frac{A_1(\chi_l)}{L} - \frac{\Gamma(1+\chi_l)}{L}\right] e^{-2l\pi\mathbf{i}\{\log n\}},\\
&\mathbb{E}(\mathcal{R}_n - \log n^*)^2 \sim U_2 - 2MU_1 - 2\frac{U_4}{L} + M^2V_1 + 2M\frac{V_2}{L} + \frac{V_3+V_4}{L^2}\\
&\qquad + \frac{p(\pi^2/6+\gamma^2)}{L^3} - \frac{2\gamma(pM+1)}{L^2} + \frac{pM^2+2M+1}{L}\\
&\qquad + \sum_{l\neq 0}\left\{B_2(\chi_l) - 2MB_1(\chi_l) - 2\frac{B_3(\chi_l)}{L} + M^2A_0(\chi_l) + 2M\frac{A_1(\chi_l)}{L} + \frac{A_2(\chi_l)+A_3(\chi_l)}{L^2}\right.\\
&\qquad \left. + \Gamma(1+\chi_l)\left[2\frac{\psi(1+\chi_l)}{L^2} + \frac{1}{L} + 2\frac{M}{L}\right]\right\} e^{-2l\pi\mathbf{i}\{\log n\}}.
\end{aligned}$$

Note that the periodic component contains $\{\log n\}$ in the exponent (and not $\{\log n^*\}$). Note also that, again, the asymptotics depend on R_i on the right-side.

To obtain the moments of $\mathcal{R}_n - \log n^*$, *given success*, we simply divide the moments given in the theorem by S_n.

5.2 Asymptotic distribution and moments of $\mathcal{R}_n - \log n^*$, failure case

The analysis of this RV (as well as the next ones, with the exception of the number of flipped coins) follows the same pattern as the previous ones. We will only present the necessary expressions. In the following we denote with $P'_R(i,k)$ the probability that, starting with i players, we fail after k rounds, and with $\tilde{P}'_R(i,k)$ the probability that, starting with i players, we fail after k rounds, given that the i players were obtained in a null round. Moreover, let $P'(i)$ denote the probability that, starting with i players, we fail, such that $P'(i) = 1 - P(i)$, and let $\tilde{P}'(i)$ denote the probability that, starting with i players, we fail, given that the i players were obtained in a null round, such that $\tilde{P}'(i) = 1 - \tilde{P}(i)$. Finally, the mean number of rounds, starting with i players, with failure at the end is denoted by R'_i and the mean number of rounds, starting with i players, with failure at the end, given that the i players were obtained in a null round, is denoted by $\tilde{R}'_i$. Note that the "$'$" notation will always be used, in the sequel, in relation with the failure case. In case of failure, the moments of $\mathcal{R}_n - \log n^*$ are computed as in [10], with some $\tilde{R}'_i$, $\tilde{R}^{(2)\prime}_i$, computed as follows. First we have

$$\begin{aligned}
P'(i) = 1 - P(i) &= (p^i)^\tau + \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'(\ell) \\
&= (p^i)^\tau + \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'(\ell),\ i \geq 2,\ P'(1) = 0.
\end{aligned}$$

Next the recurrences: we can have τ or $\tau - 1$ null rounds (all killed) at start, leading to failure. We readily obtain.

$$\begin{aligned}
P'_R(1,0) &= 0, \quad \tilde{P}'_R(1,0) = 0, \\
P'_R(1,\geq 1) &= 0, \quad \tilde{P}'_R(1,\geq 1) = 0, \\
P'_R(i,k) &= (p^i)^\tau [\![k=\tau]\!] + \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_R(\ell, k-1-s),\ i \geq 2, \\
\tilde{P}'_R(i,k) &= (p^i)^{\tau-1} [\![k=\tau-1]\!] + \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} P'_R(\ell, k-1-s),\ i \geq 2.
\end{aligned}$$

$$\begin{aligned}
R'_i &= (p^i)^\tau \tau + \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} R'_\ell + \Sigma_2(i,\tau)[P'(i) - (p^i)^\tau]/\Sigma_1(i,\tau),\ R'_1 = 0, \\
\tilde{R}'_i &= (p^i)^{\tau-1}(\tau-1) + \Sigma_1(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell} R'_\ell \\
&\quad + \Sigma_2(i,\tau-1)[P'(i) - (p^i)^\tau]/\Sigma_1(i,\tau),\ \tilde{R}'_1 = 0,
\end{aligned}$$

and similar equations for $R^{(2)\prime}_i, \tilde{R}^{(2)\prime}_i$. To obtain the moments of $\mathcal{R}_n - \log n^*$, we plug $\tilde{R}'_i, \tilde{R}^{(2)\prime}_i$ into the moments given in [10], *based only on* $f_1(\eta)$ as given by (4) of [10], with $\tilde{P}'_R(i,k)$ instead of $P(i,k)$, and again $\eta := \kappa - \log n^*$, i.e.,

$$f_1(\eta) = \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \exp\Big(-\frac{1}{p} e^{Lk} e^{-L\eta}\Big) \frac{e^{-L\eta i} e^{Lki}}{i!} \tilde{P}'_R(i,k).$$

Indeed, a maximal non-empty urn with only 1 ball leads to a success. Note that to each value $I = i \geq 2$ corresponds $\tilde{P}'(i)$. Also $A_0(\chi_l)$ is no more null here and $\tilde{P}'(1) = 0$ by convention. This gives

Theorem 5.2 *The asymptotic distribution and moments of the shifted number of rounds $\mathcal{R}_n - \log n^*$ in the asymmetric leader election algorithm with swedish stopping, failure case, satisfies*

$$\begin{aligned}
\mathbb{P}'(\mathcal{R}_n = \kappa) &\sim f_1(\eta),\\
R'_n - \log n^* = \mathbb{E}(\mathcal{R}_n - \log n^*) &\sim U_1 - MV_1 - \frac{V_2}{L}\\
&\quad + \sum_{l\neq 0}\left[B_1(\chi_l) - MA_0(\chi_l) - \frac{A_1(\chi_l)}{L}\right]e^{-2l\pi\mathbf{i}\{\log n\}},\\
\mathbb{E}(\mathcal{R}_n - \log n^*)^2 &\sim U_2 - 2MU_1 - 2\frac{U_4}{L} + M^2V_1 + 2M\frac{V_2}{L} + \frac{V_3+V_4}{L^2}\\
&\quad + \sum_{l\neq 0}\Bigg\{B_2(\chi_l) - 2MB_1(\chi_l) - 2\frac{B_3(\chi_l)}{L} + M^2A_0(\chi_l)\\
&\qquad + 2M\frac{A_1(\chi_l)}{L} + \frac{A_2(\chi_l)+A_3(\chi_l)}{L^2}\Bigg\}e^{-2l\pi\mathbf{i}\{\log n\}}.
\end{aligned}$$

To obtain the moments of $\mathcal{R}_n - \log n^*$, *given failure*, we simply divide the moments given in the theorem by F_n.

6 Asymptotic distribution and moments of $\mathcal{T}_n$ (null rounds)

6.1 Asymptotic distribution and moments of $\mathcal{T}_n$ (null rounds), with success

Let $P_T(i,t)$ denote the probability that, starting with i players, we succeed with t null rounds, and $\tilde{P}_T(i,t)$ the probability that, starting with i players, we succeed with t null rounds, given that the i players were obtained in a null round. Furthermore, we denote with $P_T(t)$ the probability that, starting with n players, we succeed with t null rounds, with T_i the average number of null rounds, starting with i players, with success at the end, and with $\tilde{T}_i$ the average number of null rounds, starting with i players, with success at the end, given that the i players were obtained in a null round.

The analysis is similar to that of $\mathcal{R}_n$. Concerning the basic recurrence relations we use the following considerations: we can have up to $\tau - 1$ or $\tau - 2$ null rounds (all killed), followed by ℓ survivors. This leads to s or $s+1$ null rounds already. Hence, we obtain:

$$\begin{aligned}
P_T(1,0) &= 1, \quad \tilde{P}_T(1,0) = 1,\\
P_T(1,\geq 1) &= 0, \quad \tilde{P}_T(1,\geq 1) = 0,\\
P_T(i,t) &= \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_T(\ell,t-s),\ i\geq 2,\\
\tilde{P}_T(i,t) &= \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_T(\ell,t-1-s),\ i\geq 2.
\end{aligned}$$

Moreover, the first two moments satisfy

$$\begin{aligned}
T_i &= \sum_t P_T(i,t)t = \sum_t \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-s)[t-s+s],\\
&= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T_\ell + \sum_t \sum_{s=0}^{\tau-1} (p^i)^s s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-s)\\
&= \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T_\ell + \Sigma_4(i,\tau) P(i)/\Sigma_1(i,\tau),\ T_1 = 0,\\
\tilde{T}_i &= \sum_t \tilde{P}_T(i,t)t = \sum_t \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-1-s)[t-1-s+s+1],\\
&= \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T_\ell + \sum_t \sum_{s=0}^{\tau-2} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-1-s)\\
&= \Sigma_1(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T_\ell + \Sigma_2(i,\tau-1) P(i)/\Sigma_1(i,\tau),\ \tilde{T}_1 = 1,\\
T_i^{(2)} &= \sum_t P_T(i,t)t^2 = \sum_t \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-s)[t-s+s]^2,\\
&= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T_\ell^{(2)} + \sum_t \sum_{s=0}^{\tau-1} (p^i)^s s^2 \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-1-s)\\
&+ 2 \sum_{s=0}^{\tau-1} (p^i)^s s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T_\ell\\
&= \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T_\ell^{(2)} + \Sigma_5(i,\tau) P(i)/\Sigma_1(i,\tau) + 2\Sigma_4(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T_\ell,\ T_1^{(2)} = 1,\\
\tilde{T}_i^{(2)} &= \sum_t \tilde{P}_T(i,t)t^2 = \sum_t \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-1-s)[t-1-s+s+1]^2,\\
&= \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T_\ell^{(2)} + \sum_t \sum_{s=0}^{\tau-2} (p^i)^s (s+1)^2 \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_T(\ell, t-1-s)\\
&+ 2 \sum_{s=0}^{\tau-2} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T_\ell\\
&= \Sigma_1(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T_\ell^{(2)} + \Sigma_3(i,\tau-1) P(i)/\Sigma_1(i,\tau)
\end{aligned}$$

$$+ 2\Sigma_2(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}T_\ell,\ \tilde{T}_1^{(2)} = 0.$$

Again, as in previous expressions, we could replace

$$\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}T_\ell \quad \text{by} \quad [T_i - \Sigma_4(i,\tau)P(i)/\Sigma_1(i,\tau)]/\Sigma_1(i,\tau).$$

Next, with (1) and $\eta := j - \log n^*$,

$$\mathbb{P}(J=j, \mathcal{T}_n = t) \sim f_3(\eta,t),$$
$$f_3(\eta,t) = \sum_{i=2}^{\infty}\exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{e^{-L\eta i}}{i!}\tilde{P}_T(i,t).$$

Hence

$$\phi_3(\alpha,t) = \int_{-\infty}^{\infty} e^{\alpha\eta}f_3(\eta,t)d\eta = \sum_{i=2}^{\infty}\frac{(1/p)^{-i+\tilde{\alpha}}}{Li!}\Gamma(i-\tilde{\alpha})\tilde{P}_T(i,t).$$

Note that there are no null rounds if the maximal non-empty urn contains only 1 ball.

The dominant component of $P_T(t)$ is given by

$$\phi_3(0,t) = \sum_{i=2}^{\infty}\frac{(1/p)^{-i}}{Li!}\Gamma(i)\tilde{P}_T(i,t) = \sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{P}_T(i,t),$$

and the periodic component by

$$\omega_{1,3}(t) = \sum_{l\neq 0}\varphi_3(\chi_l,t)e^{-2l\pi\mathbf{i}\{\log n^*\}},$$

with

$$\varphi_3(\chi_l,t) = \phi_3(\alpha,t)\big|_{\alpha=-L\chi_l}.$$

We obtain

$$\varphi_3(\chi_l,t) = \sum_{i=2}^{\infty}\frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{P}_T(i,t).$$

Hence, we have the following theorem.

Theorem 6.1 *The asymptotic distribution of the number $\mathcal{T}_n$ of null rounds, with success, in the asymmetric leader election algorithm with swedish stopping is given by*

$$P_T(t) = \mathbb{P}(\mathcal{T}_n = t) \sim \sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{P}_T(i,t) + \sum_{l\neq 0}\varphi_3(\chi_l,t)e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

The moments are given by

$$T_n \sim \sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{T}_i + \sum_{l\neq 0}\sum_{i=2}^{\infty}\frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{T}_i e^{-2l\pi\mathbf{i}\{\log n^*\}},$$

$$T_n^{(2)} \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{T}_i^{(2)} + \sum_{l \neq 0} \sum_{i=2}^{\infty} \frac{p^{i+\chi_l} \Gamma(i+\chi_l)}{Li!} \tilde{T}_i^{(2)} e^{-2l\pi \mathbf{i} \{\log n^*\}}.$$

Note that $\mathcal{T}_n = \mathcal{O}(1)$.

6.2 Asymptotic distribution and moments of $\mathcal{T}_n$ (null rounds), with failure

Let $P'_T(i,t)$ denote the probability that, starting with i players, we fail with t null rounds, and $\tilde{P}'_T(i,t)$ the probability that, starting with i players, we fail with t null rounds, given that the i players were obtained in a null round. We denote with $P'_T(t)$ the probability that, starting with n players, we fail with t null rounds, with T'_i the average number of null rounds, starting with i players, with failure at the end, and with $\tilde{T}'_i$ the average number of null rounds, starting with i players, with failure at the end, given that the i players were obtained in a null round.

Since we can have τ or $\tau-1$ null rounds (all killed) at start, leading to failure, we obtain the recurrences:

$$\begin{aligned}
P'_T(1,0) &= 0, \quad \tilde{P}'_T(1,0) = 0, \\
P'_T(1,\geq 1) &= 0, \quad \tilde{P}'_T(1,\geq 1) = 0, \\
P'_T(i,t) &= (p^i)^\tau [\![t=\tau]\!] + \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P'_T(\ell, t-s), \\
\tilde{P}'_T(i,t) &= (p^i)^{\tau-1} [\![t=\tau]\!] + \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P'_T(\ell, t-1-s).
\end{aligned}$$

Furthermore, the average numbers satisfy

$$\begin{aligned}
T'_i &= \sum_t P'_T(i,t)t = (p^i)^\tau \tau + \sum_t \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P'_T(\ell, t-s)[t-s+s] \\
&= (p^i)^\tau \tau + \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T'_\ell + \sum_t \sum_{s=0}^{\tau-1} (p^i)^s s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P'_T(\ell, t-s) \\
&= (p^i)^\tau \tau + \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T'_\ell + \Sigma_4(i,\tau)[P'(i) - (p^i)^\tau]/\Sigma_1(i,\tau), \; T'_1 = 0, \\
\tilde{T}'_i &= \sum_t \tilde{P}'_T(i,t)t = (p^i)^{\tau-1} \tau + \sum_t \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P'_T(\ell, t-1-s)[t-1-s+s+1] \\
&= (p^i)^{\tau-1} \tau + \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T'_\ell + \sum_t \sum_{s=0}^{\tau-2} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P'_T(\ell, t-1-s) \\
&= (p^i)^{\tau-1} \tau + \Sigma_1(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} T'_\ell + \Sigma_2(i,\tau-1)[P'(i) - (p^i)^\tau]/\Sigma_1(i,\tau), \; \tilde{T}'_1 = 0.
\end{aligned}$$

Next, with (1), again with $\eta := j - \log n^*$,

$$\mathbb{P}'(J=j, \mathcal{T}_n = t) \sim f_4(\eta, t),$$

$$f_4(\eta,t)=\sum_{i=2}^{\infty}\exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{e^{-L\eta i}}{i!}\tilde{P}'_T(i,t).$$

Hence

$$\phi_4(\alpha,t)=\int_{-\infty}^{\infty}e^{\alpha\eta}f_4(\eta,t)d\eta=\sum_{i=2}^{\infty}\frac{(1/p)^{-i+\tilde{\alpha}}}{Li!}\Gamma(i-\tilde{\alpha})\tilde{P}'_T(i,t).$$

Note that there are no null rounds if the maximal non-empty urn contains only 1 ball.

The dominant component of $P'_T(t)$ is given by

$$\phi_4(0,t)=\sum_{i=2}^{\infty}\frac{p^i}{Li!}\Gamma(i)\tilde{P}'_T(i,t),$$

and the periodic component by

$$\omega_{1,4}(t)=\sum_{l\neq 0}\varphi_4(\chi_l,t)e^{-2l\pi\mathbf{i}\{\log n^*\}},$$

with

$$\varphi_4(\chi_l,t)=\phi_4(\alpha,t)\big|_{\alpha=-L\chi_l}.$$

We obtain

$$\varphi_4(\chi_l,t)=\sum_{i=2}^{\infty}\frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{P}'_T(i,t).$$

Hence

Theorem 6.2 *The asymptotic distribution of the number $\mathcal{T}_n$ of null rounds, with failure, is given by*

$$P'_T(t)=\mathbb{P}'(\mathcal{T}_n=t)\sim\sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{P}'_T(i,t)+\sum_{l\neq 0}\varphi_4(\chi_l,t)e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

The moments are given by

$$T'_n\sim\sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{T}'_i+\sum_{l\neq 0}\sum_{i=2}^{\infty}\frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{T}'_i e^{-2l\pi\mathbf{i}\{\log n^*\}},$$

$$T^{(2)\prime}_n\sim\sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{T}^{(2)\prime}_i+\sum_{l\neq 0}\sum_{i=2}^{\infty}\frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{T}^{(2)\prime}_i e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

Note that $\mathcal{T}_n=\mathcal{O}(1)$.

7 Asymptotic distribution and moments of $\mathcal{L}_n$ (leftovers), with failure

Let $P'_L(i,r)$ denote the probability that, starting with i players, we fail, with r players remaining at the end (leftovers), and $\tilde{P}'_L(i,r)$ the probability that, starting with i players, we fail, with r players remaining at the end (leftovers), given that the i players were obtained in a null round. Moreover, let $P'_L(r)$ denote the probability that, starting with n players, we fail, with r players remaining at the end (leftovers). Concern the average numbers we use the notation, L'_i for the average number of leftovers, starting with i players, with failure at the end, and $\tilde{L}'_i$ for the average number of leftovers, starting with i players, with failure at the end, given that the i players were obtained in a null round. Since we have r players alive at start or r players alive before the starting null round, we obtain the following recurrences:

$$\begin{aligned}
P'_L(r,r) &= (p^r)^\tau + \Sigma_1(r,\tau)q^r P'_L(r,r), \text{ hence}\\
P'_L(r,r) &= \frac{p^{r\tau}(1-p^r)}{1-p^r-q^r+q^r p^{r\tau}},\\
\tilde{P}'_L(r,r) &= (p^r)^{\tau-1} + \Sigma_1(r,\tau-1)q^r P'_L(r,r).
\end{aligned}$$

We can have up to $\tau-1$ or $\tau-2$ null rounds (all killed), followed by ℓ survivors.

$$\begin{aligned}
P'_L(i,r) &= \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=r}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P'_L(\ell,r) = \Sigma_1(i,\tau)\sum_{\ell=r}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P'_L(\ell,r),\ i>r,\ i\geq 2,\\
\tilde{P}'_L(i,r) &= \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=r}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P'_L(\ell,r) = \Sigma_1(i,\tau-1)\sum_{\ell=r}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P'_L(\ell,r),\ i>r,\ i\geq 2,\\
&= \Sigma_1(i,\tau-1)/\Sigma_1(i,\tau)P'_L(i,r),\ i>r,\ i\geq 2.
\end{aligned}$$

Moreover, we get

$$\begin{aligned}
L'_i &= \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}}i + \sum_{r=0}^{i-1}P'_L(i,r)r\\
&= \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}}i + \Sigma_1(i,\tau)\sum_{r=0}^{i-1}\sum_{\ell=r}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P'_L(\ell,r)r,\\
&= \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}}i + \Sigma_1(i,\tau)\left[\sum_{\ell=0}^{i-1}\binom{i}{\ell}q^\ell p^{i-\ell}\sum_{r=0}^{\ell}P'_L(\ell,r)r + \sum_{r=0}^{i-1}q^i P'_L(i,r)r\right]\\
&= \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}}i + \Sigma_1(i,\tau)\left[\sum_{\ell=0}^{i-1}\binom{i}{\ell}q^\ell p^{i-\ell}L'_\ell + q^i\left[L'_i - \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}}i\right]\right],\\
\tilde{L}'_i &= \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)}\sum_{r=0}^{i-1}P'_L(i,r)r + \left[(p^i)^{\tau-1} + \Sigma_1(i,\tau-1)q^i P'_L(i,i)\right]i,
\end{aligned}$$

and similar equations for $L_i^{(2)\prime}$, $\tilde{L}_i^{(2)\prime}$.

Next, with (1) and $\eta := j - \log n^*$,

$$\mathbb{P}'(J = j, \mathcal{L}_n = r) \sim f_5(\eta, r),$$
$$f_5(\eta, r) = \sum_{i=2}^{\infty} \exp\Big(-\frac{1}{p} e^{-L\eta}\Big) \frac{e^{-L\eta i}}{i!} \tilde{P}'_L(i, r).$$

Hence

$$\phi_5(\alpha, r) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_5(\eta, r) d\eta = \sum_{i=2}^{\infty} \frac{(1/p)^{-i+\tilde{\alpha}}}{Li!} \Gamma(i - \tilde{\alpha}) \tilde{P}'_L(i, r).$$

Note that there are no leftovers if the maximal non-empty urn contains only 1 ball.

The dominant component of $P'_L(r)$ is given by

$$\phi_5(0, r) = \sum_{i=2}^{\infty} \frac{(1/p)^{-i}}{Li!} \Gamma(i) \tilde{P}'_L(i, r) = \sum_{i=2}^{\infty} \frac{p^i}{Li} \Gamma(i) \tilde{P}'_L(i, r),$$

and the periodic component by

$$\omega_{1,5}(r) = \sum_{l \neq 0} \varphi_5(\chi_l, r) e^{-2l\pi\mathbf{i}\{\log n^*\}},$$

with

$$\varphi_5(\chi_l, r) = \phi_5(\alpha)\big|_{\alpha = -L\chi_l}.$$

We obtain

$$\varphi_5(\chi_l, r) = \sum_{i=2}^{\infty} \frac{p^{i+\chi_l} \Gamma(i + \chi_l)}{Li!} \tilde{P}'_L(i, r).$$

Hence

Theorem 7.1 *The asymptotic distribution of the number $\mathcal{L}_n$ of leftovers, with failure, is given by*

$$P'_L(r) = \mathbb{P}'(\mathcal{L}_n = r) \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \frac{\Sigma_1(i, \tau - 1)}{\Sigma_1(i, \tau)} P'_L(i, r) + \sum_{l \neq 0} \varphi_5(\chi_l, r) e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

The moments are given by

$$L'_n \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{L}'_i + \sum_{l \neq 0} \sum_{i=2}^{\infty} \frac{p^{i+\chi_l} \Gamma(i + \chi_l)}{Li!} \tilde{L}'_i e^{-2l\pi\mathbf{i}\{\log n^*\}},$$
$$L_n^{(2)\prime} \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{L}_i^{(2)\prime} + \sum_{l \neq 0} \sum_{i=2}^{\infty} \frac{p^{i+\chi_l} \Gamma(i + \chi_l)}{Li!} \tilde{L}_i^{(2)\prime} e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

Note that $\mathcal{L}_n = \mathcal{O}(1)$.

8 Asymptotic distribution and moments of $\mathcal{C}_n$ (coins flipped)

8.1 Asymptotic distribution and moments of $\mathcal{C}_n$ (coins flipped), with success

This RV is more delicate to analyze. In previous cases, all interesting RV were related to urns (at high level) containing $\mathcal{O}(1)$ balls. Here *all* urns contribute to $\mathcal{C}_n$, so we must include the contribution of urns before J, which actually lead to the dominant part of $\mathcal{C}_n$. Also some correlations must be taken into account. We obtain the dominant and corrected terms of the moments as well as a central limit theorem.

Let C_i denote the Average number of coins flipped, starting with i players, with success at the end, and $\tilde{C}_i$ the average number of coins flipped, starting with i players, with success at the end, given that the i players were obtained in a null round.

8.1.1 Case $I = 1$

Note that, as explained in Section 4, this case *entails a success*. We will only deal here with the non-periodic part of our expressions. The maximal non-empty urn contains 1 ball and the position of the last non-empty urn *before* this maximal non-empty urn is denoted by J. Let us also denote by K the number of balls in urn J.

$$\begin{aligned}
&\mathbb{P}(J=j, K=k) \sim f_6(\eta,k),\ k \geq 1,\\
&\qquad f_6(\eta,k) := \exp\Big(-\frac{q}{p}e^{-L\eta}\Big)\frac{q}{p}e^{-L\eta}\exp\left(-e^{-L\eta}\right)\frac{e^{-L\eta k}}{k!},\\
&\qquad\qquad = \exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{q}{p}\frac{e^{-L\eta(k+1)}}{k!},
\end{aligned}$$

because

$$\mathbb{P}(J=j, K=k) \sim \mathcal{P}\Big(\frac{q}{p}e^{-L\eta},1\Big)\mathcal{P}\left(e^{-L\eta},k\right).$$

We have

$$\begin{aligned}
\phi_6(\alpha,k) &= \int_{-\infty}^{\infty} e^{\alpha\eta} f_6(\eta,k)d\eta = \frac{q}{Lk!}\left(\frac{1}{p}\right)^{\tilde{\alpha}-k}\Gamma(1-\tilde{\alpha}+k),\\
\Pi_4(k) &:= \phi_6(0,k) = \frac{q}{L}p^k.
\end{aligned}$$

Note that

$$Z_1 := \sum_{k=1}^{\infty}\Pi_4(k) = \frac{p}{L} \equiv \Pi_1 \quad \text{(one ball in the maximal non-empty urn)}$$

which conforms to (3).

Let us denote by Δ the *difference* between the maximal non-empty urn (containing 1 ball) and J. We have

$$\begin{aligned}
&\mathbb{P}(J=j, I=1, \Delta=\delta) \sim f_7(\eta,\delta),\\
&\qquad f_7(\eta,\delta) := \exp\Big(-\frac{q}{p}e^{-L\eta}\Big)e^{-L(\eta+\delta)}\left(1-\exp\left(-e^{-L\eta}\right)\right),
\end{aligned}$$

$$= \exp(-L\delta)\exp\Big(-\frac{q}{p}e^{-L\eta}\Big)e^{-L\eta}\left(1-\exp\left(-e^{-L\eta}\right)\right),$$

which shows that Δ is asymptotically independent of J.

This can readily be seen as follows:

$$\mathbb{P}(J=j, I=1, \Delta=\delta) \sim \mathcal{P}\left(e^{-L(\eta+1)},0\right)\mathcal{P}\left(e^{-L(\eta+2)},0\right)\dots$$
$$\dots\mathcal{P}\left(e^{-L(\eta+\delta)},1\right)\mathcal{P}\left(e^{-L(\eta+\delta+1)},0\right)\dots\left[1-\mathcal{P}\left(e^{-L\eta},0\right)\right].$$

We have

$$\begin{aligned}\phi_7(\alpha,\delta) &= \int_{-\infty}^{\infty} e^{\alpha\eta} f_7(\eta,\delta)d\eta \\ &= \exp(-L\delta)\frac{p}{Lq}\left[\left(\frac{q}{p}\right)^{\tilde{\alpha}} - q\left(\frac{1}{p}\right)^{\tilde{\alpha}}\right]\Gamma(1-\tilde{\alpha}), \\ \Pi_5(\delta) := \phi_7(0,\delta) &= e^{-L\delta}\frac{p^2}{Lq} = q^{\delta}\frac{p^2}{Lq}.\end{aligned}$$

Note that

$$\sum_{\delta=1}^{\infty}\Pi_5(\delta) = \frac{p}{L} \equiv \Pi_1$$

which again conforms to (3). We have

$$\mathbb{E}(\Delta) = \frac{1}{L} \quad \text{and} \quad \mathbb{E}(\Delta^2) = \frac{1+q}{Lp}.$$

However, note carefully that the player corresponding to $I=1$ is actually related to a *flipped coin in urn* J. So we must use a new RV G, denoting the number of flipped coins at step J: $G=K+1$, $G\geq 2$, with distribution

$$\Pi_6(g) = \frac{q}{L}p^{g-1},\ g\geq 2$$

and

$$f_8(\eta,g) = \exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{q}{p}\frac{e^{-L\eta g}}{(g-1)!}.$$

We will also need

$$Z_5 = \mathbb{E}(G) := \sum_{g=2}^{\infty}\Pi_6(g)g = \frac{p(1+q)}{Lq}.$$

Later on, we will use the following variants:

$$e^{-L\eta}f_8(\eta,g),\quad e^{-2L\eta}f_8(\eta,g),\quad \eta f_8(\eta,g),\quad e^{-L\eta}\eta f_8(\eta,g),\quad e^{-2L\eta}\eta f_8(\eta,g).$$

These variants lead respectively to $\phi_{\cdot}(0,g)$:

$$\frac{p^g qg}{L},$$

$$\begin{aligned}
&\frac{qp^{g+1}g(g+1)}{L},\\
&-\frac{p^{g-1}q[(g-1)\ln(p)+(g-1)\psi(g-1)+1]}{L^2(g-1)},\\
&-\left[qp^g[2(g-1)+(g-1)^2\ln(p)+(g-1)^2\psi(g-1)+(g-1)\ln(p)\right.\\
&\left.+(g-1)\psi((g-1))+1]\right]/[L^2(g-1)],\\
&\Omega_{15}(g) \quad \text{is too long to be displayed here.}
\end{aligned}$$

This leads to $Z_7, Z_8, Z_{11}, Z_{12}, Z_{10}, Z_{13}, Z_{15}$ as given in Appendix A: we *simply sum on* $g \geq 2$. Indeed, the case $I = 1$ immediately leads to a success.

Now we will separate the contribution of urn J (containing G balls) from that of urns $< J$.

Let us denote by $S_\Gamma(j,i)$ the sum of $(n-i)$ iid RV $\Gamma(j)$, and $\Gamma(j)$ is a truncated geometric RV $< j$. As $\Sigma_0 := \sum_{l=1}^{j-1} pq^{l-1} = 1 - q^{j-1}$, we have (we give only the terms needed in the sequel)

$$\begin{aligned}
E(j) &:= \mathbb{E}(\Gamma(j)) = \sum_{l=1}^{j-1} pq^{l-1} l/\Sigma_0 \sim \frac{1}{p} + q^{j-1} - jq^{j-1} - jq^{2(j-1)} + \mathcal{O}\big(q^{2(j-1)}\big),\\
E^{(2)}(j) &:= \mathbb{E}(\Gamma(j)^2) = \sum_{l=1}^{j-1} pq^{l-1} l^2/\Sigma_0 \sim \frac{1+q}{p^2} + q^{j-1}\frac{1+q}{p} - j\frac{2q}{p}q^{j-1}\\
&\quad - j^2 q^{j-1} + \mathcal{O}\big(j^2 q^{2(j-1)}\big).
\end{aligned}$$

Note that, with $j = \eta + \log n^*$,

$$q^j = e^{-L\eta}\frac{1}{n^*}.$$

This leads, by carefully taking into account the *correlation* between J and G (we expand the mean up to the $\log n^*/n^*$ term and the square mean up to the $\log n^*$ term) to (there are $n - G$ dead players before attaining step J)

$$\mathcal{C}_{n,1} \sim S_\Gamma(J,G) + JG, \tag{4}$$

$$\begin{aligned}
\mathbb{E}(\mathcal{C}_{n,1}) &\sim \mathbb{E}(S_\Gamma(J,G) + JG),\\
&\sim \mathbb{E}\Big[(n-G)\frac{1}{p} + \frac{n}{q}\frac{e^{-L\eta}}{n^*} - \frac{n-G}{q}\frac{e^{-L\eta}}{n^*}(\log n^* + \eta)\\
&\quad - \frac{n}{q^2}\frac{e^{-2L\eta}}{n^{*2}}\log n^* + (\log n^* + \eta)G\Big]\\
&\sim \frac{n}{p}Z_1 - \frac{1}{p}Z_5 + \frac{n}{q}\frac{Z_7}{n^*} - \frac{n}{q}\frac{Z_7}{n^*}\log n^* + \frac{1}{q}\frac{Z_8}{n^*}\log n^*\\
&\quad - \frac{n}{q}\frac{Z_{10}}{n^*} - \frac{n}{q^2}\frac{Z_{11}}{n^{*2}}\log n^* + Z_5\log n^* + Z_{13},
\end{aligned} \tag{5}$$

$$\mathbb{E}(\mathcal{C}_{n,1}^2) \sim \mathbb{E}((S_\Gamma(J,G) + JG)^2)$$

$$\begin{aligned}&\sim \mathbb{E}[n\mathbb{E}^{(2)}(J) + (n-G)(n-G-1)(\mathbb{E}(J))^2\\&+ 2\mathbb{E}[(n-G)E(J)JG] + \mathbb{E}[(\log n^* + \eta)^2 G^2].\end{aligned} \tag{6}$$

8.1.2 Case $I > 1$

First of all, we must compute the moments of $\mathcal{C}_i$ and $\tilde{\mathcal{C}}_i$. This gives

$$\begin{aligned}C_i &= \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell}[si+i+C_\ell],\ C_1=0,\\ C_i^{(2)} &= \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell}\mathbb{E}[((s+1)i+C_\ell)^2]\\ &= \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell}[((s+1)i)^2+2(s+1)iC_\ell+C_\ell^{(2)}],\end{aligned}$$

and similar expressions for $\tilde{C}_i$, $\tilde{C}_i^{(2)}$.

Next, with (1),

$$\begin{aligned}&\mathbb{P}(J=j, I=i) \sim f_9(\eta,i),\\ &\qquad f_9(\eta,i) := \exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{e^{-L\eta i}}{i!},\\ &\qquad \mathbb{P}(J=j) \sim f_{10}(\eta),\\ &\qquad f_{10}(\eta) = \sum_{i=2}^{\infty}\exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{e^{-L\eta i}}{i!} = \exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\left(\exp\left(-e^{-L\eta}\right)-1-e^{-L\eta}\right),\\ &\qquad \phi_9(\alpha,i) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_9(\eta,i)d\eta = \frac{(1/p)^{-i+\tilde{\alpha}}}{Li!}\Gamma(i-\tilde{\alpha}),\\ &\qquad \Pi_2(i) := \phi_9(0,i) = \frac{p^i}{Li},\\ &\qquad P_0 := \sum_{i=2}^{\infty}\Pi_2(i) = 1-p/L.\end{aligned}$$

Later on, we will use the following variants:

$$e^{-L\eta}f_9(\eta,i),\quad e^{-2L\eta}f_9(\eta,i),\quad \eta f_9(\eta,i),\quad e^{-L\eta}\eta f_9(\eta,i),\quad e^{-2L\eta}\eta f_9(\eta,i).$$

These variants lead respectively to $\phi_\cdot(0,i)$:

$$\begin{aligned}&\frac{p^i p}{L},\\ &\frac{p^i p^2(i+1)}{L},\end{aligned}$$

$$
\begin{aligned}
&- \frac{p^i[\ln(p)+\psi(i)]}{L^2 i},\\
&- \frac{p^i p[i\ln(p)+i\psi(i)+1]}{L^2 i},\\
&- \frac{p^i p^2[i^2\ln(p)+i^2\psi(i)+2i+i\ln(p)+i\psi(i)+1]}{L^2 i}.
\end{aligned}
$$

Multiplying by $\tilde{P}(i)$ and summing on $i \geq 2$, this leads to $V_7, V_5, V_8, V_{11}, V_{12}, V_{10}, V_{13}, V_{15}$. Indeed, the case $I > 1$ does *not* immediately lead to a success. Again we expand the mean up to the $\log n^*/n^*$ term and the square mean up to the $\log n^*$ term. We have (there are $n-I$ dead players before attaining step J)

$$
\mathcal{C}_{n,2} \sim S_\Gamma(J,I) + JI + \tilde{\mathcal{C}}_I, \tag{7}
$$

$$
\begin{aligned}
\mathbb{E}(\mathcal{C}_{n,2}) &\sim \mathbb{E}(S_\Gamma(J,I) + JI + \tilde{C}_I)\\
&\sim \mathbb{E}(S_\Gamma(J,I) + JI) + U_1\\
&\sim \mathbb{E}\Big[(n-I)\frac{1}{p} + \frac{n}{q}\frac{e^{-L\eta}}{n^*} - \frac{n-I}{q}\frac{e^{-L\eta}}{n^*}(\log n^* + \eta)\\
&\quad - \frac{n}{q^2}\frac{e^{-2L\eta}}{n^{*2}}\log n^* + (\log n^* + \eta)I\Big] + U_1\\
&\sim \frac{n}{p}V_1 - \frac{1}{p}V_5 + \frac{n}{q}\frac{V_7}{n^*} - \frac{n}{q}\frac{V_7}{n^*}\log n^* + \frac{1}{q}\frac{V_8}{n^*}\log n^*\\
&\quad - \frac{n}{q}\frac{V_{10}}{n^*} - \frac{n}{q^2}\frac{V_{11}}{n^{*2}}\log n^* + V_5\log n^* + V_{13} + U_1,
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\mathbb{E}(\mathcal{C}_{n,2}^2) &\sim \mathbb{E}((S_\Gamma(J,I)+JI)^2) + 2\mathbb{E}[(S_\Gamma(J,I)+JI)\tilde{\mathcal{C}}_I]\\
&\sim \mathbb{E}((S_\Gamma(J,I)+JI)^2) + 2\mathbb{E}\left[\left[\frac{n}{p} - \frac{n}{q}\frac{e^{-L\eta}}{n^*}\log n^* + I\log n^*\right]\tilde{C}_I\right]\\
&\sim \mathbb{E}((S_\Gamma(J,I)+JI)^2) + 2\left[\frac{n}{p}U_1 - \frac{n}{q}\frac{U_3}{n^*}\log n^* + U_5\log n^*\right]\\
&\sim \mathbb{E}[n\mathbb{E}^{(2)}(J) + (n-I)(n-I-1)(\mathbb{E}(J))^2 + 2\mathbb{E}[(n-I)E(J)JI]\\
&+ \mathbb{E}[(\log n^* + \eta)^2 I^2] + 2\left[\frac{n}{p}U_1 - \frac{n}{q}\frac{U_3}{n^*}\log n^* + U_5\log n^*\right].
\end{aligned}
$$

8.1.3 General case.

The *total mean* is given by (we provide here only two terms)

$$
\begin{aligned}
C_n = \mathbb{E}(\mathcal{C}_n) &\sim \mathbb{E}(\mathcal{C}_{n,1}) + \mathbb{E}(\mathcal{C}_{n,2})\\
&\sim n\left(\frac{p}{L} + V_1\right)\frac{1}{p} + \left(-\frac{Z_7}{p} + Z_5 - \frac{V_7}{p} + V_5\right)\log n^*.
\end{aligned}
$$

Recall that $\frac{p}{L}+V_1$ is equal to $P_d(S)$ as defined before. But the first term amounts to the mean of a sum of n GEOM pq^{l-1} RVs. (Indeed, the GEOM pq^{l-1} RV has mean $\frac{1}{p}$, second moment $\frac{1+q}{p^2}$ and variance $\frac{q}{p^2}$). This is easy to explain: from (4) and (7), the correction $\tilde{\mathcal{C}}_{I,S}$ is asymptotically $\mathcal{O}(1)$ and the correction $-\Delta$ is also asymptotically $\mathcal{O}(1)$. Similarly

$$\begin{aligned}\mathbb{E}(\mathcal{C}_n^2) &\sim \mathbb{E}(\mathcal{C}_{n,1}^2)+\mathbb{E}(\mathcal{C}_{n,2}^2)\\ &\sim n^2\left(\frac{p}{L}+V_1\right)\frac{1}{p^2}+n\left(-\frac{2(Z_7-pZ_5)}{p^2}-\frac{2(V_7-pV_5)}{p^2}\right)\log n^*\end{aligned}$$

and the *variance* is finally given by (we must adequately condition on the dominant success probability $Pd(S):=\frac{p}{L}+V_1$)

$$\mathbb{V}(\mathcal{C}_n)\sim Pd(S)\left[\frac{\mathbb{E}(\mathcal{C}_n^2)}{Pd(S)}-\left(\frac{\mathbb{E}(\mathcal{C}_n)}{Pd(S)}\right)^2\right]\sim Pd(S)n\frac{q}{p^2}.$$

So we obtain

Theorem 8.1 *The moments of $\mathcal{C}_n$ in case of success, are given by (with Maple, more terms could be provided, in particular the $\log^2 n^*$ and $\log n^*$ terms of the variance)*

$$\begin{aligned}C_n=\mathbb{E}(\mathcal{C}_n) &\sim n\left(\frac{p}{L}+V_1\right)\frac{1}{p}+\left(-\frac{Z_7}{p}+Z_5-\frac{V_7}{p}+V_5\right)\log n^*,\\ \mathbb{V}(\mathcal{C}_n) &\sim Pd(S)n\frac{q}{p^2}.\end{aligned}$$

Note again that the dominant term of the variance corresponds to a *sum of n iid GEOM pq^{l-1} RVs*. Intuitively, the asymptotic distribution should be Gaussian: again from (4) and (7), the correction $\tilde{C}_{I,S}$ is asymptotically $\mathcal{O}(1)$, but *not independent* of the dominant term and the correction $-\Delta$ is also asymptotically $\mathcal{O}(1)$, but *independent* of the dominant term. Actually we have the following theorem.

Theorem 8.2 *Conditioned on a success,*

$$\mathbb{P}\left[\frac{\mathcal{C}_n-\mathbb{E}(\mathcal{C}_n)}{\sqrt{\mathbb{V}(\mathcal{C}_n)}}\le x\right]\underset{n\to\infty}{\longrightarrow}\phi(x),$$

where $\phi(x)$ denotes the Gaussian distribution function.

The proof is given in Appendix H.

See also Kalpathy et al. [5], for a leader election scheme which stops if $I>1$. In this model, $\mathcal{C}_n$ is shown to be asymptotically Gaussian.

8.2 Distribution of $\mathcal{C}_n$ (number coins flipped), with failure

Only the case $I>1$ matters here. Proceeding as before (we omit the details), we finally derive

Theorem 8.3 *The moments of $\mathcal{C}_n$ in case of failure, are given by*

$$\begin{aligned}\mathbb{E}(\mathcal{C}_n) &\sim nV_1\frac{1}{p}+\left(-\frac{V_7}{p}+V_5\right)\log n^*,\\ \mathbb{V}(\mathcal{C}_n) &\sim V_1 n\frac{q}{p^2}.\end{aligned}$$

Again, the distribution should be asymptotically Gaussian, but we did not check the details.

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Appendix

A Some definitions and identities

Here, $\xi \in \{R, T, L, C\}$. ξ_i must be replaced by ξ_i or ξ_i' depending on the case we consider. Also $\tilde{P}(i)$ must be replaced by $\tilde{P}(i)$ or $\tilde{P}'(i)$, respectively. Note that, compared with [10], we use here $\Pi_2(i) = \frac{p^i}{Li}$ instead of $\frac{p^i}{i}$, for $V_1, \dots, V_5$. Also we have $\tilde{P}(1) = 1$, $\tilde{P}'(1) = 0$.

$$V_1 := \sum_{i=2}^{\infty} \frac{p^i}{Li}\tilde{P}(i), \quad V_2 := \sum_{i=2}^{\infty} \frac{p^i\psi(i)}{Li}\tilde{P}(i), \quad V_3 := \sum_{i=2}^{\infty} \frac{p^i\psi(i)^2}{Li}\tilde{P}(i), \quad V_4 := \sum_{i=2}^{\infty} \frac{p^i\psi(1,i)}{Li}\tilde{P}(i), \quad V_5 := \sum_{i=2}^{\infty} \frac{p^i i}{Li}\tilde{P}(i),$$

$$V_7 := \sum_{i=2}^{\infty} \frac{p^i p}{L} \tilde{P}(i) = pV_1, \quad V_8 := \sum_{i=2}^{\infty} \frac{p^i p i}{L} \tilde{P}(i) = pV_5, \quad V_{10} := \sum_{i=2}^{\infty} -\frac{p^i p[i\ln(p) + i\psi(i) + 1]}{L^2 i} \tilde{P}(i),$$

$$V_{11} := \sum_{i=2}^{\infty} \frac{p^i p^2(i+1)}{L} \tilde{P}(i) = p^2 V_5 + p^2 V_1, \quad V_{12} := \sum_{i=2}^{\infty} -\frac{p^i[\ln(p) + \psi(i)]}{L^2 i} \tilde{P}(i), \quad V_{13} := \sum_{i=2}^{\infty} -\frac{p^i[\ln(p) + \psi(i)]i}{L^2 i} \tilde{P}(i),$$

$$V_{15} := \sum_{i=2}^{\infty} -\frac{p^i p^2[-i^2\ln(p) - i^2\psi(i) - 2i + i\ln(p) - i\psi(i) + 1]}{L^2 i} \tilde{P}(i), \quad V_{16} := \sum_{i=2}^{\infty} -\frac{p^i p[i\ln(p) + i\psi(i) + 1]i}{L^2 i} \tilde{P}(i),$$

$$A_0(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!} \Gamma(i+\chi_l)\tilde{P}(i), \quad A_1(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!} \Gamma(i+\chi_l)\psi(i+\chi_l)\tilde{P}(i),$$

$$A_2(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!} \Gamma(i+\chi_l)\psi(1, i+\chi_l)\tilde{P}(i), \quad A_3(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!} \Gamma(i+\chi_l)\psi^2(i+\chi_l)\tilde{P}(i),$$

$$B_1(\chi_l) := \sum_{i=2}^{\infty} \frac{p^i}{Li!} \tilde{\xi}_i \Gamma(i+\chi_l), \quad B_2(\chi_l) := \sum_{i=2}^{\infty} \frac{p^i}{Li!} \tilde{\xi}_i^{(2)} \Gamma(i+\chi_l), \quad B_3(\chi_l) := \sum_{i=2}^{\infty} \frac{p^i}{Li!} \tilde{\xi}_i \psi(i+\chi_l)\Gamma(i+\chi_l),$$

$$U_1 := \sum_{i=2}^{\infty} \frac{p^i \tilde{\xi}_i}{Li}, \quad U_2 := \sum_{i=2}^{\infty} \frac{p^i \tilde{\xi}_i^{(2)}}{Li}, \quad U_3 := \sum_{i=2}^{\infty} \frac{pp^i \tilde{\xi}_i}{L}, \quad U_4 := \sum_{i=2}^{\infty} \frac{p^i \tilde{\xi}_i \psi(i)}{i}, \quad U_5 := \sum_{i=2}^{\infty} \frac{p^i i \tilde{\xi}_i}{Li},$$

$$Z_7 := \frac{p^2[2 - 3p + p^2]}{q^2 L}, \quad Z_8 := \sum_{g=2}^{\infty} \frac{p^g q g^2}{L} = \frac{p^2[4 - 3p + p^2]}{q^2 L}, \quad Z_{11} := \frac{2p^3(p^2 - 3p + 3)}{Lq^2},$$

$$Z_{12} := \sum_{g=2}^{\infty} -\frac{p^{g-1} q[(g-1)\ln(p) + (g-1)\psi(g-1) + 1]}{L^2(g-1)},$$

$$Z_{10} := \sum_{g=2}^{\infty} -\frac{qp^g[2(g-1) + (g-1)^2\ln(p) + (g-1)^2\psi((g-1)) + (g-1)\ln(p) + (g-1)\psi((g-1)) + 1]}{L^2(g-1)},$$

$$Z_{13} := \sum_{g=2}^{\infty} -\frac{p^{g-1} q[(g-1)\ln(p) + (g-1)\psi((g-1)) + 1]g}{L^2(g-1)}, \quad Z_{15} := \sum_{g=2}^{\infty} \Omega_{15}(g).$$

The quantities $\Sigma_1(i, \tau)$ can be expressed in the following way:

$$\Sigma_1(i,\tau) := \sum_{v=0}^{\tau-1} (p^i)^v = \frac{1 - p^{i\tau}}{1 - p^i},$$

$$\Sigma_2(i,\tau) := \sum_{v=0}^{\tau-1} (p^i)^v (v+1) = \frac{p^{i\tau}(-1 + \tau p^i - \tau) + 1}{(1-p^i)^2},$$

$$\Sigma_3(i,\tau) := \sum_{v=0}^{\tau-1} (p^i)^v (v+1)^2 = \frac{p^{i\tau}(p^i + 1 - 2\tau p^i + 2\tau + \tau^2 p^{2i} - 2\tau^2 p^i + \tau^2) - p^i + 1}{(1-p^i)^3},$$

$$\Sigma_4(i,\tau) := \sum_{v=0}^{\tau-1} (p^i)^v v = \frac{p^{i\tau}(\tau p^i - \tau - p^i) + p^i}{(1-p^i)^2},$$

$$\Sigma_5(i,\tau) := \sum_{v=0}^{\tau-1} (p^i)^v v^2 = \frac{p^{i\tau}(p^i + 2\tau p^i + \tau^2 p^{2i} - 2\tau^2 p^i + \tau^2 + p^{2i} - 2\tau p^{2i}) - p^{2i} + p^i}{(1-p^i)^3}.$$

B Success Probability

We show here that, where the results are given both, here, and in [7], they coincide. First, we look at Theorem 4.1. The constant is given by

$$\frac{1}{L}\sum_{k\ge 2} S_k \frac{p^k}{k}\frac{1-p^{(\tau-1)k}}{1-p^{\tau k}}+\frac{p}{L},$$

and it should coincide with

$$\frac{1}{L}\Big(qp^\tau+\sum_{k\ge 1}\frac{S_k}{k}\Big(p^k-\frac{q^k p^{\tau k}}{(1-q^{\tau+1})^k}\Big)\Big).$$

We use the notation S_n from [7] as in this paper. We have the recursion

$$\frac{1-p^k}{1-p^{\tau k}}S_k=\sum_{j=1}^{k}\binom{k}{j}p^{k-j}q^jS_j.$$

Therefore

$$\begin{aligned}
\sum_{k\ge 2} S_k \frac{p^k}{k}\frac{1-p^{(\tau-1)k}}{1-p^{\tau k}} &= \sum_{k\ge 2} S_k \frac{p^k}{k}\frac{1-p^{\tau k}+p^{\tau k}-p^{(\tau-1)k}}{1-p^{\tau k}}\\
&= \sum_{k\ge 2} S_k \frac{p^k}{k}-\sum_{k\ge 2} S_k \frac{p^{\tau k}}{k}\frac{1-p^k}{1-p^{\tau k}}\\
&= \sum_{k\ge 2} S_k \frac{p^k}{k}-\sum_{k\ge 2}\frac{p^{\tau k}}{k}\sum_{j=1}^{k}\binom{k}{j}p^{k-j}q^jS_j\\
&= \sum_{k\ge 1} S_k \frac{p^k}{k}-p+p^\tau q-\sum_{j\ge 1}\frac{q^jS_j}{j}\sum_{k\ge j}p^{\tau k}\binom{k-1}{j-1}p^{k-j}\\
&= \sum_{k\ge 1} S_k \frac{p^k}{k}-p+p^\tau q-\sum_{j\ge 1}\frac{q^jS_j}{j}\frac{p^{\tau j}}{(1-p^{\tau+1})^j},
\end{aligned}$$

which is the desired formula after trivial modifications.

For the Fourier coefficients, we have to prove that

$$p\Gamma(\chi_l+1)+\sum_{k\ge 2} S_k \frac{p^k}{k!}\frac{1-p^{(\tau-1)k}}{1-p^{\tau k}}\Gamma(\chi_l+k)=qp^\tau\Gamma(\chi_l+1)+\sum_{k\ge 1}\frac{S_k}{k!}\Gamma(\chi_l+k)\Big(p^k-\frac{q^kp^{\tau k}}{(1-q^{\tau+1})^{\chi_l+k}}\Big),$$

which is done in a similar way:

$$\begin{aligned}
\sum_{k\ge 2} S_k \frac{p^k}{k!}\frac{1-p^{(\tau-1)k}}{1-p^{\tau k}}\Gamma(\chi_l+k) &= \sum_{k\ge 2} S_k \frac{p^k}{k!}\frac{1-p^{\tau k}+p^{\tau k}-p^{(\tau-1)k}}{1-p^{\tau k}}\Gamma(\chi_l+k)\\
&= \sum_{k\ge 1} S_k \frac{p^k}{k!}\Gamma(\chi_l+k)-p\Gamma(\chi_l+1)-\sum_{k\ge 2}\frac{p^{\tau k}}{k!}\Gamma(\chi_l+k)\sum_{j=1}^{k}\binom{k}{j}p^{k-j}q^jS_j\\
&= \sum_{k\ge 1} S_k \frac{p^k}{k!}\Gamma(\chi_l+k)-p\Gamma(\chi_l+1)-\sum_{j\ge 1}\frac{q^jS_j}{j!}\sum_{k\ge j}p^{\tau k}\Gamma(\chi_l+k)\frac{1}{(k-j)!}p^{k-j}+qp^\tau\Gamma(\chi_l+1)
\end{aligned}$$

$$= \sum_{k\ge1} S_k \frac{p^k}{k!}\Gamma(\chi_l+k) - p\Gamma(\chi_l+1) + qp^\tau\Gamma(\chi_l+1) - \sum_{j\ge1}\frac{q^jS_j}{j!}\frac{\Gamma(\chi_l+j)p^{\tau j}}{(1-p^{\tau+1})^{\chi_j+k}},$$

which is the formula.

C Total number of rounds

Notations:

$\mathbf{P}_R(i,k) :=$ Probability that, starting with i players, we end after k rounds,

$\tilde{\mathbf{P}}_R(i,k) :=$ Probability that, starting with i players, we end after k rounds,
given that the i players were obtained in a null round, not preceded by another null round.

Next we turn to the nonfluctuating part of R_n. We must use the total number of rounds: $\tilde{\mathbf{R}}_i = \tilde{R}_i + \tilde{R}'_i$. Now from Theorems 5.1 and 5.2, we have

$$\begin{aligned}\mathbf{R}_n - \log n^* = \mathbb{E}(\mathcal{R}_n - \log n^*) &\sim \bar{U}_1 - M\bar{V}_1 - \frac{\bar{V}_2}{L} + \frac{p\gamma}{L^2} - \frac{1+pM}{L}\\ &+ \sum_{l\ne0}\left[\bar{B}_1(\chi_l) - M\bar{A}_0(\chi_l) - \frac{\bar{A}_1(\chi_l)}{L} - \frac{\Gamma(1+\chi_l)}{L}\right]e^{-2l\pi\mathbf{i}\{\log n\}}\end{aligned}$$

where, now,

$$\begin{aligned}\bar{B}_1(\chi_l) &:= \sum_{i=2}^\infty \frac{p^i}{Li!}\tilde{\mathbf{R}}_i\Gamma(i+\chi_l),\\ \bar{U}_1 &:= \sum_{i=2}^\infty \frac{p^i\tilde{\mathbf{R}}_i}{Li},\\ \bar{A}_0(\chi_l) &:= 0,\\ \bar{A}_1(\chi_l) &:= \Gamma(\chi_l),\\ \bar{V}_1 &:= \frac{L-p}{L},\\ \bar{V}_2 &:= \frac{L}{2} - \frac{\gamma(L-p)}{L}.\end{aligned}$$

We have the following recurrences, with $\mathbf{P}_R$ now combining S and F,

$$\mathbf{P}_R(1,0) = 1, \tilde{\mathbf{P}}_R(1,0) = 1,$$

$$\mathbf{P}_R(i,k) = (p^i)^\tau[\![k=\tau]\!] + \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^i\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{P}_R(\ell,k-1-s),\ i\ge2,$$

$$\tilde{\mathbf{P}}_R(i,k) = (p^i)^{\tau-1}[\![k=\tau-1]\!] + \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^i\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{P}_R(\ell,k-1-s),\ i\ge2.$$

$$\begin{aligned}\mathbf{R}_i = \sum_k \mathbf{P}_R(i,k)k &= (p^i)^\tau\tau + \sum_k\sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^i\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{P}_R(\ell,k-1-s)[k-1-s+s+1],\\ &= (p^i)^\tau\tau + \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^i\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{R}_\ell + \sum_k\sum_{s=0}^{\tau-1}(p^i)^s(s+1)\sum_{\ell=1}^i\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{P}_R(\ell,k-1-s)\end{aligned}$$

$$= (p^i)^\tau \tau + \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} \mathbf{R}_\ell + \Sigma_2(i,\tau) \Big[1-(p^i)^\tau\Big] / \Sigma_1(i,\tau),$$

$$= \frac{1-p^{i\tau}}{1-p^i} \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} \mathbf{R}_\ell + \frac{1-p^{i\tau}}{1-p^i}, \quad \mathbf{R}_1 = 0,$$

$$\begin{aligned}
\tilde{\mathbf{R}}_i &= \sum_k \tilde{\mathbf{P}}_R(i,k)k = (p^i)^{\tau-1}(\tau-1) + \Sigma_1(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} \mathbf{R}_\ell, \\
&+ \Sigma_2(i,\tau-1) \Big[1-(p^i)^\tau\Big] / \Sigma_1(i,\tau) \\
&= (p^i)^{\tau-1}(\tau-1) + \Sigma_1(i,\tau-1) \Big[\mathbf{R}_i - \Sigma_2(i,\tau)\Big[1-(p^i)^\tau\Big] / \Sigma_1(i,\tau) - (p^i)^\tau\Big] / \Sigma_1(i,\tau) \\
&+ \Sigma_2(i,\tau-1) \Big[1-(p^i)^\tau\Big] / \Sigma_1(i,\tau) \\
&= \mathbf{R}_i \frac{1-p^i-p^{i(\tau-1)}+p^{i\tau}}{(1-p^{i\tau})(1-p^i)} \\
&= \mathbf{R}_i \frac{1-p^{i(\tau-1)}}{1-p^{i\tau}}, \quad \tilde{\mathbf{R}}_1 = 0.
\end{aligned}$$

In [7], the following recursion is derived:

$$\mathbf{R}_n(\tau) = \sum_{j=1}^{n} \binom{n}{j} p^{n-j} q^j \mathbf{R}_j(\tau) + p^n \mathbf{R}_n(\tau-1) + 1, \quad \tau > 0,\ n \geq 2,$$

and the interest is in $\mathbf{R}_n = \mathbf{R}_n(\tau)$. We write

$$\mathbf{R}_n(\tau) = D_n(\tau) + p^n \mathbf{R}_n(\tau-1) = D_n(\tau) + p^n(\mathcal{R}_n(\tau) + \mathbf{R}_n(\tau-2)) = \cdots = \frac{1-p^{\tau n}}{1-p^n} D_n(\tau).$$

We find the recursion

$$\mathbf{R}_n = \frac{1-p^{\tau n}}{1-p^n} \sum_{j=1}^{n} \binom{n}{j} p^{n-j} q^j \mathbf{R}_j + \frac{1-p^{\tau n}}{1-p^n}.$$

This coincides with the recursion given here. Now

$$\begin{aligned}
&\bar{U}_1 - M\bar{V}_1 - \frac{\bar{V}_2}{L} + \frac{p\gamma}{L^2} - \frac{1+pM}{L} \\
&\quad = \sum_{i=2}^{\infty} \frac{p^i \tilde{\mathbf{R}}_i}{Li} - \frac{\ln p}{L}\frac{L-p}{L} - \frac{1}{L}\Big(\frac{L}{2} - \frac{\gamma(L-p)}{L}\Big) + \frac{p\gamma}{L^2} - \frac{1}{L} - \frac{p\ln p}{L^2} \\
&\quad = \frac{1}{L}\sum_{i=2}^{\infty} \frac{p^i(1-p^{i(\tau-1)})}{i} \frac{\mathbf{R}_i}{1-p^{i\tau}} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L} \\
&\quad = \frac{1}{L}\sum_{i=2}^{\infty} \frac{p^i(1-p^{i\tau}+p^{i\tau}-p^{i(\tau-1)})}{i} \frac{\mathbf{R}_i}{1-p^{i\tau}} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L} \\
&\quad = \frac{1}{L}\sum_{i=2}^{\infty} \frac{p^i \mathbf{R}_i}{i} - \frac{1}{L}\sum_{i=2}^{\infty} \frac{p^{i\tau}}{i} \frac{\mathbf{R}_i(1-p^i)}{1-p^{i\tau}} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{L}\sum_{i=2}^{\infty}\frac{p^i\mathbf{R}_i}{i} - \frac{1}{L}\sum_{i=1}^{\infty}\frac{p^{i\tau}}{i}\sum_{j=1}^{i}\binom{i}{j}p^{i-j}q^j\mathbf{R}_j - \frac{1}{L}\sum_{i=1}^{\infty}\frac{p^{i\tau}}{i} + \frac{p^\tau}{L} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}\\
&= \frac{1}{L}\sum_{i=2}^{\infty}\frac{p^i\mathbf{R}_i}{i} - \frac{1}{L}\sum_{j\ge 1}\frac{q^j\mathbf{R}_j}{j}\sum_{i\ge j}p^{i\tau}\binom{i-1}{j-1}p^{i-j} + \frac{1}{L}\ln(1-p^\tau) + \frac{p^\tau}{L} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}\\
&= \frac{1}{L}\sum_{i=2}^{\infty}\frac{p^i\mathbf{R}_i}{i} - \frac{1}{L}\sum_{j\ge 1}\frac{q^j\mathbf{R}_j}{j}\frac{p^{j\tau}}{(1-p^{\tau+1})^j} + \frac{1}{L}\ln(1-p^\tau) + \frac{p^\tau}{L} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}.
\end{aligned}
$$

Thus the constant term in the asymptotic expansion of $\mathbf{R}_n - \log n$ is

$$
\begin{aligned}
&\frac{\ln p}{L} + 1 + \frac{1}{L}\sum_{i=2}^{\infty}\frac{p^i\mathbf{R}_i}{i} - \frac{1}{L}\sum_{j\ge 1}\frac{q^j\mathbf{R}_j}{j}\frac{p^{j\tau}}{(1-p^{\tau+1})^j} + \frac{1}{L}\ln(1-p^\tau) + \frac{p^\tau}{L} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}\\
&= \frac{1}{L}\sum_{i\ge 2}\frac{p^i\mathbf{R}_i}{i} - \frac{1}{L}\sum_{j\ge 2}\frac{q^j\mathbf{R}_j}{j}\frac{p^{j\tau}}{(1-p^{\tau+1})^j} + \frac{1}{L}\ln(1-p^\tau) + \frac{p^\tau}{L} + \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}.
\end{aligned}
$$

This is the expansion that was also obtained in [7].

D Total number of null rounds

Notations:

$$
\begin{aligned}
\mathbf{P}_T(i,t) &:= \text{Probability that, starting with } i \text{ players, we end with } t \text{ null rounds},\\
\tilde{\mathbf{P}}_T(i,t) &:= \text{Probability that, starting with } i \text{ players, we end with } t \text{ null rounds},\\
&\qquad \text{given that the } i \text{ players were obtained in a null round.}
\end{aligned}
$$

We have the recurrences:

$$
\begin{aligned}
&\mathbf{P}_T(1,0) = 1, \quad \tilde{\mathbf{P}}_T(1,1) = 1,\\
&\mathbf{P}_T(i,t) = (p^i)^\tau [\![t=\tau]\!] + \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{P}_T(\ell,t-s),\\
&\tilde{\mathbf{P}}_T(i,t) = (p^i)^{\tau-1} [\![t=\tau]\!] + \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{P}_T(\ell,t-1-s).
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{T}_i = \sum_t \mathbf{P}_T(i,t)t &= (p^i)^\tau\tau + \sum_t\sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{P}_T(\ell,t-s)[t-s+s]\\
&= (p^i)^\tau\tau + \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{T}_\ell + \sum_t\sum_{s=0}^{\tau-1}(p^i)^s s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{P}_T(\ell,t-s),\\
&= (p^i)^\tau\tau + \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{T}_\ell + \Sigma_4(i,\tau)[1-(p^i)^\tau]/\Sigma_1(i,\tau)\\
&= \frac{1-p^{i\tau}}{1-p^i}\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\mathbf{T}_\ell + \frac{p^i\left(1-p^{i\tau}\right)}{1-p^i}, \quad \mathbf{T}_1 = 0.
\end{aligned}
$$

$$\begin{aligned}
\tilde{\mathbf{T}}_i &= \sum_t \tilde{\mathbf{P}}_T(i,t)t = (p^i)^{\tau-1}\tau + \sum_t \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell}\mathbf{P}_T(\ell, t-1-s)[t-1-s+s+1] \\
&= (p^i)^{\tau-1}\tau + \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell}\mathbf{T}_\ell + \sum_t \sum_{s=0}^{\tau-2}(p^i)^s(s+1)\sum_{\ell=1}^{i}\binom{i}{\ell} q^\ell p^{i-\ell}\mathbf{P}_T(\ell, t-1-s) \\
&= (p^i)^{\tau-1}\tau + \Sigma_1(i,\tau-1)\left[\mathbf{T}_i - (p^i)^\tau\tau - \Sigma_4(i,\tau)[1-(p^i)^\tau]/\Sigma_1(i,\tau)\right]/\Sigma_1(i,\tau) \\
&\qquad\qquad + \Sigma_2(i,\tau-1)[1-(p^i)^\tau]/\Sigma_1(i,\tau) \\
&= \frac{1-p^i+p^{i(\tau+1)}-p^{i\tau}}{(1-p^{i\tau})(1-p^i)} + \frac{1-p^i-p^{i(\tau-1)}+p^{i\tau}}{(1-p^{i\tau})(1-p^i)}\mathbf{T}_i \\
&= 1 + \frac{1-p^{i(\tau-1)}}{1-p^{i\tau}}\mathbf{T}_i, \quad \tilde{\mathbf{T}}_1 = 1.
\end{aligned}$$

The mean is given by

$$\mathbf{T}_n \sim \sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{\mathbf{T}}_i + \sum_{l\neq 0}\sum_{i=2}^{\infty}\frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{\mathbf{T}}_i e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

Indeed, $\mathbf{I}_n$ from the paper [7] satisfies the same recursion as here, after unwinding it as shown in the previous example.

And now we look at the nonfluctuating part in the asymptotic expansion of the mean:

$$\begin{aligned}
\sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{\mathbf{T}}_i &= \frac{1}{L}\sum_{n\geq 2}\frac{p^n}{n}\Big(1+\frac{1-p^{n(\tau-1)}}{1-p^{n\tau}}\mathbf{T}_n\Big) \\
&= \frac{1}{L}(-\ln(1-p)-p) + \frac{1}{L}\sum_{n\geq 2}\frac{p^n}{n}\frac{1-p^{n\tau}+p^{n\tau}-p^{n(\tau-1)}}{1-p^{n\tau}}\mathbf{T}_n \\
&= 1-\frac{p}{L}+\frac{1}{L}\sum_{n\geq 2}\frac{p^n}{n}\mathbf{T}_n - \frac{1}{L}\sum_{n\geq 2}\frac{p^{n\tau}}{n}\frac{1-p^n}{1-p^{n\tau}}\mathbf{T}_n \\
&= 1-\frac{p}{L}+\frac{1}{L}\sum_{n\geq 1}\frac{p^n}{n}\mathbf{T}_n - \frac{1}{L}\sum_{n\geq 2}\frac{p^{n\tau}}{n}\Big(\sum_{1\leq j\leq n}\binom{n}{j}q^j p^{n-j}\mathbf{T}_j + p^n\Big) \\
&= 1-\frac{p}{L}+\frac{1}{L}\sum_{n\geq 1}\frac{p^n}{n}\mathbf{T}_n - \frac{1}{L}\sum_{n\geq 1}\frac{p^{n(\tau+1)}}{n} + \frac{p^{\tau+1}}{L} - \frac{1}{L}\sum_{n\geq 1}\frac{p^{n\tau}}{n}\sum_{1\leq j\leq n}\binom{n}{j}q^j p^{n-j}\mathbf{T}_j \\
&= 1-\frac{p}{L}+\frac{1}{L}\ln(1-p^{\tau+1}) + \frac{p^{\tau+1}}{L} + \frac{1}{L}\sum_{n\geq 1}\frac{p^n}{n}\mathbf{T}_n - \frac{1}{L}\sum_{j\geq 1}\frac{q^j\mathbf{T}_j}{j}\sum_{n\geq j}\binom{n-1}{j-1}p^{n\tau}p^{n-j} \\
&= 1-\frac{p}{L}+\frac{1}{L}\ln(1-p^{\tau+1}) + \frac{p^{\tau+1}}{L} + \frac{1}{L}\sum_{n\geq 1}\frac{p^n}{n}\mathbf{T}_n - \frac{1}{L}\sum_{j\geq 1}\frac{q^j\mathbf{T}_j}{j}\frac{p^{j\tau}}{(1-\tau^{\tau+1})^j}.
\end{aligned}$$

This is the expression given in [7].

E Total number of leftovers

We have here only the failure case. This gives

$$L_i' = \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^ip^{i\tau}}i + \Sigma_1(i,\tau)\left[\sum_{\ell=0}^{i-1}\binom{i}{\ell}q^\ell p^{i-\ell}L_\ell' + q^i\left[L_i' - \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^ip^{i\tau}}i\right]\right]$$

$$\begin{aligned}\tilde{L}'_i &= \left[p^{i(\tau-1)} + \frac{1-p^{i(\tau-1)}}{1-p^i} q^i \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^ip^{i\tau}}\right] i + \frac{1-p^{i(\tau-1)}}{1-p^{i\tau}} \left[L'_i - \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^ip^{i\tau}} i\right] \\ &= \frac{1-p^{i(\tau-1)}}{1-p^{i\tau}} L'_i + \frac{p^{i(\tau-1)}(1-p^i)}{1-p^{i\tau}} i.\end{aligned}$$

The mean is given by

$$L'_n \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{L}'_i + \sum_{l\neq 0} \sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!} \tilde{L}'_i e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

The recursion derived in [7] is

$$L'_n = \frac{1-p^{n\tau}}{1-p^n} \sum_{j=1}^{n} \binom{n}{j} p^{n-j} q^j L'_j + np^{n\tau}.$$

Here, we have

$$\begin{aligned}L'_n &= \frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}} n + \frac{1-p^{n\tau}}{1-p^n} \left[\sum_{j=1}^{n-1} \binom{n}{j} q^j p^{n-j} L'_j + q^n \Big(L'_n - \frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}} n\Big)\right] \\ &= \frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}} n + \frac{1-p^{n\tau}}{1-p^n} \left[\sum_{j=1}^{n} \binom{n}{j} q^j p^{n-j} L'_j - q^n \frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}} n\right] \\ &= \frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}} n \left[1 - \frac{1-p^{n\tau}}{1-p^n} q^n\right] + \frac{1-p^{n\tau}}{1-p^n} \sum_{j=1}^{n} \binom{n}{j} q^j p^{n-j} L'_j \\ &= p^{n\tau} n + \frac{1-p^{n\tau}}{1-p^n} \sum_{j=1}^{n} \binom{n}{j} q^j p^{n-j} L'_j,\end{aligned}$$

and hence we do have the same recursion.

Now we turn to the nonfluctuating part of the mean:

$$\begin{aligned}\sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{L}'_i &= \sum_{i\ge 2} \frac{p^i}{Li} \left[\frac{1-p^{i(\tau-1)}}{1-p^{i\tau}} L'_i + \frac{p^{i(\tau-1)}(1-p^i)}{1-p^{i\tau}} i\right] \\ &= \frac{1}{L} \sum_{i\ge 2} \frac{p^i}{i} \frac{1-p^{i\tau}+p^{i\tau}-p^{i(\tau-1)}}{1-p^{i\tau}} L'_i + \frac{1}{L} \sum_{i\ge 2} \frac{p^{i\tau}(1-p^i)}{1-p^{i\tau}} \\ &= \frac{1}{L} \sum_{i\ge 2} \frac{p^i}{i} L'_i - \frac{1}{L} \sum_{i\ge 2} \frac{p^{i\tau}}{i} \left[\sum_{j=1}^{i} \binom{i}{j} p^{i-j} q^j L'_j + \frac{1-p^i}{1-p^{i\tau}} i p^{i\tau}\right] + \frac{1}{L} \sum_{i\ge 2} \frac{p^{i\tau}(1-p^i)}{1-p^{i\tau}} \\ &= \frac{1}{L} \sum_{i\ge 2} \frac{p^i}{i} L'_i - \frac{1}{L} \sum_{i\ge 2} \frac{p^{i\tau}}{i} \sum_{j=1}^{i} \binom{i}{j} p^{i-j} q^j L'_j - \frac{1}{L} \sum_{i\ge 2} \frac{1-p^i}{1-p^{i\tau}} p^{2i\tau} + \frac{1}{L} \sum_{i\ge 2} \frac{p^{i\tau}(1-p^i)}{1-p^{i\tau}} \\ &= \frac{1}{L} \sum_{i\ge 2} \frac{p^i}{i} L'_i - \frac{1}{L} \sum_{j\ge 1} \frac{q^j L'_j}{j} \sum_{i\ge j} p^{i\tau} \binom{i-1}{j-1} p^{i-j} + \frac{1}{L} \sum_{i\ge 2} p^{i\tau}(1-p^i) \\ &= \frac{1}{L} \sum_{i\ge 2} \frac{p^i}{i} L'_i - \frac{1}{L} \sum_{j\ge 1} \frac{q^j L'_j}{j} \frac{q^{j\tau}}{(1-p^{\tau+1})^j} + \frac{1}{L} \frac{1}{1-p^{\tau}} - \frac{1}{L} \frac{1}{1-p^{\tau+1}} - qp^{\tau},\end{aligned}$$

which is the same expression as in [7].

F Matrix expressions.

We will give a few explicit matrix expressions for several quantities computed before. We will not use these expressions, but we only wanted to show that some compact relations can be written down in some cases, showing an unified view of our different RV.

F.1 Success probability

Let

$$\Pi[i,u] := \Sigma_1(i,\tau)\binom{i}{u} q^u p^{i-u},\ i,u \geq 2,$$

and

$$\varphi_1(i) := \Sigma_1(i,\tau)\binom{i}{1} q^1 p^{i-1},\ i \geq 2.$$

Then we have the expression

$$P(.) = \sum_{k=0}^{\infty} \Pi^k \varphi_1 = [I - \Pi]^{-1}\varphi_1.$$

Note that, to get some precision in S_n, only finite matrices are necessary.

F.2 Number of rounds

Let

$$\varphi_2(i) := \Sigma_2(i,\tau)P(i)/\Sigma_1(i,\tau),\ i \geq 2.$$

Then we have the expression

$$R_. = \Pi R_. + \varphi_2 = [I - \Pi]^{-1}\varphi_2.$$

F.3 Number of null rounds

Let

$$\varphi_3(i) := \Sigma_4(i,\tau)P(i)/\Sigma_1(i,\tau),\ i \geq 2.$$

Then we have the expression

$$T_. = [I - \Pi]^{-1}\varphi_3.$$

F.4 Number of leftovers

Fix r. Let

$$\Pi_1[i,u] := \Sigma_1(i,\tau)\binom{i}{u} q^u p^{i-u},\ i,u > r,$$

and

$$\varphi_3(i) := \Sigma_1(i,\tau)\binom{i}{r} q^r p^{i-r} P'_L(r,r).$$

Then we have the expression

$$P'_L(.,r) = [I - \Pi_1]^{-1}\varphi_3.$$

G Model 2

We will only briefly mention the modifications related to the main expressions. A supplementary last index will indicate how many null rounds are allowed before failure. Only the mean in the success case will be given, all other cases can be similarly computed; we leave the details for any research student who is interested.

$$
\begin{aligned}
P(i,\tau) &= \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P(\ell,\tau-s),\ i\geq 2,\\
\tilde{P}(i,\tau) &= \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P(\ell,\tau-s-1) = P(i,\tau-1),\\
P_R(i,k,\tau) &= \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_R(\ell,k-1-s,\tau-s),\ i\geq 2,\\
\tilde{P}_R(i,k,\tau) &= \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_R(\ell,k-1-s,\tau-s-1) = P_R(i,k,\tau-1),\\
R_i(\tau) &= \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell(\tau-s) + \sum_{s=0}^{\tau-1}(p^i)^s(s+1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P(\ell,\tau-s),\\
\tilde{R}_i(\tau) &= R_i(\tau-1),\\
C_i(\tau) &= \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}[si+i+C_\ell(\tau-s)],\\
\tilde{C}_i(\tau) &= C_i(\tau-1),\\
P_T(i,t,\tau) &= \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_T(\ell,t-s,\tau-s),\ i\geq 2,\\
\tilde{P}_T(i,t) &= \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_T(\ell,t-1-s,\tau-s-1),\ i\geq 2,\\
T_i(\tau) &= \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}T_\ell(\tau-s) + \sum_{s=0}^{\tau-1}(p^i)^s s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P(\ell,\tau-s)\\
\tilde{T}_i &= \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}T_\ell(\tau-s-1) + \sum_{s=0}^{\tau-2}(p^i)^s(s+1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P(\ell,\tau-s-1),\\
P'_L(r,r,\tau) &= (p^r)^\tau + \Sigma_1(r,\tau)q^r P'_L(r,r,\tau),\\
\tilde{P}'_L(r,r,\tau) &= P'_L(r,r,\tau-1),\\
P'_L(i,r,\tau) &= \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=r}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P'_L(\ell,r,\tau-s), i>r,\ i\geq 2,\\
\tilde{P}'_L(i,r,\tau) &= \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=r}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P'_L(\ell,r,\tau-s-1) = P'_L(i,r,\tau-1),\\
L'_i(\tau) &= \sum_{s=0}^{\tau-1}\sum_{\ell=r}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}\sum_{r=0}^{i-1}P'_L(\ell,r,\tau-s)r + [(p^i)^\tau + q^i P'_L(i,i,\tau)]i,
\end{aligned}
$$

$$\tilde{L}'_i = L'_i(\tau - 1).$$

One can now proceed as in Model 1.

H Proof of Theorem 8.2

We start from

$$\mathcal{C}_{n,S} = [\![I = 1]\!][S_{\Gamma,1}(J,G) + JG] + [\![I > 1]\!][S_{\Gamma,2}(J,I) + JI + \tilde{\mathcal{C}}_{I,S}].$$

Here, $S_{\Gamma,1}, \eta_1$ are related to the case $I = 1$ and $S_{\Gamma,2}, \eta_2$ are related to the case $I > 1$. In the sequel, with some abuse of notation, $\mathcal{O}_V(1)$ will denote a RV, asymptotically independent of n, with finite moments. Rewriting,

$$\mathcal{C}_{n,S} = \sum_j \left(\mathbb{P}[J = j, I = 1][S_{\Gamma,1}(j,G) + jG] + \sum_{i \geq 2} \mathbb{P}[J = j, I = i] \left[[S_{\Gamma,2}(j,i) + ji]\tilde{P}(i) + \tilde{C}_{i,S} \right] \right).$$

We have, *conditioned on a success*, (we use the dominant success probability $Pd(S)$)

$$\begin{aligned}
\frac{\mathcal{C}_{n,S}}{Pd(S)} &\sim \frac{\Pi_1}{Pd(S)} \sum_j \frac{\mathbb{P}[J = j, I = 1]}{\Pi_1}[S_{\Gamma,1}(j,G) + jG] \\
&+ \frac{V_1}{Pd(S)} \sum_j \sum_{i \geq 2} \frac{\mathbb{P}[J = j, I = i]\tilde{P}(i)}{V_1}[S_{\Gamma,2}(j,i) + ji] \\
&+ \frac{V_1}{Pd(S)} \sum_j \sum_{i \geq 2} \frac{\mathbb{P}[J = j, I = i]}{V_1}\tilde{\mathcal{C}}_{i,S} \\
&= \frac{\Pi_1}{Pd(S)} \sum_j \frac{\mathbb{P}[J = j, I = 1]}{\Pi_1}[S_{\Gamma,1}(j,G) + jG] \\
&+ \frac{V_1}{Pd(S)} \sum_j \sum_{i \geq 2} \frac{\mathbb{P}[J = j, I = i]\tilde{P}(i)}{V_1}[S_{\Gamma,2}(j,i) + ji] + \mathcal{O}_V(1).
\end{aligned}$$

Again we will separate the contribution of urn J from that of urns $< J$. So, conditioning on $J = j$ and $\Gamma_k(j)$ denoting a sequence of iid truncated geometric RV $< j$,

$$\begin{aligned}
S_{\Gamma,1}(j,G) + jG &= S_{\Gamma,1}(j,0) - \sum_{j=1}^{G} \Gamma_k(j) + jG \\
&= S_{\Gamma,1}(j,0) + \mathcal{O}_V(1) + (\log n^* + \eta_1)G \\
&= S_{\Gamma,1}(j,0) + \mathcal{O}_V(1) + \log n^* \cdot \mathcal{O}_V(1),
\end{aligned}$$

and similarly for $S_{\Gamma,2}(j,i) + ji$. So

$$\begin{aligned}
\frac{\mathcal{C}_{n,S}}{Pd(S)} &\sim \frac{\Pi_1}{Pd(S)} \sum_j \frac{\mathbb{P}[J = j, I = 1]}{\Pi_1} S_{\Gamma,1}(j,0) \\
&+ \frac{V_1}{Pd(S)} \sum_j \sum_{i \geq 2} \frac{\mathbb{P}[J = j, I = i]\tilde{P}(i)}{V_1} S_{\Gamma,2}(j,0) + \mathcal{O}_V(1) + \log n^* \cdot \mathcal{O}_V(1).
\end{aligned}$$

Now we must show that $S_\Gamma(j,0)$ is asymptotically Gaussian. We could simply use Feller [2, example *IX,1,a* on triangular arrays], but we want an error estimation. We will provide the first terms of our expansions, but Maple "knows" more. The standard deviation of $\Gamma(j)$ will be denoted by $\sigma(j)$. We have

$$\Sigma_0 = 1 - \frac{e^{-L\eta}}{np},$$
$$E(j) \sim \frac{1}{p} - \frac{e^{-L\eta}(j-1)}{np},$$
$$\sigma(j) \sim \frac{\sqrt{q}}{p} - \frac{e^{-L\eta}(j-1)^2}{2n\sqrt{q}}.$$

Now the probability generating function (PGF) of $\Gamma(j)$ is given by

$$F(z) = \frac{1}{\Sigma_0}\sum_{l=1}^{j-1} pq^{l-1}z^l = \frac{1}{\Sigma_0}\left[\frac{pz}{1-qz} - \frac{e^{-L\eta}z^j}{n(1-qz)}\right],$$

and the PGF of $S_\Gamma(j,0)$ is given by $[F(z)]^n$. We will now use the Saddle point method (for a good introduction to this method, see Flajolet and Sedgewick [3, chapter VIII]). By Cauchy's theorem,

$$\mathbb{P}(S_\Gamma(j,0)=k) = \frac{1}{2\pi\mathbf{i}}\int_\Omega \frac{[F(z)]^n}{z^{k+1}}dz = \frac{1}{2\pi\mathbf{i}}\int_\Omega e^{H(z)}dz,$$

where Ω is inside the domain of analyticity of the integrand and encircles the origin and

$$H(z) = n\left(\ln\left[\frac{pz}{1-qz} - \frac{e^{-L\eta}z^j}{n(1-qz)}\right] - \ln(\Sigma_0)\right) - (k+1)\ln(z).$$

Set

$$H^{(i)} := \frac{d^iH}{dz^i}.$$

First we must find the solution of

$$H^{(1)}(\tilde{z}) = 0 \tag{H.1}$$

with smallest modulus.

Set $\tilde{z} := z^* - \varepsilon$, where $z^* = \lim_{n\to\infty}\tilde{z}$. Here, it is easy to check that $z^* = 1$. Set $k = nE(j) + \sqrt{n}\sigma(j)x$, x fixed. We will soon see that $\varepsilon = \mathcal{O}\big(\frac{1}{\sqrt{n}}\big)$, so we can expand z^j in $F(z)$ as

$$z^j = 1 - j\varepsilon + \frac{j(j-1)}{2}\varepsilon^2 + \cdots$$

Also $j = \log n^* + \eta$. This leads, to first order (keeping only the ε term in (H.1)), to

$$\varepsilon := \frac{-px}{\sqrt{nq}} + \mathcal{O}\Big(\frac{\log n^*}{n}\Big).$$

This shows that, asymptotically, ε is given by a series of powers of $n^{-1/2}$, where each coefficient is given by a series of powers of $\log n^*$. To obtain more precision, we set again $k = nE(j) + \sqrt{n}\sigma(j)x$, expand in powers of $n^{-1/2}$, and equate each coefficient to 0. We have, with $\tilde{z} := z^* - \varepsilon = 1 - \varepsilon$,

$$\mathbb{P}(S_\Gamma(j,0)=k) = \frac{1}{2\pi\mathbf{i}}\int_\Omega \exp\Big[H(\tilde{z}) + H^{(2)}(\tilde{z})(z-\tilde{z})^2/2! + \sum_{l=3}^{\infty} H^{(l)}(\tilde{z})(z-\tilde{z})^l/l!\Big]dz.$$

Note that the linear term vanishes. Set $z = \tilde{z} + \mathbf{i}\tau$. This gives

$$\mathbb{P}(S_\Gamma(j,0)=k) \sim \frac{1}{2\pi}\exp[H(\tilde{z})]\int_{-\infty}^{\infty}\exp\left[H^{(2)}(\tilde{z})(\mathbf{i}\tau)^2/2! + \sum_{l=3}^{\infty} H^{(l)}(\tilde{z})(\mathbf{i}\tau)^l/l!\right]d\tau. \tag{H.2}$$

Let us first analyze $H(\tilde{z})$. We obtain

$$H(\tilde{z}) = -x^2/2 + \mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big).$$

Also,

$$H^{(2)}(\tilde{z}) = n\frac{q}{p^2} + \mathcal{O}(\sqrt{n}\,),$$
$$H^{(4)}(\tilde{z}) = \mathcal{O}(n).$$

We can now compute (H.2), for instance by using the classical trick of setting

$$S^{(2)}(\tilde{z})(i\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(i\tau)^l/l! = -u^2/2.$$

Computing τ as a truncated series in u, this gives, by inversion,

$$\tau = \frac{u}{\sqrt{nq/p^2}} + u^2\mathcal{O}\Big(\frac{1}{n}\Big).$$

Setting $d\tau = \frac{d\tau}{du}du$, and integrating on $-\infty < u < \infty$, this gives

$$\frac{1}{\sqrt{2\pi nq/p^2}}\left[1+\mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big)\right].$$

Finally (H.2) leads to

$$\mathbb{P}(S_\Gamma(j,0)=k) \sim \frac{1}{\sqrt{2\pi nq/p^2}}e^{-x^2/2}\left[1+\mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big)\right].$$

Now we consider

$$\begin{aligned}
&\mathbb{P}\left(\frac{\frac{\mathcal{C}_{n,S}}{Pd(S)} - \frac{\mathbb{E}(\mathcal{C}_{n,S})}{Pd(S)}}{\sqrt{nq/p^2}} \le x\right)\\
&\sim \mathbb{P}\left(\frac{\frac{\Pi_1}{Pd(S)}\sum_j \frac{\mathbb{P}[J=j,I=1]}{\Pi_1}[S_{\Gamma,1}(j,0)-nE_1(j)] + \frac{V_1}{Pd(S)}\sum_j\sum_{i\ge 2}\frac{\mathbb{P}[J=j,I=i]\tilde{P}(i)}{V_1}[S_{\Gamma,2}(j,0)-nE_2(j)]}{\sqrt{nq/p^2}}\right.\\
&\qquad\qquad\left. + \frac{\mathcal{O}_V(1) + \log n^*\cdot\mathcal{O}_V(1)}{\sqrt{nq/p^2}} \le x\right)\\
&\sim \mathbb{P}\left(\frac{\frac{\Pi_1}{Pd(S)}\sum_j \frac{\mathbb{P}[J=j,I=1]}{\Pi_1}[S_{\Gamma,1}(j,0)-nE_1(j)]}{\sqrt{n}\sigma_1(j)}\right.\\
&\qquad\qquad\left. + \frac{\frac{V_1}{Pd(S)}\sum_j\sum_{i\ge 2}\frac{\mathbb{P}[J=j,I=i]\tilde{P}(i)}{V_1}[S_{\Gamma,2}(j,0)-nE_2(j)]}{\sqrt{n}\sigma_2(j)} \le x\right)
\end{aligned}$$

as

$$\frac{\sigma(j)}{\sqrt{q/p^2}} \underset{n\to\infty}{\longrightarrow} 1.$$

Now

$$\frac{\mathbb{V}(\mathcal{C}_n)}{Pd(S)nq/p^2} \underset{n\to\infty}{\longrightarrow} 1,$$

which concludes the proof.