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The determining number of Kneser graphs[†]

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A set of vertices S is a *determining set* of a graph G if every automorphism of G is uniquely determined by its action on S. The *determining number* of G is the minimum cardinality of a determining set of G. This paper studies the determining number of Kneser graphs. First, we compute the determining number of a wide range of Kneser graphs, concretely $K_{n:k}$ with $n \geq \frac{k(k+1)}{2} + 1$. In the language of group theory, these computations provide exact values for the base size of the symmetric group S_n acting on the k-subsets of $\{1, \ldots, n\}$. Then, we establish for which Kneser graphs $K_{n:k}$ the determining number is equal to $n - k$, answering a question posed by Boutin. Finally, we find all Kneser graphs with fixed determining number 5, extending the study developed by Boutin for determining number 2, 3 or 4.

Keywords: Determining set, determining number, Kneser graph, hypergraph.

1 Introduction

The *determining number* of a graph $G = (V(G), E(G))$ is the minimum cardinality of a set $S \subseteq V(G)$ such that the automorphism group of the graph obtained from G by fixing every vertex in S is trivial. In this paper, we continue the study on the determining number of Kneser graphs carried out by Boutin (2006), introducing a different technique from the tools used in that article.

An *automorphism* f of G is a bijective mapping of $V(G)$ onto itself such that $f(u)f(v) \in E(G)$ whenever $uv \in E(G)$. As usual $\mathrm{Aut}(G)$ denotes the automorphism group of G. A subset $S \subseteq V(G)$ is said to be a *determining set* of G if whenever $g, h \in \mathrm{Aut}(G)$ so that $g(s) = h(s)$ for all $s \in S$, then $g(v) = h(v)$ for all $v \in V(G)$. The minimum cardinality of a determining set of G, denoted by $\mathrm{Det}(G)$, is the *determining number* of G; a set of that cardinality is said to be a *minimum determining set*. Observe

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that $0 \leq \mathrm{Det}(G) \leq |V(G)| - 1$ since any subset of $V(G)$ containing all but one vertex is a determining set of G. It is easy to see that the only connected graphs G with $\mathrm{Det}(G) = |V(G)| - 1$ are the complete graphs. A graph G with $\mathrm{Det}(G) = 0$ is called *asymmetric* or *rigid*. Erdős and Rényi (1963) proved that almost all graphs are rigid.

The Kneser graph $K_{n:k}$ has vertices associated with the k-subsets of the n-set $[n] = \{1, \ldots, n\}$ and edges connecting disjoint sets. This family of graphs is usually considered for $n \geq 2k$ but here we shall assume that $n > 2k$ since the case $n = 2k$ gives a set of disconnected edges and its determining number is half the number of vertices.

Determining sets are particular cases of *bases* which were introduced by Sims (1971) in the context of computational group theory. Indeed, a *base* of a permutation group Γ acting on a set X is a subset B of X such that the pointwise stabilizer of B in Γ is trivial, i.e., the only element of Γ which fixes each point of B is the identity element. The *base size* of Γ is the minimum cardinality of a base. When X is the vertex-set of a graph G and $\Gamma = \mathrm{Aut}(G)$, a base B is precisely a determining set of G and the base size of $\mathrm{Aut}(G)$ is $\mathrm{Det}(G)$.

There exists an extensive literature on determining the base sizes of primitive permutation groups. See Bailey and Cameron (2011) for a number of references on this topic, and also Burness et al. (2011) where primitive actions on symmetric groups are considered. Erwin and Harary (2006) introduced the term *fixing set* to refer to a base for $\mathrm{Aut}(G)$, and Boutin (2006) used the term *determining set* for the same concept. Here we shall follow the terminology of Boutin (2006) (see also Albertson and Boutin (2007)) since in that paper, a study on Kneser graphs was performed. Concretely, the author proved that their determining number is tightly bound by $\log_2(n+1) \leq \mathrm{Det}(K_{n:k}) \leq n - k$ computing also the exact value $\mathrm{Det}(K_{2^r-1:2^{r-1}-1}) = r$. Further, all Kneser graphs with determining number 2, 3 or 4 were provided. The main tools used in Boutin (2006) to find determining sets or to bound the determining number of Kneser graphs are based on characteristic matrices and vertex orbits.

This paper contains two main results. In Section 2, we compute the determining number of all Kneser graphs $K_{n:k}$ with $n \geq \frac{k(k+1)}{2} + 1$. Our technique is based on two key ideas. First, every subset of vertices of $K_{n:k}$ has an associated k-regular hypergraph. The fact of being determining is achieved by imposing conditions on the edges of the corresponding hypergraph. Second, the relevant hypergraphs that let us compute $\mathrm{Det}(K_{n:k})$ for $n \geq \frac{k(k+1)}{2} + 1$ are constructed by using perfect matchings and Hamiltonian cycles in a complete graph. These hypergraphs are, in fact, graphs with loops or they have only one edge of cardinality bigger than 2.

We want to stress that since the automorphism group of the Kneser graph $K_{n:k}$ is the action of the symmetric group S_n on the k-subsets of $[n]$ (see Godsil and Royle (2001)), our results in Section 2 provide exact values for the base size of S_n acting on the k-subsets of $[n]$ for $n \geq \frac{k(k+1)}{2} + 1$. The problem of computing base sizes for the action of S_n on k-subsets was first studied in the (unpublished) D. Phil. thesis of Maund (1989). His approach was given in the context of group theory and ours using the language of graphs, but both approaches obtain similar results. We have also recently learnt of a related result by Halási (2012) which provides the values for the base size of S_n acting on k-subsets for $n = 2k$ and $n \geq k^2$.

Our second main result concerns the question of whether there exists an infinite family of Kneser graphs $K_{n:k}$ with $k \geq 2$ and determining number $n - k$, which was posed by Boutin (2006). The above approach by hypergraphs is our main tool to answer this question which is done in Section 3. Concretely, we show that $K_{n:1}$ for any n, $K_{5:2}$ and $K_{6:2}$ are the only Kneser graphs with determining number $n - k$.

Finally, as an application of our technique and so following with the series of results presented in Boutin (2006) providing all Kneser graphs with fixed determining number 2, 3 or 4, we list all Kneser graphs with fixed determining number 5; this is the content of Section 4. We conclude the paper with some remarks and open problems.

2 Computing the determining number of Kneser graphs

In this section, we compute the determining number of a wide range of Kneser graphs, obtaining a characterization of their determining sets in terms of hypergraphs. In order to do this, we shall apply the following result from Boutin (2006).

Lemma 2.1 (Boutin (2006)) *The set $\{V_1, \dots V_r\}$ is a determining set of $K_{n:k}$ if and only if there exists no pair of distinct elements $a, b \in [n]$ so that for each i either $\{a, b\} \subseteq V_i$ or $\{a, b\} \subseteq V_i^c$.*

Observe that the above lemma implies that every determining set of $K_{n:k}$ has to contain all the elements of $[n]$ but at most one.

Recall that a *hypergraph* is a generalization of a graph, where edges can connect any number of vertices. Formally, a hypergraph $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$ where $V(\mathcal{H})$ is the set of *vertices*, and $E(\mathcal{H})$ is a set of non-empty subsets of $V(\mathcal{H})$ called *hyperedges* or simply *edges*. When edges appear only once, the hypergraph is called *simple*. The *order* of a hypergraph is the number of its vertices, denoted by $|V(\mathcal{H})|$, and the *size* is the number of its edges $|E(\mathcal{H})|$. A hyperedge containing r vertices is said to be an *edge of cardinality* r. Thus, given a hypergraph $\mathcal{H}$ with n vertices there are edges of cardinality ranging over the set $\{1, \dots, n\}$. The *edge-cardinality-sequence* of $\mathcal{H}$, written as $r_1 \geq \dots \geq r_{|E(\mathcal{H})|}$, gives the cardinality r_i of the edges of $\mathcal{H}$. The *degree* $\delta(v)$ of a vertex v is the number of hyperedges containing v. A hypergraph is called k-*regular* if every vertex has degree k. In all the figures in this paper, hyperedges of cardinality bigger than 2 are illustrated as shadowed regions.

For any set of vertices $S \subseteq V(K_{n:k})$ denote by $\mathcal{H}_S$ the k-regular hypergraph obtained as follows: The vertex-set $V(\mathcal{H}_S)$ is equal to S, and two vertices belong to the same hyperedge whenever they contain a common element. When an element of $[n]$ appears only once in the vertices of S, we have a loop in the corresponding vertex of $\mathcal{H}_S$. Figure 1 shows an instance of a hypergraph associated to a set $S \subseteq V(K_{6:3})$.

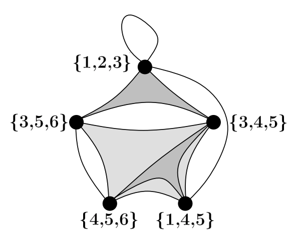


Fig. 1: Hypergraph associated to $S = \{\{1,2,3\}, \{3,4,5\}, \{1,4,5\}, \{4,5,6\}, \{3,5,6\}\}$. Number 2 appears only once in S so there is a loop attached to $\{1,2,3\}$. There are two edges of cardinality 2 determined by numbers 1 and 6, two edges of cardinality 3 since numbers 3 and 4 appear in three vertices of S, and one edge of cardinality 4 determined by number 5.

The two following results state that the condition of being determining set can be captured from the structure of the associated hypergraph.

Lemma 2.2 *A set $S \subseteq V(K_{n:k})$ is a determining set of $K_{n:k}$ if and only if the associated k-regular hypergraph $\mathcal{H}_S$ is simple and has either n or $n-1$ edges.*

Proof: Consider a determining set S of $K_{n:k}$ and the hypergraph $\mathcal{H}_S$. By Lemma 2.1, there exists no pair of distinct elements $a, b \in [n]$ so that for each vertex $V \in S$ either $\{a, b\} \subseteq V$ or $\{a, b\} \subseteq V^c$. Hence, $\mathcal{H}_S$ is simple. Indeed, having multiple edges in $\mathcal{H}_S$ is equivalent to have at least two elements of $[n]$ in exactly the same vertices of S, which implies that they are not distinguishable by any vertex of S.

It is clear that $\mathcal{H}_S$ has either n or $n-1$ edges since Lemma 2.1 says that S has to contain all the elements of $[n]$ but at most one. Thus, elements of $[n]$ in S correspond to edges in $\mathcal{H}_S$.

Suppose now that $\mathcal{H}_S$ is simple and has either n or $n-1$ edges. Then, for every $a, b \in [n]$ the corresponding edges in $\mathcal{H}_S$ are different and at most one element of $[n]$ is contained in no vertex of S. By Lemma 2.1 yields the result. $\square$

Lemma 2.3 *For every k-regular simple hypergraph $\mathcal{H}$ with either n or $n-1$ edges and $n \geq 2k+1$, there exists a determining set S of $K_{n:k}$ such that $\mathcal{H}$ is isomorphic to $\mathcal{H}_S$.*

Proof: We first label every edge of $\mathcal{H}$ with the elements of either $[n]$ or $[n-1]$ depending on the number of edges. The vertices of $\mathcal{H}$ are given the labels of their incident edges, giving rise to $|V(\mathcal{H})|$ different k-subsets of $[n]$. Take S as the set formed by these k-subsets. Clearly, $\mathcal{H}$ is isomorphic to $\mathcal{H}_S$ and by Lemma 2.2 the result follows. $\square$

When a determining set S is minimum, Lemma 2.3 guarantees that there does not exist a k-regular simple hypergraph $\mathcal{H}$ with either n or $n-1$ edges and $|V(\mathcal{H})| < |S| = |V(\mathcal{H}_S)|$. Thus, we say that among the k-regular simple hypergraphs with either n or $n-1$ edges, $\mathcal{H}_S$ is of *minimum order*. More generally, a k-regular simple hypergraph $\mathcal{H}$ with either n or $n-1$ edges is said to be of *minimum order* if there does not exist a k-regular simple hypergraph $\mathcal{H}'$ with either n or $n-1$ edges and $|V(\mathcal{H}')| < |V(\mathcal{H})|$.

Lemma 2.4 *A set $S \subset V(K_{n:k})$ is a minimum determining set of $K_{n:k}$ with $n \geq 2k+1$ if and only if the hypergraph $\mathcal{H}_S$ is simple, has either n or $n-1$ edges and minimum order.*

Remark 2.5 *The above characterization ensures us that the number of Kneser graphs $K_{n:k}$ with fixed determining number, say d, is finite. Indeed, the hypergraph associated to a minimum determining set must have a fixed number of vertices d which implies that neither k nor n can take infinite values.*

Lemmas 2.2 and 2.3 let us compute the determining number of all Kneser graphs $K_{n:k}$ with $n \geq \frac{k(k+1)}{2} + 1$, which is done in Theorems 2.6 and 2.7 below. We shall use, as a main tool to construct an appropriate hypergraph, the fact that every complete graph K_d has a 1-factorization whenever d is even (i.e., a collection of $d-1$ pairwise disjoint perfect matchings), and a Hamiltonian decomposition whenever d is odd (i.e., a collection of $\frac{d-1}{2}$ pairwise disjoint Hamiltonian cycles). (See Lucas (1892) and Baranyai (1975) or the recent paper by Bryant (2007).)

Theorem 2.6 *Let k and d be two positive integers such that $k \leq d$ and $d > 2$. Then,*

$$\mathrm{Det}\left(K_{\lfloor \frac{d(k+1)}{2} \rfloor + 1:k}\right) = d.$$

Proof: We first show that $\mathrm{Det}(K_{\lfloor \frac{d(k+1)}{2} \rfloor + 1:k}) \leq d$. By Lemma 2.3, it suffices to prove that there exists a k-regular simple hypergraph, say $\mathcal{H}_{k,d}$, with order d and either $\lfloor \frac{d(k+1)}{2} \rfloor$ or $\lfloor \frac{d(k+1)}{2} \rfloor + 1$ edges (in this proof hypergraphs will be, in fact, graphs with loops). We distinguish three cases according to the parity of k and d.

Case 1. d even: $\mathcal{H}_{k,d}$ is the hypergraph formed by d vertices, a loop attached at each vertex, and the edges of $k-1$ pairwise disjoint perfect matchings of the complete graph K_d (see Figure 2(a)). Clearly, this hypergraph is k-regular and has $\frac{d(k+1)}{2}$ edges. Note that its construction does not depend on the parity of k.

Case 2. d odd and k odd: $\mathcal{H}_{k,d}$ is formed by d vertices with one loop attached at each vertex, and $\frac{k-1}{2}$ pairwise disjoint Hamiltonian cycles of K_d (see Figure 2(b)). It is easy to check that $\mathcal{H}_{k,d}$ is a k-regular hypergraph with $\frac{d(k+1)}{2}$ edges.

Case 3. d odd and k even: Consider the hypergraph $\mathcal{H}_{k+1,d}$ obtained from Case 2. Take a Hamiltonian cycle, say $C = (e_1, e_2, \ldots, e_d)$. Now, delete the edges with even subindex, i.e., $e_2, e_4, \ldots, e_{d-1}$, and the loop attached at the common vertex of e_1 and e_d. This construction gives rise to a k-regular hypergraph $\mathcal{H}_{k,d}$ with $\frac{d(k+1)-1}{2} = \lfloor \frac{d(k+1)}{2} \rfloor$ edges.

To complete the proof, it remains to show that $\mathrm{Det}(K_{\lfloor \frac{d(k+1)}{2} \rfloor + 1:k})$ is exactly equal to d. By Lemma 2.3, it suffices to prove that every k-regular hypergraph with either $\lfloor \frac{d(k+1)}{2} \rfloor$ or $\lfloor \frac{d(k+1)}{2} \rfloor + 1$ edges has at least d vertices. Suppose on the contrary that there exists a k-regular hypergraph $\mathcal{H}$ with $\lfloor \frac{d(k+1)}{2} \rfloor$ edges (analogous for $\lfloor \frac{d(k+1)}{2} \rfloor + 1$ edges) and $d' < d$ vertices. By Theorem 2.8 of Duchet (1995) it follows that $d' = |V(\mathcal{H})| \geq \lceil \frac{2|E(\mathcal{H})|}{k+1} \rceil$. Hence,

$$d' \geq \left\lceil \frac{2}{k+1} \left\lfloor \frac{d(k+1)}{2} \right\rfloor \right\rceil = d$$

which is a contradiction. $\square$

(a) (b)

Fig. 2: Hypergraphs (graphs with loops in both cases) constructed by using perfect matchings and Hamiltonian cycles: (a) $H_{4,6}$ has 6 vertices, a loop attached at each vertex, and the edges of 3 pairwise disjoint perfect matchings of the complete graph K_6, (b) $H_{5,7}$ has 7 vertices with loops attached at each vertex, and 2 pairwise disjoint Hamiltonian cycles of K_7.

Our next aim is to extend Theorem 2.6 to Kneser graphs $K_{n:k}$ where $d \geq k$, $d > 2$ and $\lfloor \frac{(d-1)(k+1)}{2} \rfloor < n - 1 < \lfloor \frac{d(k+1)}{2} \rfloor$. For our purpose, we need to introduce an operation on the edge-set of a hypergraph $\mathcal{H}$.

Let $e_1, e_2 \in E(\mathcal{H})$. We say that e_1 and e_2 *are merged* obtaining a new hypergraph $\mathcal{H}'$ if $V(\mathcal{H}') = V(\mathcal{H})$ and $E(\mathcal{H}') = (E(\mathcal{H}) \setminus \{e_1, e_2\}) \cup \{e_1 \cup e_2\}$. Now, we can extend this operation to *merge a finite set of edges*, say $\{e_1, e_2, \ldots, e_t\}$ obtaining the hypergraph $\mathcal{H}'$ with $E(\mathcal{H}') = (E(\mathcal{H}) \setminus \{e_1, e_2, \ldots, e_t\}) \cup \{e_1 \cup e_2 \cup \ldots \cup e_t\}$. Note that $|E(\mathcal{H}')| = |E(\mathcal{H})| - t + 1$. We shall apply the operation of *merging edges* in regular hypergraphs and for $e_i \neq e_j$ whenever $i \neq j$. Observe that if $\mathcal{H}$ is k-regular and $e_1, e_2, \ldots, e_t$ are pairwise disjoint edges, i.e., they have no vertex in common, then $\mathcal{H}'$ is k-regular (see Figure 3).

Fig. 3: The edge $\{a, b, c, d, e, f\}$ of the hypergraph in (b) is the result of merging three edges of the hypergraph in (a): $\{a\}$, $\{b, c\}$ and $\{d, e, f\}$. Both hypergraphs are 5-regular.

Theorem 2.7 *Let k and d be two positive integers where $3 \leq k + 1 \leq d$. For every $n \in \mathbb{N}$ such that $\lfloor \frac{(d-1)(k+1)}{2} \rfloor < n < \lfloor \frac{d(k+1)}{2} \rfloor$ it holds that Det$(K_{n+1:k}) = d$.*

Proof: Since $\lfloor \frac{(d-1)(k+1)}{2} \rfloor < n < \lfloor \frac{d(k+1)}{2} \rfloor$ then there exists $r \in \mathbb{N}$ such that $n = \lfloor \frac{d(k+1)}{2} \rfloor - r$ with $r \leq \lfloor \frac{k-1}{2} \rfloor$ whenever d is odd or d is even and k is odd, and $r \leq \frac{k}{2}$ whenever k is even and d is even. We first prove that $\text{Det}(K_{n+1:k}) \leq d$ by distinguishing four cases according to the parity of d and k. By Lemma 2.3, it suffices to show that there exists a k-regular simple hypergraph with d vertices and n edges. We shall construct hypergraphs (all with only one edge of cardinality bigger than 2) by using the constructions of Theorem 2.6 and the above operation of merging.

Case 1. d even and k even: Consider the hypergraph $\mathcal{H}_{k,d}$ constructed in Case 1 of the proof of Theorem 2.6. Since $k \leq d - 1$ and k and d are even, then $k \leq d - 2$ and so $r \leq \frac{k}{2} \leq \frac{d-2}{2} < \frac{d}{2}$. Hence we can take $r + 1 \leq \frac{d}{2}$ edges of any perfect matching on the vertices of $\mathcal{H}_{k,d}$ and merge them obtaining the hypergraph $\mathcal{H}'_{k,d}$. Since the edges of a perfect matching are disjoint, $\mathcal{H}'_{k,d}$ is k-regular. Moreover, by construction $d = |V(\mathcal{H}_{k,d})| = |V(\mathcal{H}'_{k,d})|$ and

$$|E(\mathcal{H}'_{k,d})| = |E(\mathcal{H}_{k,d})| - r = \frac{d(k+1)}{2} - r = n.$$

Case 2. d even and k odd: Analogous to the previous case but considering,

$$r \leq \left\lfloor \frac{k-1}{2} \right\rfloor \leq \left\lfloor \frac{d-2}{2} \right\rfloor < \frac{d}{2}$$

and so $r + 1 \leq \frac{d}{2}$.

Case 3. d odd and k odd: Let $\mathcal{H}_{k,d}$ be the hypergraph constructed by considering d vertices with one loop attached at each vertex, and $\frac{k-1}{2}$ pairwise disjoint Hamiltonian cycles of K_d (see Case 2 of the proof of Theorem 2.6). Each cycle has d edges and $r \leq \frac{k-1}{2} \leq \frac{d-3}{2}$, then we can merge r disjoint edges of a Hamiltonian cycle, say $C = (e_1, \ldots, e_d)$. Merge those edges with even subindex plus the loop attached at the common vertex of e_1 and e_d; $r+1$ disjoint edges in total. Thus, we obtain a k-regular simple hypergraph $\mathcal{H'}_{k,d}$ with d vertices and $n = \lfloor \frac{d(k+1)}{2} \rfloor - r$ edges.

Case 4. d odd and k even: Consider the hypergraph $\mathcal{H}_{k,d}$ constructed in Case 3 of the proof of Theorem 2.6. This hypergraph is k-regular, has d vertices and $\frac{d(k+1)-1}{2}$ edges. Note that we can merge $r+1$ disjoint edges of a Hamiltonian cycle C of order d since $r \leq \lfloor \frac{k-1}{2} \rfloor = \frac{k-2}{2}$ and so $r+1 \leq \frac{k}{2} \leq \frac{d-1}{2} < \lceil \frac{d}{2} \rceil$. It suffices to consider $r+1$ pairwise disjoint edges among the odd labeled edges of C. The resulting hypergraph $\mathcal{H'}_{k,d}$ is simple, k-regular, has d vertices and $n = \lfloor \frac{d(k+1)}{2} \rfloor - r$ edges.

It remains to prove that $\mathrm{Det}(K_{n+1:k}) = d$. Suppose on the contrary that $\mathrm{Det}(K_{n+1:k}) \leq d-1$, then by Lemma 2.2 there exists a k-regular simple hypergraph $\mathcal{H}$ with $d-1$ vertices and either $n+1$ or n edges. Assume first that $\mathcal{H}$ has $n+1$ edges. The edge-cardinality-sequence $r_1 \geq \ldots \geq r_{n+1}$ of $\mathcal{H}$ satisfies (see Theorem 2.8 of Duchet (1995)):

$$\sum_{i=1}^{n+1} r_i = \sum_{v \in V(\mathcal{H})} \delta(v) = k(d-1).$$

Note that the number of loops in $\mathcal{H}$ is at most $d-1$, so there are $n+1-(d-1) = n-d+2$ edges of cardinality at least 2. Hence, we obtain the following inequalities about the sum on the edge cardinalities:

$$\left.\begin{array}{l} \displaystyle\sum_{i=n+1-(d-2)}^{n+1} r_i \geq d-1 \\ \displaystyle\sum_{i=1}^{n+1-(d-2)-1} r_i \geq 2(n-d+2) \end{array}\right\} \Rightarrow k(d-1) = \sum_{i=1}^{n+1} r_i \geq d-1+2(n-d+2) = 2n-d+3.$$

Therefore, $n \leq \frac{(d-1)(k+1)}{2} - 1$ which is a contradiction since $\lfloor \frac{(d-1)(k+1)}{2} \rfloor < n$.
Suppose now that $\mathcal{H}$ has n edges. Then,

$$\left.\begin{array}{l} \displaystyle\sum_{i=n-(d-2)}^{n} r_i \geq d-1 \\ \displaystyle\sum_{i=1}^{n-(d-2)-1} r_i \geq 2(n-d+1) \end{array}\right\} \Rightarrow k(d-1) = \sum_{i=1}^{n} r_i \geq d-1+2(n-d+1) = 2n-d+1.$$

Hence, $n \leq \frac{(d-1)(k+1)}{2}$ which contradicts $\lfloor \frac{(d-1)(k+1)}{2} \rfloor < n$. $\square$

3 Kneser graphs $K_{n:k}$ with determining number $n-k$

In this section, we answer a question posed by Boutin (2006): *"We know that* $\mathrm{Det}(K_{n:k}) = n-k$ *for* $K_{n:1}$ *for any* n*,* $K_{5:2}$ *and* $K_{6:2}$*. However,* $\mathrm{Det}(K_{7:2}) = 4 \neq 7-2$*. Is there an infinite family of Kneser graphs with* $k \geq 2$ *for which* $\mathrm{Det}(K_{n:k}) = n-k$*?"* Theorem 3.3 below says that $K_{n:1}$, $K_{5:2}$ and $K_{6:2}$ are the only Kneser graphs with that property.

Lemma 3.1 *Let* k *and* n *be two positive integers such that* $2k \leq n < \frac{k(k+1)}{2}$*. Then,* $\mathrm{Det}(K_{n+1:k}) \leq k$*.*

Proof: By Lemma 2.3, it suffices to prove that there exists a k-regular simple hypergraph $\mathcal{H}$ with k vertices and n edges. Consider the k-regular simple hypergraph $\mathcal{H}_{k,d}$ constructed in the proof of Theorem 2.6. Recall that, independently of the parity of k and d, the hypergraph $\mathcal{H}_{k,d}$ is k-regular, has d vertices and $\lfloor \frac{d(k+1)}{2} \rfloor$ edges. Take $k=d$ and the hypergraph $\mathcal{H}_{k,k}$ with k vertices and $\frac{k(k+1)}{2}$ edges.

Let $\{v_0, v_1, ..., v_{k-1}\}$ be the vertex-set of $\mathcal{H}_{k,k}$. We distinguish two cases according to the parity of k. Note that all the indices below are taken modulo k.

Case 1. k odd: $\mathcal{H}_{k,k}$ is the hypergraph formed by k vertices with one loop attached at each vertex, and a Hamiltonian decomposition of K_k (see Case 2 of Theorem 2.6). Assign the following set of $\frac{k-1}{2}$ edges to each vertex $v_i \in V(\mathcal{H}_{k,k})$ (see Figure 4):

$$\mathcal{E}_i = \{\{v_{i-1}, v_{i+1}\}, \{v_{i-2}, v_{i+2}\}, ..., \{v_{i-\frac{k-1}{2}}, v_{i+\frac{k-1}{2}}\}\}.$$

Note that the edges of $\mathcal{E}_i$ are disjoint and two sets $\mathcal{E}_i$, $\mathcal{E}_j$ have no edges in common whenever $i \neq j$. Thus, the k-regularity is preserved in the process of merging that we are going to describe next. We consider again two cases.

Case 1.1. If $\frac{k(k+1)}{2} - \frac{k-1}{2} + 1 < n < \frac{k(k+1)}{2}$ then merge a subset of $\frac{k(k+1)}{2} - n + 1$ edges of $\mathcal{E}_0$, obtaining a k-regular hypergraph $\mathcal{H}'_{k,k}$ with $\frac{k(k+1)}{2} - (\frac{k(k+1)}{2} - n + 1) + 1 = n$ edges.

Case 1.2. If $2k \leq n \leq \frac{k(k+1)}{2} - \frac{k-1}{2} + 1$ then we can merge the edges of at least one set $\mathcal{E}_i$ obtaining a k-regular hypergraph with at least n edges. If the number of edges is equal to n then the process is concluded. Otherwise, suppose that we can merge the edges of s subsets with $0 \leq s \leq k-1$, say $\mathcal{E}_0, \mathcal{E}_2, \ldots, \mathcal{E}_{s-1}$, obtaining a k-regular hypergraph $\mathcal{H}'_{k,k}$ with $\frac{k(k+1)}{2} - \frac{s(k-1)}{2} + s$ edges which satisfies

$$\frac{k(k+1)}{2} - \frac{(s-1)(k-1)}{2} + (s-1) < n < \frac{k(k+1)}{2} - \frac{s(k-1)}{2} + s.$$

Then the edges of $\mathcal{E}_s$ cannot be merged since if so the resulting hypergraph would have less than n edges. Hence, we proceed as in Case 1.1 merging $\frac{k(k+1)}{2} - \frac{s(k-1)}{2} + s - n + 1$ edges of $\mathcal{E}_s$. This process leads to a k-regular simple hypergraph $\mathcal{H}$ with k vertices and n edges.

Case 2. k even: $\mathcal{H}_{k,k}$ is a hypergraph with k vertices, a loop attached at each vertex, and the edges of a 1-factorization of the complete graph K_k (see Case 1 of Theorem 2.6). We distinguish three cases.

Case 2.1. If $\frac{k(k+1)}{2} - \frac{k}{2}(\frac{k}{2} - 1) + \frac{k}{2} \leq n < \frac{k(k+1)}{2}$ then we can follow an analogous process of merging than in Case 1 preserving also the k-regularity, but instead of assigning the sets $\mathcal{E}_i$ to each vertex v_i, we now assign the following set of edges to v_i for $i = 0, \ldots, \frac{k}{2} - 1$ (see Figure 5(a)):

$$\mathcal{F}_i = \{\{v_{i-1}, v_{i+1}\}, \{v_{i-2}, v_{i+2}\}, ..., \{v_{i-\frac{k}{2}+1}, v_{i+\frac{k}{2}-1}\}\}.$$

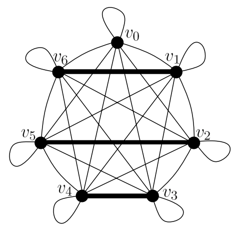


Fig. 4: The selected edges form the set $\mathcal{E}_0$ in $\mathcal{H}_{7,7}$.

Note that the assignment is done to half of the vertices since $\mathcal{F}_i = \mathcal{F}_{i+\frac{k}{2}}$ because of the parity of k. Observe also that the edges of $\mathcal{F}_i$ are disjoint and two sets $\mathcal{F}_i$, $\mathcal{F}_j$ have no edges in common whenever $i \neq j$.

This process of merging leads to a k-regular simple hypergraph $\mathcal{H}'_{k,k}$ resulting from merging at most $\sum_{i=0}^{\frac{k}{2}-1} |\mathcal{F}_i| = \frac{k}{2}(\frac{k}{2}-1)$ edges in $\mathcal{H}_{k,k}$. In this case, we obtain a hypergraph with $\frac{k(k+1)}{2} - \frac{k}{2}(\frac{k}{2}-1) + \frac{k}{2}$ edges. Note that the edges obtained by this procedure are all of cardinality $k-2$ except at most one of smaller cardinality.

Case 2.2. If $\frac{k(k+1)}{2} - \frac{k}{2}(\frac{k}{2}-1) + \frac{k}{2} - \frac{k}{2}(\frac{k}{2}-2) + \frac{k}{2} = 3k \leq n < \frac{k(k+1)}{2} - \frac{k}{2}(\frac{k-2}{2}-1)$ then we first merge all the sets of edges $\mathcal{F}_i$, obtaining a hypergraph $\mathcal{H}'_{k,k}$ with $\frac{k(k+1)}{2} - \frac{k}{2}(\frac{k}{2}-1) + \frac{k}{2}$ edges. We now assign the following set of edges to v_i for $i = 0, \ldots, \frac{k}{2}-1$ (see Figure 5(b)):

$$\mathcal{F}'_i = \{\{v_{i-1}, v_{i+2}\}, \{v_{i-2}, v_{i+3}\}, ..., \{v_{i-\frac{k}{2}+2}, v_{i+\frac{k}{2}-1}\}\}.$$

Again, we follow the procedure described in Case 1 which gives a k-regular simple hypergraph that is the result of merging at most $\sum_{i=0}^{\frac{k}{2}-1} |\mathcal{F}'_i| = \frac{k}{2}(\frac{k}{2}-2)$ edges in $\mathcal{H}'_{k,k}$. In this case, a hypergraph with $\frac{k(k+1)}{2} - \frac{k}{2}(\frac{k}{2}-1) + \frac{k}{2} - \frac{k}{2}(\frac{k}{2}-2) + \frac{k}{2}$ edges is obtained. Observe that the edges obtained by this process are all of cardinality $k-4$ except at most one of smaller cardinality.

Case 2.3. If $2k \leq n < 3k$ then merge the sets $\mathcal{F}_i$ and $\mathcal{F}'_i$, obtaining a hypergraph with $3k$ edges. These edges are: k loops, $\frac{k}{2}$ edges of cardinality $k-2$, $\frac{k}{2}$ edges of cardinality $k-4$ and k edges of cardinality 2 forming the cycle $\{v_0, ...v_{k-1}\}$. For every vertex v_i, consider now the set of edges (see Figure 5(c)):

$$\mathcal{F}''_i = \{\{v_i\}, \{v_{i+1}, v_{i+2}\}\}$$

and merge the required sets $\mathcal{F}''_i$ to obtain a hypergraph with n edges.

□

Remark 3.2 *The process of merging described in the proof of Lemma 3.1 leads to hypergraphs which have in most cases a number of edges of cardinality bigger than 2.*

Now, we can formulate our main result in this section.

Theorem 3.3 Det$(K_{n+1:k}) = n+1-k$ *if and only if either* $k=1$ *or* $k=2$ *and* $n = 4, 5$.

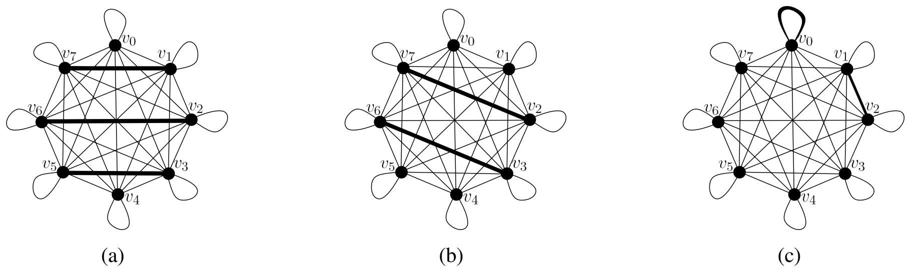


Fig. 5: The selected edges form the set: (a) $\mathcal{F}_0$, (b) $\mathcal{F}_0'$, (c) $\mathcal{F}_0''$ in $\mathcal{H}_{8,8}$.

Proof: The values $k = 1$ or $k = 2$ and $n = 4, 5$ give the Kneser graphs $K_{n+1:1}$ for any n, $K_{5:2}$ and $K_{6:2}$ whose determining numbers are, respectively, n, 3 and 4.

Let $K_{n+1:k}$ be a Kneser graph with $\mathrm{Det}(K_{n+1:k}) = n + 1 - k$. By Lemma 3.1, there are no values of n and k such that $2k \leq n < \frac{k(k+1)}{2}$ and $\mathrm{Det}(K_{n+1:k}) = n + 1 - k$ since $\mathrm{Det}(K_{n+1:k}) \leq k < n + 1 - k$. Then, we can assume that $n \geq \frac{k(k+1)}{2}$ and $k \geq 2$ (for $k = 1$ the Kneser graph $K_{n+1:1}$ is isomorphic to the complete graph K_{n+1} and so $\mathrm{Det}(K_{n+1:k}) = n$).

Suppose first that there exists $d \in \mathbb{N}$ with $d > 2$ such that $n = \lfloor \frac{d(k+1)}{2} \rfloor$. Then $d \geq k$ and by Theorem 2.6 we have $\mathrm{Det}(K_{n+1:k}) = d$. Thus, it suffices to prove that $d < \lfloor \frac{d(k+1)}{2} \rfloor + 1 - k$ except for $d = 3$ and $k = 2$ which gives the graph $K_{5:2}$. Suppose on the contrary that either $d \neq 3$ or $k \neq 2$ and $d \geq \lfloor \frac{d(k+1)}{2} \rfloor + 1 - k$. We distinguish two cases.

Case 1. d even or k odd: The contradiction follows since $(2 - k - 1)d \geq 2(1 - k)$ and so $d \leq 2$.

Case 2. d odd and k even: We have $2d \geq d(k+1) - 2k + 1$ which easily implies that $(k-1)(d-2) \leq 1$. Clearly, the inequality only holds for $k = 2$ and $d = 3$.

Assume now that there exists $d \in \mathbb{N}$ with $3 \leq k + 1 \leq d$ such that $\lfloor \frac{(d-1)(k+1)}{2} \rfloor < n < \lfloor \frac{d(k+1)}{2} \rfloor$. By Theorem 2.7, $\mathrm{Det}(K_{n+1:k}) = d$ and it suffices to show that $d < n + 1 - k$ except for $k = 2$ and $d = 4$ which leads to the Kneser graph $K_{6:2}$. The following expression holds for all positive integers d, k and n satisfying the above conditions except for $k = 2$ and $d = 4$:

$$d - 1 < \lfloor \frac{(d-1)(k+1)}{2} \rfloor + 1 - k < n + 1 - k.$$

Hence, the result follows. $\square$

4 Kneser graphs with fixed determining number

As an application of our technique and following the list of cases studied in Boutin (2006) characterizing all Kneser graphs with fixed determining number 2, 3 or 4, we provide all Kneser graphs with fixed determining number 5. Our method also works for the cases presented in Boutin (2006).

Lemma 4.1 *Let $\mathcal{H}$ be a k-regular simple hypergraph with d vertices and m edges. Then the following statements hold:*

(a) $k \leq 2^{d-1}$ *and* $m \leq 2^d - 1$.

(b) *If* $m > d + \binom{d}{2}$ *then* $kd \geq 3m - 2d - \binom{d}{2}$.

(c) *If* $m > d + \binom{d}{2} + \binom{d}{3}$ *then* $kd \geq 4m - 3d - 2\binom{d}{2} - \binom{d}{3}$.

Proof: Statement (a) follows since the cardinality of the power set $\mathcal{P}(V(\mathcal{H}))$ of the vertex-set $V(\mathcal{H})$ is 2^d, and a hyperedge is a non-empty subset of vertices.

To prove Statement (b), assume that $m > d + \binom{d}{2}$ and consider the edge-cardinality-sequence $r_1 \geq r_2 \geq \ldots \geq r_{m-1} \geq r_m$ of $\mathcal{H}$ which satisfies $kd = \sum_{i=1}^{m} r_i$ (see Theorem 2.8 of Duchet (1995)). Then,

$$\begin{aligned} kd &= \sum_{i=m-d+1}^{m} r_i + \sum_{i=m-\left(d+\binom{d}{2}\right)+1}^{m-d} r_i + \sum_{i=1}^{m-\left(d+\binom{d}{2}\right)} r_i \\ &\geq d + 2\binom{d}{2} + 3\left(m - d - \binom{d}{2}\right) \\ &= 3m - 2d - \binom{d}{2}. \end{aligned}$$

Suppose now that $m > d + \binom{d}{2} + \binom{d}{3}$. Since $kd = \sum_{i=1}^{m} r_i$ we have,

$$\begin{aligned} kd &= \sum_{i=m-d+1}^{m} r_i + \sum_{i=m-\left(d+\binom{d}{2}\right)+1}^{m-d} r_i + \sum_{i=m-\left(d+\binom{d}{2}+\binom{d}{3}\right)+1}^{m-\left(d+\binom{d}{2}\right)} r_i \sum_{i=1}^{m-\left(d+\binom{d}{2}+\binom{d}{3}\right)} r_i \\ &\geq d + 2\binom{d}{2} + 3\binom{d}{3} + 4\left(m - d - \binom{d}{2} - \binom{d}{3}\right) \\ &= 4m - 3d - 2\binom{d}{2} - \binom{d}{3}. \end{aligned}$$

Hence, Statement (c) holds. □

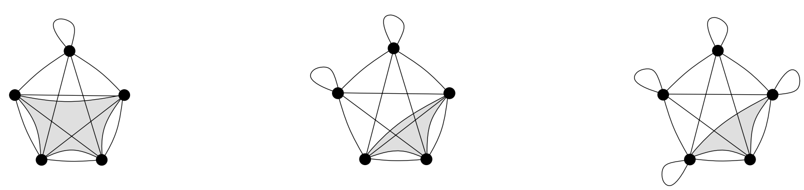
Proposition 4.2 *The Kneser graphs with determining number 5 are* $K_{6:1}$, $K_{8:2}$, $K_{10:3}$, $K_{11:3}$, $K_{12:4}$, $K_{13:4}$, $K_{13:5}$, $K_{14:5}$, $K_{15:5}$, $K_{16:5}$, $K_{14:6}$, $K_{15:6}$, $K_{16:6}$, $K_{17:6}$, $K_{16:7}$, $K_{17:7}$, $K_{18:7}$, $K_{19:7}$, $K_{17:8}$, $K_{18:8}$, $K_{19:8}$, $K_{20:8}$, $K_{21:8}$, $K_{19:9}$, $K_{20:9}$, $K_{21:9}$, $K_{22:9}$, $K_{21:10}$, $K_{22:10}$, $K_{23:10}$, $K_{24:10}$, $K_{23:11}$, $K_{24:11}$, $K_{25:11}$, $K_{26:11}$, $K_{25:12}$, $K_{26:12}$, $K_{27:12}$, $K_{27:13}$, $K_{28:13}$, $K_{29:14}$, *and* $K_{31:15}$.

k	6	7	8	9	10	11	12	13	14	15
n	13, 14, 15, 16	15, 16, 17, 18	16, 17, 18, 19, 20	18, 19, 20, 21	20, 21, 22, 23	22, 23, 24, 25	24, 25, 26	26, 27	28	30

Tab. 1: Values of n for $k \geq 6$ for which $\mathrm{Det}(K_{n+1:k}) = 5$.

Proof: Since every Kneser graph $K_{n+1:1}$ is isomorphic to the complete graph K_{n+1} then only $K_{6:1}$ has determining number 5. Consider now the graph $K_{n+1:k}$ with $k \geq 2$ and suppose that $\mathrm{Det}(K_{n+1:k}) = 5$. By Lemma 2.4, there exists a k-regular simple hypergraph $\mathcal{H}$ with minimum order 5 and either n or $n+1$ edges. By Lemma 4.1 it follows that $2k \leq n \leq 31$ and $2 \leq k \leq 16$. Thus, we obtain a list of 196 candidate Kneser graphs to have determining number 5. When $n \geq \frac{k(k+1)}{2}$ (which happens for 157 graphs among the 196) we can apply Theorems 2.6 and 2.7 obtaining that only $K_{8:2}$, $K_{10:3}$, $K_{11:3}$, $K_{12:4}$, $K_{13:4}$ and $K_{16:5}$ have determining number 5.

Assume now that $2k \leq n < \frac{k(k+1)}{2}$ which implies that $4 \leq k \leq 16$. When $k = 4$, the possible values of n are 8 or 9 but they correspond to Kneser graphs with determining number 4 (see Proposition 14 of Boutin (2006)). When $k = 5$, Lemma 4.1 gives $10 \leq n \leq 14$. However for n equal to either 10 or 11, we obtain Kneser graphs already studied in Boutin (2006), whose determining numbers are equal to 4. The remaining values of n correspond to the Kneser graphs $K_{13:5}$, $K_{14:5}$ and $K_{15:5}$ whose associated hypergraphs with 5 vertices are illustrated in Figure 6.



(a) (b) (c)

Fig. 6: Hypergraphs associated to Kneser graphs with determining number 5: (a) $K_{13:5}$, (b) $K_{14:5}$, (c) $K_{15:5}$.

Table 1 shows the values of n obtained for $k \geq 6$ for which $\mathrm{Det}(K_{n+1:k}) = 5$. In all the cases, it is easy to construct the associated hypergraph. On the other hand, note that for $k = 6$ and $n = 12$ or $k = 7$ and $n = 14$, the corresponding Kneser graph has determining number 4. For the remaining available values of n and k, we use Statements (b) and (c) of Lemma 4.1 in order to show that the hypergraph $\mathcal{H}$ does not exist and hence the determining number in those cases cannot be equal to 5. For instance, if $k = 6$ and $n = 18$ then it is straightforward to check that Statement (b) of Lemma 4.1 does not hold when $d = 5$. $\square$

5 Concluding Remarks

We have introduced a new technique to find determining sets and compute the determining number of Kneser graphs. Concretely, we have shown that every subset S of vertices of $K_{n:k}$ has an associated k-regular hypergraph $\mathcal{H}_S$. The set S is a determining set if and only if $\mathcal{H}_S$ is simple and has either n or $n-1$ edges.

Our main results use perfect matchings and Hamiltonian cycles in a complete graph to construct the appropriate hypergraphs to obtain the determining number of all Kneser graphs $K_{n:k}$ with $n \geq \frac{k(k+1)}{2}+1$. Also, we find all Kneser graphs $K_{n:k}$ with $k \geq 2$ and determining number $n-k$, answering a question posed by Boutin (2006). Finally, as an application of our approach and following with the study developed in Boutin (2006), we give all Kneser graphs with fixed determining number 5.

Figure 7 illustrates values of n and k for which $\text{Det}(K_{n:k})$ has been computed. The determining number of $K_{n:k}$ remains to be computed for the integer points (n,k) on the line $n = 2k+1$ with $n \neq 2^r - 1$ and those in between the line $n = 2k+1$ and the curve $n = \frac{k(k+1)}{2} + 1$. It appears that our technique should be generalized to be applied for these points, constructing hypergraphs from other structures instead of perfect matchings and Hamiltonian cycles in complete graphs.

Proposition 4.2 gives all Kneser graphs with determining number 5. As was said before, our method can also be applied for determining number 2, 3 or 4 (see Propositions 12, 13 and 14 of Boutin (2006)). We believe that our technique lets us go further to determining number 6 or larger, but the list of candidate graphs increases rapidly and so a computational implementation would be required.

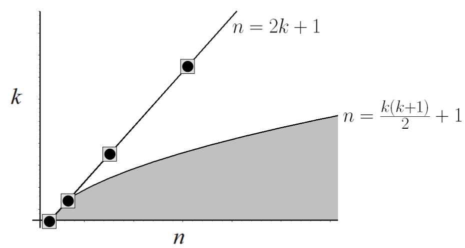


Fig. 7: The integer points (n,k) in the shadowed region correspond to values of n and k for which the determining number is provided by Theorems 2.6 and 2.7. The squared points represent values $(2^r - 1, 2^{r-1} - 1)$ studied in Boutin (2006).

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