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Upper k-tuple domination in graphs

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For a positive integer k, a k-tuple dominating set of a graph G is a subset S of $V(G)$ such that $|N[v] \cap S| \geq k$ for every vertex v, where $N[v] = \{v\} \cup \{u \in V(G): uv \in E(G)\}$. The upper k-tuple domination number of G, denoted by $\Gamma_{\times k}(G)$, is the maximum cardinality of a minimal k-tuple dominating set of G. In this paper we present an upper bound on $\Gamma_{\times k}(G)$ for r-regular graphs G with $r \geq k$, and characterize extremal graphs achieving the upper bound. We also establish an upper bound on $\Gamma_{\times 2}(G)$ for claw-free r-regular graphs. For the algorithmic aspect, we show that the upper k-tuple domination problem is NP-complete for bipartite graphs and for chordal graphs.

Keywords: Upper k-tuple domination, r-regular graph, bipartite graph, split graph, chordal graph, NP-completeness.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. In a graph G with vertex set $V(G)$ and edge set $E(G)$, the *open neighborhood* of a vertex v is $N(v) = \{u \in V(G): uv \in E(G)\}$ and the *closed neighborhood* is $N[v] = \{v\} \cup N(v)$. The *degree* of v, denoted by $d(v)$, is the cardinality of $N(v)$. Denote by $\delta(G)$ the minimum degree of a vertex in G. A graph is r-regular if $d(v) = r$ for all $v \in V$. A *stable set* (respectively, *clique*) of G is a subset S of $V(G)$ in which every two vertices are not adjacent

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(respectively, are adjacent). For two disjoint subsets A and B of $V(G)$, let $e[A, B]$ denote the number of edges between A and B.

For $S \subseteq V(G)$, the *subgraph induced* by S is the graph $G[S]$ with vertex set S and edge set $\{uv \in E(G)\colon u, v \in S\}$. A *bipartite* graph is a graph whose vertex set can be partitioned into two sets such that every two distinct vertices may be adjacent only if they are in different sets. A *split* graph is a graph whose vertex set can be partitioned into a stable set and a clique. A *chord* of a cycle is an edge joining two vertices on the cycle that are not adjacent on the cycle. A *chordal graph* is a graph in which every cycle of length at least four has a chord. Split graphs are chordal. A graph G is called *claw-free* if it does not contain the bipartite complete graph $K_{1,3}$ as an induced subgraph.

For positive integer k, a *k-tuple dominating set* of G is a subset S of $V(G)$ such that $|N[v] \cap S| \geq k$ for all $v \in V(G)$. For a k-tuple dominating set S, any vertex in $N[v] \cap S$ is said to *dominate* v. Notice that a graph has a k-tuple dominating set if and only if $\delta(G) \geq k - 1$. The *k-tuple domination number* $\gamma_{\times k}(G)$ of G is the minimum cardinality of a k-tuple dominating set of G, while the *upper k-tuple domination number* $\Gamma_{\times k}$ of G is the maximum cardinality of a minimal k-tuple dominating set. A *$\Gamma_{\times k}(G)$-set* of G is a minimal k-tuple dominating set of G of cardinality $\Gamma_{\times k}(G)$. An application of k-tuple domination for fault tolerance networks is presented in [9, 12]. For more results on k-tuple domination, we refer to [1, 2, 3, 4, 5, 13, 14, 15, 16, 17, 18, 19].

In this paper we first give an upper bound on $\Gamma_{\times k}$ for r-regular graphs, and characterize the extremal graphs achieving the upper bound. We also establish a sharp upper bound on $\Gamma_{\times 2}(G)$ for claw-free r-regular graphs. Finally, we show that the upper k-tuple domination problem is NP-complete for bipartite graphs and chordal graphs.

2 Upper k-tuple domination for r-regular graphs

This section establishes a sharp upper bound for upper k-tuple domination on r-regular graphs.

First, a k-tuple dominating set S is minimal if and only if every vertex in S is not *avoidable*, that is, it has a closed neighbor that is dominated by exactly k vertices in S. Hence, we have the following property.

Lemma 1 *In a graph G with $\delta(G) \geq k - 1$, a k-tuple dominating set S is minimal if and only if each vertex in S has some closed neighbor u with $|N[u] \cap S| = k$.*

For integers $r \geq k \geq 1$, let $\mathcal{H}_{r,k}$ be the family of r-regular graphs H whose vertex set is the disjoint union $F_1 \cup F_2 \cup F_3$, where F_1 induces an $(r-1)$-regular graph of which each vertex has exactly one neighbor in F_2, F_2 is a stable set of which each vertex has exactly $k-1$ neighbors in F_1 and exactly $r+1-k$ neighbors in F_3, and F_3 is a stable set of which each vertex has exactly r neighbors in F_2, see Figure 1. Since $(r+1-k)|F_2| = e[F_2, F_3] = r|F_3|$, there is some integer $m \geq 1$ such that $|F_2| = rm/g$ and $|F_3| = (r+1-k)m/g$, where $g = \gcd(r+1-k, r) = \gcd(r, k-1)$. And then $|F_1| = (k-1)rm/g$. The total number of vertices in H is $n = (kr + r + 1 - k)m/g$. According to Lemma 1, $F_1 \cup F_2$ is a minimal k-upper dominating set of H and so $\Gamma_{\times k}(H) \geq \frac{krn}{kr+r+1-k}$.

Theorem 2 *If G is a r-regular graph of order n with $r \geq k \geq 2$, then $\Gamma_{\times k}(G) \leq \frac{krn}{kr+r+1-k}$ with equality if and only if $G \in \mathcal{H}_{r,k}$.*

Proof: Let S be a $\Gamma_{\times k}(G)$-set of G and $S' = V(G) \setminus S$. For $k \leq i \leq r+1$, we define

$$S_i = \{u \in S : |N[u] \cap S| = i\} \quad \text{and} \quad S'_i = \{u \in S' : |N[u] \cap S| = i\}.$$

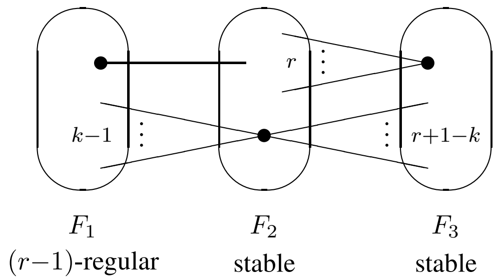

Fig. 1: An r-regular graph H in $\mathcal{H}_{r,k}$.

Notice that $S'_{r+1} = \emptyset$ and S'_r is stable as every vertex of S'_r has neighbors only in S. Since every vertex has at least k closed neighbors in S, it is the case that $S = \bigcup_{i=k}^{r+1} S_i$ and $S' = \bigcup_{i=k}^{r} S'_i$ are disjoint unions. Therefore, $|S| = \sum_{i=k}^{r+1} |S_i|$ and $|S'| = \sum_{i=k}^{r} |S'_i|$.

According to Lemma 1, every vertex in S_{r+1} has at least one neighbor in S_k, while every vertex in S_k has at most $k-1$ neighbors in S_{r+1}. Therefore,

$$|S_{r+1}| \le e[S_{r+1}, S_k] \le (k-1)|S_k| \tag{1}$$

or equivalently

$$|S_{r+1}| - (k-1)|S_k| \le 0. \tag{2}$$

By Lemma 1 again, every vertex in $X = \bigcup_{i=k+1}^{r+1} S_i$ has at least one neighbor in $S_k \cup S'_k$ and every vertex in S_k (respectively, S'_k) has at most $k-1$ (respectively, k) neighbors in X. Therefore,

$$\sum_{i=k+1}^{r+1} |S_i| \le e[X, S_k] + e[X, S'_k] \le (k-1)|S_k| + k|S'_k| \tag{3}$$

or equivalently

$$\sum_{i=k+1}^{r+1} |S_i| - (k-1)|S_k| \le k|S'_k|. \tag{4}$$

We then use a double counting of $e[S, S']$ to get

$$\sum_{i=k}^{r} (r+1-i)|S_i| = e[S, S'] = \sum_{i=k}^{r} i|S'_i|. \tag{5}$$

Let $p = \min(\frac{r+1-k}{k}, 1)$ and $q = \max(\frac{r+1-k}{k}, 1) - 1$. Then, $p > 0$, $q \ge 0$ and $q+1 \ge \frac{r+1-k}{k} = p+q$. Consequently, $(r+1-k) - (p+q)(k-1) = p+q$ and $q+r+1-i \ge q+1 \ge p+q$ for $k+1 \le i \le r$, which gives (6). Adding p times (2), q times (4) and (5) gives the first inequality in (7). And $(q+1)k = \max\{r+1-k, k\} \le r$ with equality only when $r = k$, which gives the second inequality in (7).

$$(p+q)|S| \le ((r+1-k) - (p+q)(k-1))|S_k| + (p+q)|S_{r+1}| + \sum_{i=k+1}^{r} (q+r+1-i)|S_i| \tag{6}$$

$$\leq (q+1)k|S'_k| + \sum_{i=k+1}^{r} i|S'_i| \leq r|S'| = rn - r|S|. \tag{7}$$

Hence, $\frac{kr+r+1-k}{k}|S| = (r+p+q)|S| \leq rn$ and so $\Gamma_{\times k}(G) \leq \frac{krn}{kr+r+1-k}$, which proves the first part of the theorem.

Suppose $\Gamma_{\times k}(G) = \frac{krn}{kr+r+1-k}$. By the proof above, the inequalities in (1), the inequalities in (3) when $q > 0$, and the inequalities in (6) and (7) are equalities. The equality in (6) gives that $S = S_k \cup S_{r+1}$ when $p + q < q + 1$ and $S = S_k \cup S_{r+1} \cup S_r$ when $p + q \geq q + 1$, while the second equality in (7) gives that $S' = S'_r$ which is stable. The first equality in (1) gives that every vertex in S_{r+1} has exactly one neighbor in S_k, while the second equality gives that every vertex in S_k has exactly $k-1$ neighbors in S_{r+1} and so exactly $r+1-k$ neighbors in S'. We claim that $p+q \geq q+1$ is impossible, for otherwise $p \geq 1$ implying $\frac{r+1-k}{k} \geq 1$ and so $r > k$. Then by Lemma 1, every vertex in S_r is adjacent to some vertex in $S'_k = \emptyset$ as $S' = S'_r$, a contradiction. Hence, $p + q < q + 1$ and so $S = S_k \cup S_{r+1}$. Letting $F_1 = S_{r+1}$, $F_2 = S_k$ and $F_3 = S'$, we have that G is in $\mathcal{H}_{r,k}$. On the other hand, any graph G in $\mathcal{H}_{r,k}$ satisfies $\Gamma_{\times k}(G) \geq \frac{krn}{kr+r+1-k}$ and so $\Gamma_{\times k}(G) = \frac{krn}{kr+r+1-k}$. □

The extremal graphs in $\mathcal{H}_{r,k}$ for Theorem 2 contain claws. For claw-free r-regular graphs, the upper bound in Theorem 2 for upper 2-tuple domination can be improved.

Theorem 3 *If G is a claw-free r-regular graph of order n with $r \geq 3$, then $\Gamma_{\times 2}(G) \leq \frac{2n}{3}$.*

Proof: Let S, S', S_i and S'_i be defined as in the proof of Theorem 2 with $k = 2$. For $2 \leq i \leq r+1$, we further write

$$S_{2,i} = \{u \in S_2 \colon |N(u) \cap S_i| = 1\} \quad \text{and} \quad S_{i,2} = \{u \in S_i \colon |N(u) \cap S_2| \geq 1\}.$$

Then $|S_2| = \sum_{i=2}^{r+1} |S_{2,i}|$. By the definition of $S_{2,i}$ and $S_{i,2}$, for $3 \leq i \leq r+1$,

$$|S_{i,2}| \leq e[S_{i,2}, S_{2,i}] = |S_{2,i}|. \tag{8}$$

By Lemma 1, $S_{r+1,2} = S_{r+1}$ and so

$$|S_{r+1}| \leq |S_{2,r+1}| \leq |S_2|. \tag{9}$$

We then consider $e[S'_r, S]$. Since S'_r is stable and G is claw-free, every vertex in S has at most 2 neighbors in S'_r. First, every vertex $u \in S_{2,r+1}$ has no neighbor in S'_r. Otherwise, if such a neighbor v exists, then u has another neighbor w in S' and a neighbor x in S_{r+1}. Since x has only neighbors in S, it is not adjacent to v nor to w. Also, v is not adjacent to w since it has only neighbors in S. Hence a claw occurs at u, a contradiction. Second, for $3 \leq i \leq r$, every vertex $u \in S_i \setminus S_{i,2}$ has no neighbor in S_2 and hence, by Lemma 1, it has a neighbor in S'_2. Thus, if u is in S_r ($i = r$), then this neighbor in S'_2 is the only neighbor of u in S', and so u has no neighbor in S'_r. Else, if $3 \leq i \leq r-1$, then u has at most one neighbor in S'_r, or it would be the center of a claw. Finally, by definition, every vertex in S_{r+1} has no neighbor in S'_r. These give the first inequality in (10), while (8) and (9) give the second inequality in (10).

$$r|S'_r| = e[S'_r, S] \leq \sum_{i=2}^{r} 2|S_{2,i}| + \sum_{i=3}^{r} |S_{i,2}| + \sum_{i=3}^{r-1} |S_i| \leq 3|S_2| - 3|S_{r+1}| + \sum_{i=3}^{r-1} |S_i|. \tag{10}$$

Formula (5) with $k = 2$ gives

$$\sum_{i=2}^{r}(r+1-i)|S_i| = \sum_{i=2}^{r} i|S'_i|. \tag{11}$$

Formula (3) with $k = 2$ gives

$$\sum_{i=3}^{r+1}|S_i| \leq |S_2| + 2|S'_2|. \tag{12}$$

Adding $\frac{r-3}{r}\cdot(9)$, $\frac{1}{r}\cdot(10)$, (11) and $\frac{r-3}{2}\cdot(12)$ gives

$$\tfrac{r-1}{2}(|S_2| + |S_r| + |S_{r+1}|) + \sum_{i=3}^{r-1}(\tfrac{3r-1}{2} - i - \tfrac{1}{r})|S_i| \leq (r-1)(|S'_2| + |S'_r|) + \sum_{i=3}^{r-1} i|S'_i| \tag{13}$$

The left side is bounded below by $\frac{r-1}{2}|S|$, the right above by $(r-1)|S'|$. Thus we get $|S| \leq 2|S'|$ and finally, $\Gamma_{\times 2}(G) \leq \frac{2n}{3}$. □

3 Complexity results

The upper domination problem was shown to be NP-complete by Cheston, Fricke, Hedetniemi and Jacobs [6]. However, Cockayne, Favaron, Payan and Thomason [7] proved that $\Gamma(G) = \beta_0(G)$ for any bipartite graph G, and so $\Gamma(G)$ can be computed for bipartite graphs in polynomial time. It was also shown by Jacobson and Peters [11] that $\Gamma(G) = \beta_0(G)$ for any chordal graph G, and so $\Gamma(G)$ can be computed for chordal graphs in polynomial time. Besides, Hare, Hedetniemi, Laskar, Peters and Wimer [10] also established a polynomial algorithm for determining $\Gamma(G)$ on generalized series-parallel graphs.

On the other hand, we shall prove that the k-tuple domination problem, with $k \geq 2$ fixed, is NP-complete for bipartite graphs and for chordal graphs. The proofs are separated into the cases of $k = 2$ and of $k \geq 3$. We consider the decision problem version as follows.

Upper k-tuple domination problem (UkTD)

Instance: A graph $G = (V, E)$ and a positive integer $s \leq |V|$.

Question: Does G have a minimal upper k-dominating set of cardinality at least s?

To show that U2TD is NP-complete, we will make use of the well-known NP-complete problem 3-SAT [8].

One-in-three 3SAT (OneIn3SAT)

Instance: A set $U = \{u_1, \ldots, u_n\}$ of n variables and a collection $\mathcal{C} = \{c_1, \ldots, c_m\}$ of m clauses over U such that each clause $c \in \mathcal{C}$ has $|c| = 3$ and no clause contains a negated variable.

Question: Is there a truth assignment $\mathscr{A}: U \to \{\text{true, false}\}$ for U such that each clause in $\mathcal{C}$ has exactly one true literal?

Theorem 4 *The upper* 2*-tuple domination problem is NP-complete even restricted on bipartite graphs or on split graphs, and hence also on chordal graphs.*

Proof. Obviously, U2TD is in NP. We shall show the NP-completeness of U2TD for bipartite graphs by reducing OneIn3SAT to it in polynomial time. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{C} = \{c_1, c_2, \ldots, c_m\}$ be an instance I of OneIn3SAT. We transform I to the instance (G_I, s) of U2TD in which $s = 3n + m$ and G_I is the bipartite graph formed as follows.

Corresponding to each variable u_i we associate a cycle $C_i = u_i x_i y_i z_i u_i$. Corresponding to each 3-element clause c_j we associate a vertex named w_j. Joining the vertex u_i to the vertex w_j if and only if the literal u_i belongs to the clause c_j. According to the above construction, it is easy to see that G_I is a bipartite graph and the construction is accomplished in polynomial time. The graph G_I associate with $(u_1 \vee u_2 \vee u_3) \wedge (u_2 \vee u_3 \vee u_4)$ is shown in Figure 2.

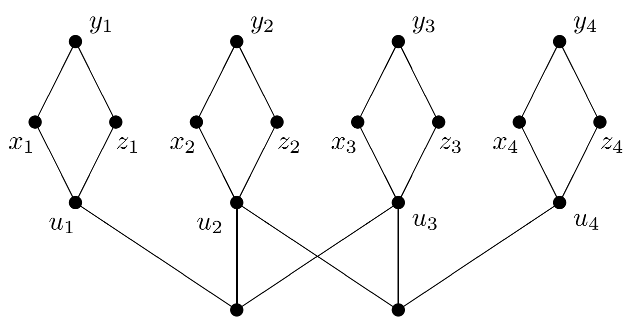

Fig. 2: The graph G_I for $(u_1 \vee u_2 \vee u_3) \wedge (u_2 \vee u_3 \vee u_4)$.

We next show that I has a satisfying truth assignment if and only if G_I has a minimal upper 2-tuple dominating set of cardinality at least $s = 3n + m$.

Suppose first I has a satisfying truth assignment $\mathscr{A}$. A minimal upper 2-tuple dominating set S of G_I of cardinality s is constructed as follows. Let w_j belong to S for all $1 \le j \le m$. For each $1 \le i \le n$, if $\mathscr{A}(u_i) = \{\text{true}\}$, then let u_i, x_i and z_i be in S; otherwise, let x_i, y_i and z_i be in S. Clearly, $|N[v] \cap S| \ge 2$ for each vertex $v \in V(G_I)$ and S satisfies the conditions of Lemma 1, and so S is a minimal upper 2-tuple dominating set of G_I with cardinality $s = 3n + m$.

On the other hand, assume that S is a minimal upper 2-tuple dominating set of G_I with cardinality at least $s = 3n + m$. It follows from Lemma 1 that $|S \cap V(C_i)| \le 3$ for $1 \le i \le n$. Further, $|S \cap \{w_1, w_2, \ldots, w_m\}| \le m$ and so $|S| \le 3n + m$. Notice that $|S| \ge 3n + m$ by the assumption. Hence, $|S \cap V(C_i)| = 3$ for $1 \le i \le n$ and $\{w_1, w_2, \ldots, w_m\} \subseteq S$. Let $\{u_{j_1}, u_{j_2}, u_{j_3}\}$ be the open neighborhood of w_j in G_I for $1 \le j \le m$. Since S is a minimal upper 2-tuple dominating set, $|S \cap \{u_{j_1}, u_{j_2}, u_{j_3}\}| \ge 1$. We claim that $|S \cap \{u_{j_1}, u_{j_2}, u_{j_3}\}| = 1$. Otherwise, $|S \cap \{u_{j_1}, u_{j_2}, u_{j_3}\}| \ge 2$. Then the degree of w_j in $G_I[S]$ is more than 1. However, w_j does not satisfy the conditions of Lemma 1 because $|S \cap V(C_i)| = 3$ for $1 \le i \le n$, a contradiction. Let $\mathscr{A}\colon U \to \{\text{true, false}\}$ be defined by $\mathscr{A}(u_i) = \{\text{true}\}$ if $u_i \in S$ and $\mathscr{A}(u_i) = \{\text{false}\}$ if $u_i \notin S$. By the construction of G_I, we have each clause c_j of I contains only one variable u_i belonging to S. So $\mathscr{A}$ is a satisfying truth assignment for I. Consequently, I has a satisfying truth assignment if and only if G_I has a minimal upper 2-tuple dominating set of cardinality at least $s = 3n + m$. This completes the proof for bipartite graphs.

To deal with split graphs, we add edges to make $\{u_1, y_1, u_2, y_2, \ldots, u_n, y_n\}$ a clique. The same arguments hold, and show the NP-completeness of the upper 2-tuple domination problem for split graphs. $\square$

Theorem 5 *For any fixed integer $k \geq 3$, the k-tuple domination problem is NP-complete for bipartite graphs and for chordal graphs.*

Proof: For any bipartite graph G on n vertices, consider the bipartite graph G' obtained by the following process. For each vertex v of G, we add a copy of $K_{k-1,k-1}$, denoted G_v with bipartition denoted (X_v, Y_v). Then we link by an edge the vertex v to $k-2$ vertices in X_v. The widget added to each vertex v is drawn in Figure 3.

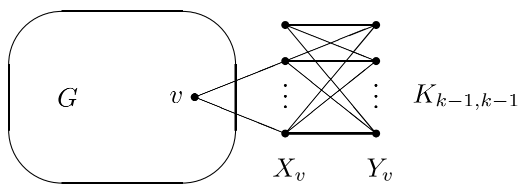

Fig. 3: The widget G_v added to each vertex v in G.

We claim that G has a minimal upper 2-tuple dominating set of size at least s if and only if G' has a minimal upper k-tuple dominating set of size at least $s + 2(k-1)n$.

Clearly, if S is a minimal upper 2-dominating set of G with size at least s, then $S \cup \big(\bigcup_{v \in V(G)} (X_v \cup Y_v)\big)$ is a minimal upper k-tuple dominating set of G' with size at least $s + 2(k-1)n$.

On the other hand, suppose S' is a minimal upper k-tuple dominating set of G' with size at least $s + 2(k-1)n$. Since every vertex in Y_v is of degree $k-1$, S' necessarily includes $\bigcup_{v \in V(G)} (X_v \cup Y_v)$. Let $S = S' \setminus \big(\bigcup_{v \in V} (X_v \cup Y_v)\big)$. Every vertex v in $V(G)$ is dominated precisely $k-2$ times by vertices from $S' \setminus S$. Therefore, v is dominated by at least two vertices in S and so S is a 2-tuple dominating set of G. By Lemma 1, v is dominated by some $u \in V(G')$ with $N_{G'}[u] = k$. This vertex u must be in $V(G)$ and $N_G[u] = 2$. By Lemma 1 again, S is a minimal upper 2-tuple dominating set of G with size at least s.

The NP-completeness of the upper k-tuple domination problem for bipartite graphs then follows from the NP-completeness of the upper 2-tuple domination problem for bipartite graphs.

To deal with the chordal case, we start with a chordal graph G and for each $v \in V(G)$ we add to G_v edges joining any pair of vertices in X_v, forming a clique. The same arguments hold, and show the NP-completeness of the upper k-tuple domination problem for chordal graphs as a consequence of the NP-completeness of the upper 2-tuple domination problem for chordal graphs. $\square$

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