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On bipartite powers of bigraphs

Yoshio Okamoto[1†] Yota Otachi[2‡] Ryuhei Uehara[2§]

[1]*Department of Communication Engineering and Informatics, Graduate School of Informatics and Engineering, University of Electro-Communications, Chofugaoka 1-5-1, Chofu, Tokyo, 182-8585 Japan.*
[2]*School of Information Science, Japan Advanced Institute of Science and Technology, Asahidai 1-1, Nomi, Ishikawa 923-1292, Japan.*

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The notion of graph powers is a well-studied topic in graph theory and its applications. In this paper, we investigate a bipartite analogue of graph powers, which we call *bipartite powers of bigraphs*. We show that the classes of bipartite permutation graphs and interval bigraphs are closed under taking bipartite power. We also show that the problem of recognizing bipartite powers is NP-complete in general.

Keywords: Bipartite power, Interval bigraph, Bipartite permutation graph, Closure property, NP-completeness

1 Introduction

For a positive integer k, the *kth power* G^k of a graph G has the same vertex set as G, and two vertices are adjacent in G^k if and only if their distance in G is at most k. If $H^k = G$, then H is called a *kth root* of G. (In general, kth roots of a graph are not unique and do not necessarily exist.) The concept of graph powers has been extensively studied in graph theory and its algorithmic applications (see Prisner (1995) and the references therein). It is known that several important graph classes are closed under the power operation. That is, for some graph class $\mathcal{C}$, $G \in \mathcal{C}$ implies $G^k \in \mathcal{C}$ as well. The recognition problem has also been studied for graph powers. It was shown that recognizing graph powers is NP-complete in general while it is solvable in polynomial time for some special graph classes (Chang et al. (2006); Farzad et al. (2009); Kearney and Corneil (1998); Lau (2006); Lau and Corneil (2004); Le and Nguyen (2010); Lin and Skiena (1995); Motwani and Sudan (1994)).

A graph is *bipartite* if its vertices can be partitioned into two parts in such a way that the endpoints of every edge belong to different parts. Bipartite graphs are also known as *bigraphs*. It is known that the class of bigraphs coincides with the class of graphs with no cycle of odd length. If a bigraph is connected and has at least three vertices, then clearly its kth power (for $k \geq 2$) has a cycle of length three. Hence, the class of bigraphs is not closed under the power operation. Furthermore, any non-trivial subclass of the class of bigraphs is not closed under the power operation.

†Email: `okamotoy@uec.ac.jp`.
‡Email: `otachi@jaist.ac.jp`.
§Email: `uehara@jaist.ac.jp`.

Chandran et al. (2011) introduced a bipartite analogue of graph powers. For odd $k \in \mathbb{Z}^+$, the *kth bipartite power* $G^{[k]}$ of a bigraph G has the same vertex set as G, and two vertices are adjacent in $G^{[k]}$ if and only if their distance in G is at most k and odd. It is easy to see that $G^{[k]}$ is also bipartite. They showed that for any tree T and for any odd k, $T^{[k]}$ is a chordal bipartite graph. They used this fact to prove that chordal bipartite graphs can have arbitrarily large boxicity. Recently, Chandran and Mathew (2012) have strengthened the closure-property result by showing that the class of chordal bipartite graphs is closed under the bipartite power operation.

In this paper, we complement the known closure-property results by proving that some subclasses of the class of chordal bipartite graphs are closed under the bipartite power operation. Note that the closure property of a graph class $\mathcal{C}$ under the bipartite power operation is not implied by the closure property of a superclass of $\mathcal{C}$, and vice versa. We also focus on the computational complexity of the problem for recognizing the kth bipartite power graphs. We prove that given a bipartite graph G, the problem of determining whether there exists a bipartite graph H such that $G = H^{[k]}$ is NP-complete for any fixed odd $k \geq 3$.

2 Preliminaries

In this paper, graphs are finite, simple, and undirected. We denote by $N_G(v)$ the neighborhood of a vertex v in a graph G. We denote by $\mathrm{dist}_G(u, v)$ the distance between u and v in G.

A graph $G = (V, E)$ with $V = \{1, 2, \ldots, n\}$ is a *permutation graph* if there is a permutation π over V such that $\{i, j\} \in E(G)$ if and only if $(i - j)(\pi(i) - \pi(j)) < 0$. A graph is a *bipartite permutation graph* if it is bipartite and a permutation graph. A *bi-interval representation* of a bigraph $G = (U, V; E)$ is a pair (I_U, I_V) of sets of closed intervals such that $I_U = \{[\ell_u, r_u] \mid u \in U\}$ and $I_V = \{[\ell_v, r_v] \mid v \in V\}$, and $\{u, v\} \in E(G)$ if and only if $[\ell_u, r_u] \cap [\ell_v, r_v] \neq \emptyset$ for $u \in U$ and $v \in V$. A *bi-interval representation* (I_U, I_V) is *unit* if for each interval $[\ell, r] \in I_U \cup I_V$, $r - \ell = 1$. A bigraph is an *interval bigraph* if it has a bi-interval representation. An interval bigraph is a *unit interval bigraph* if it has a unit bi-interval representation. A bigraph is a *chordal bipartite graph* if every induced cycle is of length four. We denote the classes of bipartite permutation graphs, unit interval bigraphs, interval bigraphs, and chordal bipartite graphs by BP, UIB, IB, and CB, respectively. The following relations among these classes are known.

Theorem 2.1 (Hell and Huang (2004); Müller (1997)) *BP = UIB ⊂ IB ⊂ CB.*

3 Closure properties of bipartite powers of bigraphs

It is known that several important (non-bipartite) graph classes are closed under the power operation. For example, the classes of unit interval graphs, interval graphs, strongly chordal graphs are closed under the kth power for any k (Chen and Chang (2001); Dahlhaus and Duchet (1987); Lubiw (1987); Raychaudhuri (1987, 1992)). On the other hand, few things are known on bipartite powers of bigraphs. To the best of our knowledge, the following is the only known closure property.

Theorem 3.1 (Chandran and Mathew (2012)) *If G is a chordal bipartite graph, then so is $G^{[k]}$ for any odd $k \in \mathbb{Z}^+$.*

In this section, we show that the closure property holds also for smaller graph classes.

Theorem 3.2 *If G is an interval bigraph, then so is $G^{[k]}$ for any odd $k \in \mathbb{Z}^+$.*

Theorem 3.3 *If G is a bipartite permutation graph, then so is $G^{[k]}$ for any odd $k \in \mathbb{Z}^+$.*

We note that the closure property is not hereditary for the graph classes. That is, Theorem 3.1 does not imply Theorem 3.2, and Theorem 3.2 does not imply Theorem 3.3.

3.1 Bipartite powers of interval bigraphs

Now we prove Theorem 3.2. Let $G = (U, V; E)$ be an interval bigraph, and (I_U, I_V) be a bi-interval representation of G. For odd $k \in \mathbb{Z}^+$ and $I_w \in I_U \cup I_V$, we form a new interval $I_w(k) = [\ell_w, r_w(k)]$, where

$$r_w(k) = \max\{l_x \mid \mathrm{dist}_G(w, x) \text{ is odd and at most } k\}.$$

This extension of intervals is inspired by the proof of Agnarsson et al. (2000), who showed that the kth power of an interval graph is also an interval graph. Note that $I_u \cap I_v \neq \emptyset$ if and only if $\max\{\ell_u, \ell_v\} \leq \min\{r_u, r_v\}$. The following two lemmas show that for $u \in U$, $v \in V$, and odd $k \in \mathbb{Z}^+$, $I_u(k) \cap I_v(k) \neq \emptyset$ if and only if $\mathrm{dist}_G(u, v) \leq k$. Since $u \in U$ and $v \in V$, clearly $\mathrm{dist}_G(u, v)$ is odd. Thus we have the theorem by the two lemmas.

In what follows, we assume $u \in U$ and $v \in V$. We also assume that G is connected. It is easy to see that if Theorem 3.2 is valid for connected graphs, then it is also valid for disconnected graphs.

Lemma 3.4 *If $\mathrm{dist}_G(u, v) \leq k$, then $I_u(k) \cap I_v(k) \neq \emptyset$.*

Proof: Without loss of generality, we assume $\ell_u \leq \ell_v$. Since $\mathrm{dist}_G(u, v)$ is odd and at most k, it holds that $\ell_v \leq r_u(k)$ from the definition of $r_u(k)$. Hence $\ell_v \in [\ell_u, r_u(k)] = I_u(k)$, and thus $I_u(k) \cap I_v(k) \neq \emptyset$. □

Lemma 3.5 *If $I_u(k) \cap I_v(k) \neq \emptyset$, then $\mathrm{dist}_G(u, v) \leq k$.*

Proof: Without loss of generality, we assume $\ell_u \leq \ell_v$. By this assumption and $I_u(k) \cap I_v(k) \neq \emptyset$, it holds that $\ell_u \leq \ell_v \leq r_u(k)$. Let $v' \in V$ be a vertex such that $\mathrm{dist}_G(u, v') \leq k$ and $r_u(k) = \ell_{v'}$. Let $P = (u_0, v_1, u_2, \ldots, u_{2d}, v_{2d+1})$ denote a shortest u–v' path in G, where $u_0 = u$ and $v_{2d+1} = v'$. Clearly, $2d + 1 \leq k$. If $\{v, u_i\} \in E(G)$ for some i, there is a u–v path $(u_0, v_1, u_2 \ldots, v_{i-1}, u_i, v)$ of length $i + 1 \leq 2d + 1 \leq k$ in G and this implies that $\mathrm{dist}_G(u, v) \leq k$. Hence, we assume there is no such index i. Then, the following claim holds.

Claim 3.6 *There is some index j such that $I_v \subseteq I_{v_j}$.*

Proof (Claim 3.6): Since I_v does not intersect with any I_{u_i}, we can partition $U \cap V(P)$ into two parts $L(P) = \{u_i \mid r_{u_i} < \ell_v\}$ and $R(P) = \{u_i \mid r_v < \ell_{u_i}\}$. By the assumption $\ell_u \leq \ell_v$ and $\{u, v\} = \{u_0, v\} \notin E(G)$, it holds that $r_u < \ell_v$. Furthermore, since $\ell_v \leq r_u(k) = \ell_{v'}$ and $\{u_{2d}, v'\} = \{u_{2d}, v_{2d+1}\} \in E(G)$, we have $\ell_v \leq \ell_{v'} \leq r_{u_{2d}}$. Because $\{u_{2d}, v\} \notin E(G)$, it holds that $r_v < \ell_{u_{2d}}$. Hence, both $L(P)$ and $R(P)$ are non-empty since $u \in L(P)$ and $u_{2d} \in R(P)$. Since P is a connected subgraph of G, there is a vertex v_j that has neighbors both in $L(P)$ and in $R(P)$. Then, the corresponding interval I_{v_j} contains I_v (see Fig. 1). □

Since G is connected, v has a neighbor u' in G. Thus, we have $I_v \cap I_{u'} \neq \emptyset$, which implies $I_{u'} \cap I_{v_j} \neq \emptyset$. It is easy to see that $(u_0, v_1, \ldots, u_{j-1}, v_j, u', v)$ is a u–v path of length $j + 2$, where $j - 1 \leq 2d - 2$. Therefore, the lemma holds. □

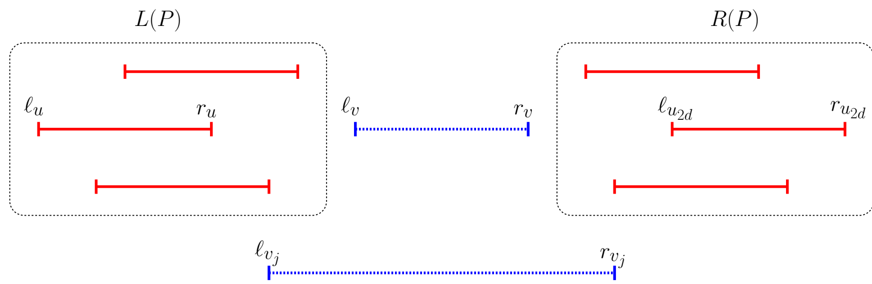

Fig. 1: The bridge interval I_{v_j} between $L(P)$ and $R(P)$.

3.2 Bipartite powers of bipartite permutation graphs

Next we prove Theorem 3.3. An *asteroidal triple* is a set of three pairwise non-adjacent vertices such that there is a path between any pair of them avoiding the neighbors of the third. If a graph has no asteroidal triple, then it is *AT-free*. We use the following characterization of bipartite permutation graphs.

Theorem 3.7 (Hell and Huang (2004)) *A bipartite graph is a permutation graph if and only if it is AT-free.*

By the theorem above, it is sufficient to show the following lemma. Our proof is inspired by the proof of Raychaudhuri (1987), who proved that the kth power of an AT-free graph is also AT-free.

Lemma 3.8 *If G is bipartite and AT-free, then so is $G^{[k]}$ for any odd $k \in \mathbb{Z}^+$.*

Proof: Let v_1, v_2, v_3 be any pairwise non-adjacent triple of vertices in G. Without loss of generality, assume that any v_1–v_2 path in G passes through a neighbor of v_3. We shall prove that any v_1–v_2 path in $G^{[k]}$ passes through a neighbor of v_3.

Let $P = (u_1, u_2, \dots, u_p)$ be a v_1–v_2 path in $G^{[k]}$ such that $u_1 = v_1$ and $u_p = v_2$. For $1 \le i \le p-1$, let $P(u_i, u_{i+1})$ denote a shortest u_i–u_{i+1} path in G. Replacing every edge $\{u_i, u_{i+1}\}$ in P by the corresponding path $P(u_i, u_{i+1})$, we obtain a walk W_P in G. Clearly, W_P contains a v_1–v_2 path in G, and hence, $N_G(v_3) \cap V(W_P) \neq \emptyset$. Let $w \in N_G(v_3) \cap V(W_P)$. Assume, without loss of generality, $w \in V(P(u_i, u_{i+1}))$ for some i, $1 \le i \le p-1$. Since $\{u_i, u_{i+1}\} \in E(G^{[k]})$, $\mathrm{dist}_G(u_i, u_{i+1})$ is odd and at most k. If $w \in \{u_i, u_{i+1}\}$, then $N_{G^{[k]}}(v_3) \cap V(P) \neq \emptyset$. Thus, we assume $w \notin \{u_i, u_{i+1}\}$. Since $P(u_i, u_{i+1})$ is a shortest u_i–u_{i+1} path in G and $w \in V(P(u_i, u_{i+1}))$, we have $\mathrm{dist}_G(u_i, w) + \mathrm{dist}_G(w, u_{i+1}) = \mathrm{dist}_G(u_i, u_{i+1})$. Thus, $\mathrm{dist}_G(u_i, u_{i+1}) \le k$ and $w \notin \{u_i, u_{i+1}\}$ together imply $\mathrm{dist}_G(u_i, w) < k$ and $\mathrm{dist}_G(w, u_{i+1}) < k$. Hence, $\mathrm{dist}_G(v_3, u_i) \le k$ and $\mathrm{dist}_G(v_3, u_{i+1}) \le k$. Since $\mathrm{dist}_G(u_i, u_{i+1})$ is odd, one of $\mathrm{dist}_G(u_i, w)$ and $\mathrm{dist}_G(w, u_{i+1})$ is also odd.

Without loss of generality, we assume $\mathrm{dist}_G(u_i, w)$ is odd, and $\mathrm{dist}_G(w, u_{i+1})$ is even. Then, G has a v_3–u_{i+1} path with length $\mathrm{dist}_G(w, u_{i+1}) + 1$, which is odd and at most k. Since G is bipartite, G cannot have any v_3–u_{i+1} path of even length. Hence $\mathrm{dist}_G(v_3, u_{i+1})$ is odd and at most k. This implies that $u_{i+1} \in N_{G^{[k]}}(v_3)$ and therefore, $N_{G^{[k]}}(v_3) \cap P \neq \emptyset$. $\square$

4 Recognizing bipartite powers of bigraphs is hard

Powers of graphs are studied also in computational complexity aspects. Motwani and Sudan (1994) first showed that given a graph G, it is NP-complete to decide whether there exists a graph H such that $H^2 = G$. Following their result, several related results have been obtained (Chang et al. (2006); Farzad et al. (2009); Kearney and Corneil (1998); Lau (2006); Lau and Corneil (2004); Le and Nguyen (2010); Lin and Skiena (1995)). Recently, Le and Nguyen (2010) have shown the following hardness result.

Theorem 4.1 (Le and Nguyen (2010)) *For any fixed $k \geq 2$ and given a graph G, it is NP-complete to determine whether there exists a graph H such that $H^k = G$.*

In this section, we prove that the problem of recognizing bipartite powers is also hard.

Theorem 4.2 *For any fixed odd $k \geq 3$ and given a bipartite graph G, it is NP-complete to determine whether there exists a bipartite graph H such that $H^{[k]} = G$.*

The problem is clearly in NP. Thus, it is sufficient to show that the problem is NP-hard. We reduce SET SPLITTING, a well-known NP-complete problem, to ours.

Problem: SET SPLITTING (Garey and Johnson, 1979, SP4)

Instance: Collection C of non-empty subsets of a finite set S.

Question: Is there a partition of S into two subsets S_1 and S_2 such that no subset in C is entirely contained in either S_1 or S_2?

Without loss of generality, we assume that every element of S belongs to at least one set in C.

4.1 An auxiliary lemma

Before presenting the reduction, we show a useful lemma. In their proof of NP-hardness, Le and Nguyen (2010) used the following *tail structure* presented first by Motwani and Sudan (1994) and generalized later by Lau (2006). Note that for two sets A and B, we mean by $A \subset B$ that $A \subseteq B$ and $A \neq B$; that is, A is properly included in B.

Lemma 4.3 (Lau (2006)) *Let G be a connected graph with $\{v_1, \ldots, v_{k+1}\} \subset V(G)$ where $N_G(v_1) = \{v_2, \ldots, v_{k+1}\}$, and $N_G(v_i) \subset N_G[v_{i+1}]$ for all $1 \leq i \leq k$. If $H^k = G$ for some graph H, then*

1. $N_H(v_1) = \{v_2\}$, $N_H(v_i) = \{v_{i-1}, v_{i+1}\}$ *for all* $2 \leq i \leq k$, *and*
2. $N_H(v_{k+1}) \setminus \{v_k\} = N_G(v_2) \setminus \{v_1, v_2, \ldots, v_{k+1}\}$.

The tail structure played an important role in the proofs of NP-hardness of recognizing powers of graphs (Lau (2006); Le and Nguyen (2010); Motwani and Sudan (1994)). Although we cannot use the structure directly, we show the following similar structure for bipartite powers.

Lemma 4.4 *Let $k \geq 3$ be odd, and G be a connected bipartite graph with $\{v_0, \ldots, v_{k+1}\} \subset V(G)$ where $N_G(v_0) = \{v_1, v_3, \ldots, v_{k-2}, v_k\}$, $N_G(v_1) = \{v_0, v_2, \ldots, v_{k-1}, v_{k+1}\}$, and $N_G(v_i) \subset N_G(v_{i+2})$ for $0 \leq i \leq k-1$. If $H^{[k]} = G$ for some bipartite graph H, then*

1. $N_H(v_0) = \{v_1\}$, $N_H(v_i) = \{v_{i-1}, v_{i+1}\}$ *for* $1 \leq i \leq k$, *and*
2. $N_H(v_{k+1}) \setminus \{v_k\} = N_G(v_2) \setminus \{v_0, v_1, \ldots, v_{k+1}\}$.

To prove Lemma 4.4, we need some definitions. A vertex $u \in U$ in a bipartite graph $G = (U, V; E)$ is *bipartite universal* if $N_G(u) = V$. We denote by $N_H^d(v)$ the set of vertices in H such that the distance from v is exactly d; that is, $N_H^d(v) = \{u \in V(H) \mid \mathrm{dist}_H(u, v) = d\}$.

Proof (Lemma 4.4): First, observe that the vertices v_0 and v_1 are not bipartite universal in G.

Since v_0 is not bipartite universal in $G = H^{[k]}$ and G is connected, there is a vertex $u \in V(H)$ such that $\mathrm{dist}_H(v_0, u)$ is finite and greater than k. This implies that $N_H^{2i+1}(v_0) \neq \emptyset$ for all $0 \leq i \leq (k-1)/2$. Moreover, since $|N_G(v_0)| = |\{v_1, v_3, \ldots, v_{k-2}, v_k\}| = (k+1)/2$, it follows $|N_H^{2i+1}(v_0)| = 1$ for all $0 \leq i \leq (k-1)/2$. Then, the assumption $N_G(v_i) \subset N_G(v_{i+2})$ implies that $N_H^{2i+1}(v_0) = \{v_{2i+1}\}$ for all $0 \leq i \leq (k-1)/2$ (see Fig. 2).

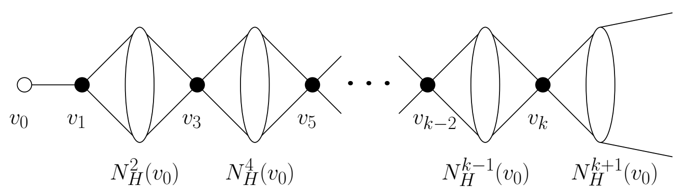

Fig. 2: Partial tail structure of vertices.

Next we apply a similar argument to v_1. Since v_1 is not bipartite universal in G, $N_H^{2i+1}(v_1) \neq \emptyset$ for all $0 \leq i \leq (k-1)/2$. Since $N_H(v_0) = \{v_1\}$ and $|N_G(v_1)| = |\{v_0, v_2, \ldots, v_{k-1}, v_{k+1}\}| = (k+1)/2 + 1$, it follows $|N_H(v_1)| = 2$ and $|N_H^{2i+1}(v_1)| = 1$ for all $1 \leq i \leq (k-1)/2$. By the assumption $N_G(v_i) \subset N_G(v_{i+2})$, we have $N_H(v_1) = \{v_0, v_2\}$ and $N_H^{2i+1}(v_1) = \{v_{2i+2}\}$ for $1 \leq i \leq (k-1)/2$ (see Fig. 3). Thus the first property holds. The second property immediately follows from the first one. $\square$

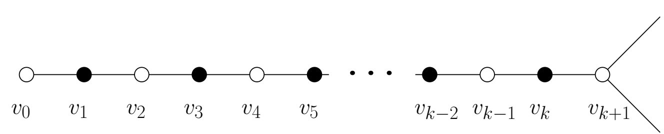

Fig. 3: Tail structure of vertices.

4.2 Reduction

Let (S, C) be an instance of SET SPLITTING, where $S = \{u_1, \ldots, u_n\}$ is the ground set, and $C = \{c_1, \ldots, c_m\}$ is a set of non-empty subsets of S. For any fixed odd $k \geq 3$, we construct a graph G from (S, C). The vertex set $V(G)$ of G consists of the following vertices:

- Element vertices u_i for all element $u_i \in S$;
- Subset vertices $c_j^0, \ldots, c_j^{k+1}$ for all $c_j \in C$; and
- Partition vertices $S_1^0, \ldots, S_1^{k-2}$ and $S_2^0, \ldots, S_2^{k-2}$.

The edge set $E(G)$ of G consists of the following edges: For each subset $c_j \in C$,

- c_j^p is adjacent to c_j^q if $p \not\equiv q \pmod 2$,
- c_j^p is adjacent to $c_{j'}^q$ if $p+q \leq k-4$ and $p \not\equiv q \pmod 2$;
- c_j^p is adjacent to $c_{j'}^q$ if $c_j \cap c_{j'} \neq \emptyset$ and $p+q = k-2$,
- c_j^p is adjacent to all $u_i \in S$ if p is even and $0 \leq p \leq k-3$,
- c_j^{k-1} is adjacent to each $u_i \in c_j$,
- c_j^p is adjacent to S_1^q and S_2^q if $p+q \leq k-2$ and $p \not\equiv q \pmod 2$,

For each element $u_i \in S$,

- u_i is adjacent to $S_1^0, S_1^2, \ldots, S_1^{k-3}$ and $S_2^0, S_2^2, \ldots, S_2^{k-3}$;

For each $i \in \{1, 2\}$,

- S_i^p is adjacent to S_i^q if $p \not\equiv q \pmod 2$,
- S_1^p and S_2^q are adjacent if $p+q \leq k-4$ and $p \not\equiv q \pmod 2$.

4.3 Equivalence

Now we prove that (S, C) has a desired partition if and only if there exists a bipartite graph H such that $H^{[k]} = G$.

Lemma 4.5 *If there exists a partition of S into two disjoint subsets S_1 and S_2 such that each subset in C intersects both S_1 and S_2, then there exists a bipartite graph H such that $G = H^{[k]}$.*

Proof: Let $V(H) = V(G)$. We define $E(H)$ as follows (see Fig. 4).

- For each $c_j \in C$, $\{c_j^i, c_j^{i+1}\} \in E(H)$ for $0 \leq i \leq k$, and $\{c_j^0, u_i\} \in E(H)$ for each $u_i \in c_j$.
- For each $u_i \in S$, $\{u_i, S_j^0\} \in E(H)$ if $u_i \in S_j$.
- For $j \in \{1, 2\}$, $\{S_j^i, S_j^{i+1}\} \in E(H)$ for $0 \leq i \leq k-3$.

It is a routine exercise to verify that $G = H^{[k]}$. □

Lemma 4.6 *If there is a bigraph H such that $H^{[k]} = G$, then there exists a partition of S into two disjoint subsets S_1 and S_2 such that each subset in C intersects both S_1 and S_2.*

Proof: We construct a partition (S_1, S_2) of S such that $S_1 = \{u_i \mid \{u_i, S_1^0\} \in E(H)\}$ and $S_2 = S \setminus S_1$. We show that (S_1, S_2) is a good partition; that is, $c_j \cap S_i \neq \emptyset$ for each $c_j \in C$ and $i \in \{1, 2\}$. To this end, we show that

1. for any $c_j \in C$ and $\ell \in \{1, 2\}$, some $u_i \in c_j$ has S_ℓ^0 as a neighbor in H, and

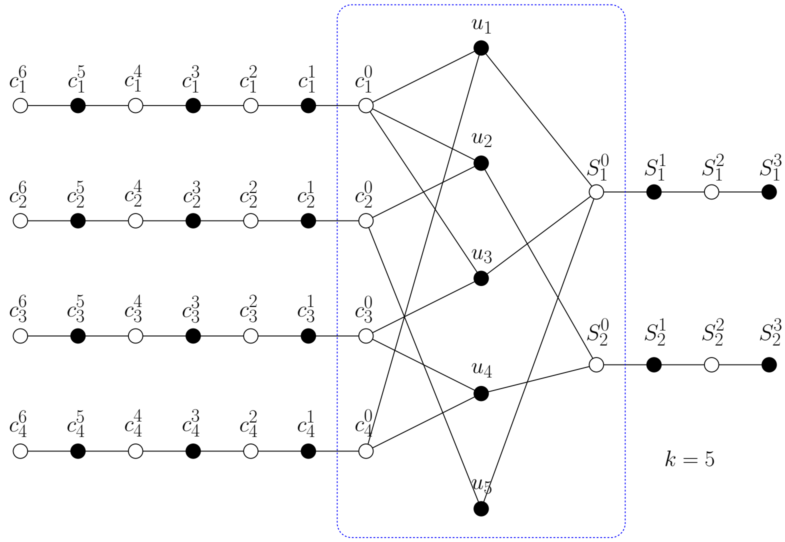

Fig. 4: An example ($k = 5$).

2. no $u_i \in S$ has both S_1^0 and S_2^0 as neighbors in H.

They will conclude the proof. Let $c_j \in C$. From Lemma 4.4, $N_H(c_j^{k+1}) = \{c_j^k\}$, $N_H(c_j^i) = \{c_j^{i+1}, c_j^{i-1}\}$ for $1 \leq i \leq k$, and $N_H(c_j^0) = \{c_j^1\} \cup \{u_i \mid u_i \in c_j\}$. Obviously, $\mathrm{dist}_H(c_j^p, c_j^q) = |p - q|$. For $v \notin \{c_j^0, \ldots, c_j^{k+1}\}$, every c_j^p–v path in H has c_j^q as an internal node if $p > q$. Hence, $\mathrm{dist}_H(c_j^p, v) = p - q + \mathrm{dist}_H(c_j^q, v)$. Note that this also holds when $p \leq q$.

Claim 4.7 *For* $\ell \in \{1, 2\}$ *and* $a, b \geq 0$, $\mathrm{dist}_H(c_j^a, S_\ell^b) = a + b + 2$.

Proof (Claim 4.7): First, we show that $\mathrm{dist}_H(c_j^{k-b-2}, S_\ell^b) = k$.
Since $\{c_j^{k-b-2}, S_\ell^b\} \in E(G)$, $\mathrm{dist}_H(c_j^{k-b-2}, S_\ell^b)$ is odd and at most k. Clearly, $\mathrm{dist}_H(c_j^{k-b}, S_\ell^b)$ is also odd. Thus $\{c_j^{k-b}, S_\ell^b\} \notin E(G)$ implies that

$$\mathrm{dist}_H(c_j^{k-b}, S_\ell^b) = 2 + \mathrm{dist}_H(c_j^{k-b-2}, S_\ell^b) > k.$$

Hence, $\mathrm{dist}_H(c_j^{k-b-2}, S_\ell^b) = k$ since k is odd. Recall that $\mathrm{dist}_H(c_j^p, v) = p - q + \mathrm{dist}_H(c_j^q, v)$. Therefore,

$$\mathrm{dist}_H(c_j^a, S_\ell^b) = a - (k - b - 2) + \mathrm{dist}_H(c_j^{k-b-2}, S_\ell^b) = a + b + 2.$$

Thus, the claim holds. $\square$

Let $\ell \in \{1, 2\}$. From Claim 4.7, we have $\mathrm{dist}_H(c_j^0, S_\ell^0) = 2$. Since $N_H(c_j^0) = \{c_j^1\} \cup \{u_i \mid u_i \in c_j\}$ and $N_H(c_j^1) = \{c_j^2, c_j^0\}$, some $u_i \in c_j$ must be adjacent to S_ℓ^0 in H. Thus, we have the first property. To show the second property, we use the following fact.

Claim 4.8 $\mathrm{dist}_H(S_2^0, S_2^{k-2}) = k-2$.

Proof (Claim 4.8): By Claim 4.7, any shortest c_j^0–S_2^{k-2} path in H is of length k. Let $P = (p_0, p_1, \ldots, p_k)$ be such a path, where $p_0 = c_j^0$ and $p_k = S_2^{k-2}$. It is easy to see that $p_1 = u_i$ for some $u_i \in c_j$. Also, it is not difficult to see that either $p_2 = c_{j'}^0$ for some $c_{j'} \in C$, or $p_2 = S_\ell^x$ for some $x \in \{0, \ldots, k-2\}$ and $\ell \in \{1, 2\}$. If $p_2 = c_{j'}^0$, then $\mathrm{dist}_H(c_{j'}^0, S_2^{k-2}) = k-2$. This contradicts Claim 4.7. Hence, we have $p_2 = S_\ell^x$. Since (c_j^0, u_i, S_ℓ^x) is a shortest path in H, $\mathrm{dist}_H(c_j^0, S_\ell^x) = 2$. Thus we have $x = 0$ from Claim 4.7. If $p_2 = S_1^0$, then $(p_2, \ldots, p_k)$ is a S_1^0–S_2^{k-2} path of length $k-2$. So, $\mathrm{dist}_H(S_1^0, S_2^{k-2}) = k-2$. This contradicts $\{S_1^0, S_2^{k-2}\} \notin E(G) = E(H^{[k]})$. Hence, we can conclude that $p_2 = S_2^0$, and thus $(p_2, \ldots, p_k)$ is a shortest S_2^0–S_2^{k-2} path of length $k-2$ in H. □

Now we prove the second property. Suppose some $u_i \in c_j$ has both S_1^0 and S_2^0 as neighbors in H. Then, $\mathrm{dist}_H(S_1^0, S_2^0) = 2$. Since $\mathrm{dist}_H(S_2^0, S_2^{k-2}) = k-2$ by Claim 4.8,

$$\mathrm{dist}_H(S_1^0, S_2^{k-2}) \leq \mathrm{dist}_H(S_1^0, S_2^0) + \mathrm{dist}_H(S_2^0, S_2^{k-2}) = k.$$

Since there is a S_1^0–S_2^{k-2} path of odd length in G and because G is bipartite, $\mathrm{dist}_H(S_1^0, S_2^{k-2})$ is odd. Hence, $\{S_1^0, S_2^{k-2}\} \in E(H^{[k]})$. This contradicts $\{S_1^0, S_2^{k-2}\} \notin E(G) = E(H^{[k]})$. □

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