A Note on Planar Ramsey Numbers for a Triangle Versus Wheels

Guofei Zhou[1†] Yaojun Chen[1‡] Zhengke Miao[2] Shariefuddin Pirzada[3]

[1] *Department of Mathematics, Nanjing University, China.*
[2] *Department of Mathematics, Jiangsu Normal University, Xuzhou, China*
[3] *Department of Mathematics, University of Kashmir, Srinagar*

received 12th April 2012, revised 1st October 2012, accepted 12th October 2012.

For two given graphs G and H, the planar Ramsey number $PR(G, H)$ is the smallest integer n such that every planar graph F on n vertices either contains a copy of G, or its complement contains a copy of H. In this paper, we determine all planar Ramsey numbers for a triangle versus wheels.

Keywords: Planar Ramsey Numbers,Triangle,Wheels

1 Introduction

We assume that the reader is familiar with standard graph-theoretic terminology and refer the readers to Bondy and Murty (2008) for any concept and notation that is not defined here. In this paper, we consider simple, undirected graphs.

Given two graphs G and H, the Ramsey number $R(G, H)$ is the smallest integer n such that every graph F on n vertices contains a copy of G, or its complement $\overline{F}$ contains a copy of H. The determination of Ramsey numbers is notoriously difficult in general. A variant considered here is the concept of planar Ramsey numbers, introduced by Walker (1967) and rediscovered by Steinberg and Tovey (1993).

For two given graphs G and H the planar Ramsey number $PR(G, H)$ is the smallest integer n such that every planar graph F on n vertices either contains a copy of G, or its complement contains a copy of H. It is easy to see that $PR(G, H) \leq R(G, H)$. Moreover, since the complement of a planar graph F may not be planar, the planar Ramsey number is not symmetric with respect to G and H.

The planar Ramsey numbers for all pairs of complete graphs was determined in Steinberg and Tovey (1993). Meanwhile, planar Ramsey numbers for several other pairs of graphs were determined, see e.g. Dudek and Ruciński (2005) and Gorgol and Ruciński (2008).

Let G be a graph and $\overline{G}$ the complement of G. Let $U \subseteq V(G)$, denote by $G[U]$ the subgraph induced by U in G. We call U a cut set of a connected graph G if $G - U$ is not connected. Let v be a vertex in

[†]Email: `gfzhou@nju.edu.cn`
[‡]Email: `yaojunc@nju.edu.cn`

G, denote by $N_G(v)$ the neighbors of v in G, and denote by $d_G(v)$ the degree of v in G. Further, denote by $N_G[v]$ the closed neighborhood of v in G, i.e., $N_G[v] = N_G(v) \cup \{v\}$. The independence number, the connectivity and the minimum degree of G, are denoted by $\alpha(G)$, $\kappa(G)$ and $\delta(G)$ respectively. A wheel $W_n = \{x\} + C_n$ is the graph of order $n+1$ consisting of a cycle C_n and an extra vertex x, the hub, which is joined to every vertex of C_n. A graph G of order n is said to be Hamiltonian if it contains an n-cycle; and G is said to be pancyclic if G contains cycles of length k, for all $k = 3, 4, \cdots, n$. A graph H is called a minor of the graph G if H is isomorphic to a graph that can be obtained by zero or more edge contractions on a subgraph of G.

In this paper, we determine $PR(C_m, W_n)$ for all pairs of $m = 3$ and $n \geq 3$. That is, our main result is the following.

Theorem 1 *The Planar Ramsey numbers of a triangle versus wheels are*

$$PR(C_3, W_n) = \begin{cases} 9, & if\ n \in \{3,4\}, \\ 10, & if\ n = 5, \\ 11, & if\ n = 6, \\ n+4, & if\ n \geq 7. \end{cases}$$

2 Preliminaries

In order to prove Theorem 1, we need some auxiliary results.

We refer the following observations about Hamiltonian and pancyclic graphs due to Chvátal and Erdős (1972) and Bondy (1971), respectively.

Theorem 2 (Chvátal and Erdős (1972)) *If $\alpha(G) \leq k(G)$, then G is Hamiltonian.*

Theorem 3 (Bondy (1971)) *If G is Hamiltonian and $\varepsilon(G) \geq \frac{n^2}{4}$, then G is pancyclic unless $G = K_{\frac{n}{2},\frac{n}{2}}$.*

Brandt (1997) proved the following result about the existence of cycles in non-bipartite graphs.

Theorem 4 (Brandt (1997)) *Every non-bipartite graph of order n with more than $(n-1)^2/4+1$ edges contains cycles of every length between 3 and the length of a longest cycle.*

Gorgol and Ruciński (2008) proved that

Theorem 5 (Gorgol and Ruciński (2008)) $PR(C_3, C_3) = 6$, $PR(C_3, C_4) = 7$ *and* $PR(C_3, C_n) = n+2$ *for* $n \geq 5$.

The following lemma is used several times in the proof of our main result.

Lemma 1 *If G is a planar triangle-free graph and U is a cut set of $\overline{G}$ then*

(a) $\overline{G} - U$ has exactly two components.

(b) one of the components has at most two vertices, and

(c) both components of $\overline{G} - U$ are complete graphs.

Proof:

(a) If the number of components of $\overline{G} - U$ is at least 3, then by taking one vertex in each component, we get a C_3 in G, a contradiction to the initial hypothesis.

(b) If there are two components of $\overline{G} - U$ such that the order of each component is at least 3, then there is a $K_{3,3}$ in G, which contradicts the fact that G is a planar graph.

(c) Suppose that U_1 and U_2 are two components of $\overline{G} - U$ such that $|U_1| \leq 2$ and $|U_2| \geq 3$. If U_2 does not induce a complete graph in $\overline{G}$, then there are two vertices u and v in U_2 such that u and v are not adjacent in $\overline{G}$, and hence they are adjacent in G. Take one vertex of U_1, say w, then in G, w is adjacent to u and v. This implies that uvw is a C_3 in G, a contradiction. A similar argument will deduce that U_1 is also a complete graph.

□

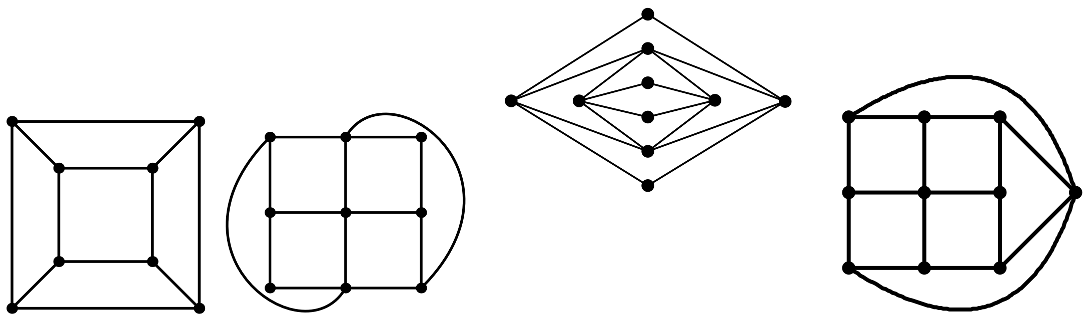
Now we are prepared for the proof of Theorem 1.

3 Proof of Theorem 1

If $n = 3$, since $W_3 = K_4$, and $PR(K_3, K_l) = 3l - 3$ for $l \geq 3$ (Steinberg and Tovey (1993)), so $PR(C_3, W_3) = 9$.

For $n = 4$ we obtain that $8 < PR(C_3, W_4) \leq PR(C_3, K_5 - e) = 9$ from Table 10 of Dudek and Ruciński (2005).

For $5 \leq n \leq 7$, the upper bound can be verified along the lines of the proof below. Anyway, it needs some extra reasoning and in the case $n = 7$, so we omit it here. We also checked these values by the computer program “Planram” due to Dudek.



(a) (b) (c) (d)

Fig. 1: Triangle-free planar graphs whose complement contains no W_n for $n = 4, 5, 6, 7$ respectively.

In Figure 1 we illustrate four triangle-free planar graphs whose complement contains no W_4, W_5, W_6 and W_7 respectively, this implies that $PR(C_3, W_4) \geq 9$, $PR(C_3, W_5) \geq 10$, $PR(C_3, W_6) \geq 11$, $PR(C_3, W_7) \geq 11$. As a matter of fact, by using a program "Planram" due to Dudek, we can easily check that the above inequalities can be reversed, so we have

Lemma 2 $PR(C_3, W_4) = 9$, $PR(C_3, W_5) = 10$, $PR(C_3, W_6) = 11$, $PR(C_3, W_7) = 11$.

Now it is left to consider the case that $n \geq 8$.

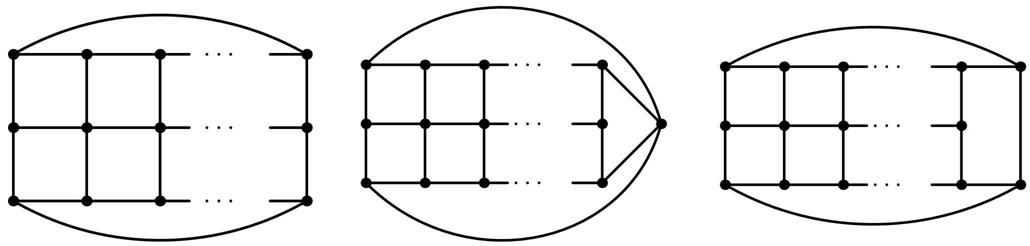
Theorem 6 $PR(C_3, W_n) \leq n + 4$ *for* $n \geq 8$.

Proof: Let G be a triangle-free planar graph of order $n + 4$ with $n \geq 8$. Let $v \in V(G)$such that $d_G(v) = \delta(G)$. If $\delta(G) \leq 1$ then by Theorem 2 the complement of $G - v$ has an n-cycle so $\overline{G}$ contains W_n with hub v. Since by Euler's formula we know that every triangle-free planar graph G has at most $2|V(G)|-4$ edges we may assume that $2 \leq \delta(G) \leq 3$. Among the edges $vw \in E(G)$ with $d_G(v) = \delta(G)$ we choose one edge such that $d_G(w)$ as large as possible.

We first show that for $H = G - N_G[v]$, $\overline{H}$ contains a cycle of length at least n. Indeed, since $H \subseteq G$, H is also triangle free, so $\alpha(\overline{H}) \leq 2$, moreover, since $|V(\overline{H})| \geq n$, by Theorem 2 , $\overline{H}$ has a cycle at least n if $\overline{H}$ is 2-connected. So we assume that $\kappa(\overline{H}) \leq 1$. Choose a vertex $u \in V(H)$ such that $\overline{H} - u$ is disconnected and the order of the largest component is as small as possible. By Lemma 1 there are exactly two components of $\overline{H} - u$ with vertex sets U_1, U_2, where $|U_1| \leq |U_2|$ and $|U_1| \leq 2$, and both sets induce complete subgraphs of $\overline{H}$. By the choice of u, we know that u has at least two neighbors in U_2 and therefore $\overline{H}[\{u\} \cup U_2]$ is Hamiltonian and $|\overline{H}[\{u\} \cup U_2]| \geq |U_2| - 1$. So we may assume that $|U_2| \leq n - 2$.

First assume that $|U_1| = 2$. Then $|U_2| \geq n - 3$ and both vertices of U_1 have in G degree at least $n - 3$.

If $d_G(w) \geq n - 3$ as well, then since $n \geq 8$ w has a neighbor in U_2. Since G is triangle-free, w has no neighbor in U_1 and therefore all its neighbors except v and possibly u belong to U_2. But then G contains a $K_{3,3}$ with one partite set being $\{w\} \cup U_1$, contradicting the planarity of G.



(a) $n+3 \equiv 0\ (mod\ 3)$ (b) $n+3 \equiv 1\ (mod\ 3)$ (c) $n+3 \equiv 2\ (mod\ 3)$

Fig. 2: The planar Ramsey graphs of order $n+3$ for $n \geq 8$.

If $d_G(w) < n-3$, then by the choice of vw every vertex of U_2 has degree at least 3, and, since U_2 is an independent set of G, every vertex of U_2 has an neighbor outside $U_1 \cup U_2$. To avoid a $K_{3,3}$ subgraph, u can be adjacent to at most two vertices of U_2 so three vertices of U_2 must have an neighbor in $N_G(v)$. Deleting u and the other vertices of U_2 and contracting $\{v\} \cup N_G(v)$ into one vertex v^* we obtain a $K_{3,3}$ with one partite set $\{v^*\} \cup U_1$. Therefore G has a $K_{3,3}$ minor, contradicting to the planarity of G.

Finally, assume that $|U_1| = 1$. Then $|U_2| = n-2$ and $\delta(G) = 3$. Since $\{v\} \cup U_2$ forms an independent set in G of cardinality $n-1$, G contains at least $3(n-1)$ edges, which exceeds $2(n+4)-4$ for $n \geq 8$. This again contradicts the fact that G is triangle-free and planar.

Therefore, $\overline{H}$ contains a cycle C of length at least $n \geq 8$. Now H contains at most $2|V(H)|-4$ edges of G. Since $n \geq 8$ the graph $\overline{H}$ is non-bipartite and has at least $\binom{|V(H)|}{2} - 2|V(H)| + 4 > (|V(H)|-1)^2/4 + 1$ edges. By Theorem 4 it contains a cycle C' of length exactly n. Now the subgraph of $\overline{G}$ induced by $\{v\} \cup V(C')$ contains a wheel W_n with hub v. □

The lower bounds for $PR(C_3, W_n)$ follows by considering the Ramsey graphs illustrated in Figure 2 and noticing that they are all planar graphs without triangles, and each of them has minimum degree 3, thus its complement contains no W_n (the hub of the wheel need to have its degree at least n). Therefore, $PR(C_3, W_n) \geq n+4$ for $n \geq 8$. This completes the proof of Theorem 1. □

Acknowledgements

(1) The authors are grateful to the referees whose valuable suggestions resulted in an improved paper.

(2) This research is supported by NSFC under grant number 61073198, 11071115 and 11171288.

References

J. A. Bondy. Pancyclic graphs I. *Journal of Comb. Theory B*, 11(1):80–84, 1971.

J. A. Bondy and U. S. R. Murty. *Graph Theory*. Springer, 2008.

S. Brandt. A sufficient condition for all short cycles. *Discrete Applied Mathematics*, 79(1-3):63–66, 1997.

V. Chvátal and P. Erdős. A note on Hamiltonian circuits. *Discrete Mathematics*, 2:111–113, 1972.

A. Dudek. `http://homepages.wmich.edu/~zyb1431/papers/planram.c`.

A. Dudek and A. Ruciński. Planar Ramsey numbers for small graphs. *Congr. Numer.*, 176:201–220, 2005.

I. Gorgol and A. Ruciński. Planar Ramsey numbers for cycles. *Discrete Mathematics*, 308(19):4389–4395, 2008.

R. Steinberg and C. A. Tovey. Planar Ramsey numbers. *Journal of Comb. Theory B*, 59(2):288–296, 1993.

K. Walker. The analog of Ramsey numbers for planar graphs. *Bull. London Math. Soc.*, 1(2):187–190, 1967.