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Further results on Maximal Nontraceable graphs of smallest size

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Let $g(n)$ denote the minimum number of edges of a maximal nontraceable (MNT) graph of order n. In 2005 Frick and Singleton (Lower bound for the size of maximal nontraceable graphs, *Electronic Journal of Combinatorics*, 12(1) R32, 2005) proved that $g(n) = \lceil \frac{3n-2}{2} \rceil$ for $n \geq 54$ as well as for $n \in I$, where $I = \{12, 13, 22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$ and they determined $g(n)$ for $n \leq 9$. We determine $g(n)$ for 18 of the remaining 26 values of n, showing that $g(n) = \lceil \frac{3n-2}{2} \rceil$ for $n \geq 54$ as well as for $n \in I \cup \{18, 19, 20, 21, 24, 25, 26, 27, 28, 29, 32, 33\}$ and $g(n) = \lceil \frac{3n}{2} \rceil$ for $n \in \{10, 11, 14, 15, 16, 17\}$. We give results based on "analytic" proofs as well as computer searches.

Keywords: maximal nontraceable, xy–maximal nontraceable, z–maximal nontraceable, maximal nonhamiltonian

1 Introduction

We consider only simple, finite graphs G and denote the vertex set, the edge set and the size (number of edges) of G by $V(G)$, $E(G)$ and $e(G)$, respectively. We denote the complement of G by $\overline{G}$. The degree of a vertex v in a graph G is denoted by $d_G(v)$. If no confusion can result we simply write $d(v)$. A vertex $v \in V(G)$ is a *universal* vertex of G if it is adjacent to all vertices in $G - v$. The *open neighbourhood of a vertex* v in G is the set $N_G(v) = \{x \in V(G) : vx \in E(G)\}$. The *open neighbourhood of a subgraph* H of G is the set $N_G(H) = \{x \in V(G) - V(H) : vx \in E(G)$ for at least one $v \in V(H)\}$. If S is a nonempty subset of $V(G)$, then $\langle S \rangle$ denotes the subgraph of G induced by S.

A graph G is *hamiltonian* if it has a *Hamilton cycle* (a cycle containing all the vertices of G), and *traceable* if it has a *Hamilton path* (a path containing all the vertices of G). A graph G is *maximal nonhamiltonian* (MNH) if G is not hamiltonian, but $G + e$ is hamiltonian for each $e \in E(\overline{G})$. A graph G is *maximal nontraceable* (MNT) if G is not traceable, but $G + e$ is traceable for each $e \in E(\overline{G})$. Let x and y be distinct vertices in G. Then G is *xy–traceable* if G is traceable between x and y, i.e. has a

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Hamilton path with x and y as endvertices. Also, G is *xy–maximal nontraceable* (xy–MNT) if G is not xy–traceable (*xy–nontraceable*), but $G+e$ is xy–traceable for each $e \in E(\overline{G})$. A graph G is *z–traceable* if it is traceable from z (i.e. there exists a Hamilton path with z as an endvertex), and it is *z–maximal nontraceable* (z–MNT) if G is not z–traceable, but $G + e$ is z–traceable for each $e \in E(\overline{G})$. A graph G is hypohamiltonian if it is not hamiltonian, but $G - v$ is hamiltonian for every $v \in V(G)$.

In [1] Bollobás posed the problem of finding the least number of edges in an MNH graph of given order. This problem was solved by combined results given in [2, 5, 6, 9]. A similar problem was investigated for MNT graphs (also referred to as hamiltonian path saturated (HPS) graphs) as well as for m–path cover saturated (mPCS) graphs, which are generalizations of HPS graphs, in [7, 8, 12, 13]. Suppose $g(n)$ denotes the minimum number of edges of an MNT graph of order n. Frick and Singleton [8] showed that $g(n) = \lceil \frac{3n-2}{2} \rceil$ for $n \geq 54$ as well as for $n \in I$, where $I = \{12, 13, 22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$ and determined $g(n)$ for $n \leq 9$. In this paper we show that $g(n) = \lceil \frac{3n}{2} \rceil$ for $n = 10, 11, 14, 15, 16, 17$ and $g(n) = \lceil \frac{3n-2}{2} \rceil$ for $n \in \{18, 19, 20, 21, 24, 25, 26, 27, 28, 29, 32, 33\}$ and supply graphs having the minimum number of edges.

Many MNT graphs have either xy–MNT graphs or z–MNT graphs as subgraphs and thus we have a section devoted to xy–MNT graphs and z–MNT graphs of smallest size.

2 Preliminary results

For easy reference we include certain results from [8].

Lemma 2.1 [8] *If G is a connected MNT graph and $v \in V(G)$ with $d(v) = 2$, then the neighbours of v are adjacent. Also, one of the neighbours has degree at least 4 and the other neighbour has degree 2 or at least 4.*

Theorem 2.2 [8] *Suppose G is a connected MNT graph without vertices of degree 1 or adjacent vertices of degree 2. If G has order $n \geq 7$ and m vertices of degree 2, then $e(G) \geq \frac{1}{2}(3n + m)$.*

We include the main steps of the proof of the following theorem, as the proofs of our main results relate to the construction and cases used in this proof.

Theorem 2.3 [8] *If G is an MNT graph of order n, then*

$$e(G) \geq \frac{3n-2}{2} \textit{ for } n \geq 10.$$

Proof: If G is not connected, then $G = K_k \cup K_{n-k}$, for some positive integer $k < n$ and then, clearly, $e(G) > \frac{3n-2}{2}$ for $n \geq 10$. Thus we assume that G is connected.

We need to prove that the sum of the degrees of the vertices of G is at least $3n - 2$. In view of Theorem 2.2, we let

$$M = \{v \in V(G) \mid d(v) = 2 \text{ and no neighbour of } v \text{ has degree } 2\}.$$

The remaining vertices of degree 2 can be dealt with simultaneously with the vertices of degree 1. We let

$$S = \{v \in V(G) - M \mid d(v) = 2 \text{ or } d(v) = 1\}.$$

Let $m = |M|$ and $s = |S|$. If $S = \emptyset$, then it follows from Theorem 2.2 that $e(G) \geq \frac{1}{2}(3n + m)$. Thus we assume $S \neq \emptyset$.

We observe that, if H is a component of the graph $\langle S \rangle$, then either $H \cong K_1$ or $H \cong K_2$ and $N_G(H)$ consists of a single vertex, which is a cut-vertex of G.

An example of such a graph G is depicted in Figure 1.

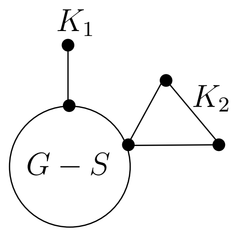


Fig. 1: Example of G

Since G is MNT the graph $\langle S \rangle$ has at most three components. We consider three cases:

Case 1. $\langle S \rangle$ has exactly three components, say H_1, H_2, H_3:

In this case the neighbourhoods of H_1, H_2, H_3 are pairwise disjoint, $G - S$ is a complete graph of order at least 4 and hence

$$e(G) = \binom{n-s}{2} + 2s - 3 \text{ for } s = 3, 4, 5 \text{ or } 6.$$

This case is a Zelinka Type II construction, cf. [14].

Case 2. $\langle S \rangle$ has exactly two components, say H_1, H_2:

There are two subcases.

Case 2.1. $N_G(H_1) = N_G(H_2)$:

Then $G - S$ is a complete graph and hence

$$e(G) = \binom{n-s}{2} + 2s - 2 \text{ for } s = 2, 3 \text{ or } 4.$$

This case is a Zelinka Type I construction, cf. [14].

Case 2.2. $N_G(H_1) \neq N_G(H_2)$:

Let $N_G(H_i) = y_i$, $i = 1, 2$ and $y_1 \neq y_2$. Then $y_1 y_2 \in E(G)$.

Now $G - S$ is not complete. If v and w are nonadjacent vertices in $V(G - S)$, then $(G - S) + vw$ has a Hamilton cycle containing $y_1 y_2$. Hence $G - S$ is either hamiltonian or MNH. We consider these two cases separately:

Case 2.2.1. $G - S$ is hamiltonian:

We note that no Hamilton cycle in $G - S$ contains $y_1 y_2$, otherwise G would be traceable. Also $d_{G-S}(y_i) \geq 3$ for $i = 1, 2$. We have

$$e(G) = e(G - M - S) + 2m + 2s - 2 \geq \frac{1}{2}(3n + m + s - 4). \tag{1}$$

Case 2.2.2. $G - S$ is nonhamiltonian, i.e. $G - S$ is MNH:

We have

$$e(G) = e(G-S) + 2s - 2 \geq \frac{1}{2}(3n + s - 4). \tag{2}$$

Case 3. $\langle S \rangle$ has exactly one component, say H:
Since

$$\sum_{v \in S} d_G(v) = 3s - 2, \text{ for } s = 1, 2$$

it follows that

$$\begin{aligned} e(G) &= e(G-M) + 2m \\ &= \frac{1}{2}\left(\sum_{v \in V(G-M)-S} d_{G-M}(v) + \sum_{v \in S} d_{G-M}(v)\right) + 2m \\ &\geq \frac{1}{2}\left(3\left(n - m - s\right) + 3s - 2\right) + 2m. \end{aligned}$$

Hence

$$e(G) \geq \frac{1}{2}\left(3n + m - 2\right). \tag{3}$$

$\square$

Remark 2.4 *We note the following:*

(a) *If G is an MNT graph as described in Case 1 of Theorem 2.3, then*

$$e(G) > \frac{3n}{2} \text{ if } n > 10.$$

(b) *If G is an MNT graph as described in Case 2.1 of Theorem 2.3, then*

$$e(G) > \frac{3n}{2} \text{ if } n > 8.$$

(c) *The subgraphs $G - S$ in Cases 2.2.1 and 2.2.2 are y_1y_2–MNT graphs. The subgraph $G - S$ in Case 3 is a z–MNT graph, where $z = N_G(H)$.*

We introduce the following terminology which we require in proving certain results in order to show that $g(10) = 15$.

A vertex v in a subgraph H of a graph G is an *attachment vertex* of H if v has a neighbour in $V(G) - V(H)$. The *circumference* $c(G)$ of a graph G is the order of a longest cycle in G. We suppose that a cycle C has an orientation. If $u, v \in V(C)$ we denote the path on C from u to v (with same orientation as C) by $C[u, v]$ and the other path on C from u to v by $\overline{C}[u, v]$. The paths obtained by deleting the endvertices u, v are denoted by $C(u, v)$ and $\overline{C}(u, v)$, respectively.

Lemma 2.5 *Suppose C is a circumference cycle of a graph G and $G - V(C)$ has a path $P = yLz$ with endvertices y and z, where we also consider $V(L) = \emptyset$ as well as $y = z$. If y is adjacent to $u \in V(C)$ and z is adjacent to $v \in V(C)$, $u \neq v$, then*

$$|V(C(u, v))| \geq |V(P)| \text{ and } |V(\overline{C}(u, v))| \geq |V(P)|.$$

Proof: Suppose $|V(C(u,v))| < |V(P)|$. Then $C' = \overline{C}[u,v]zLyu$ is a cycle such that $|V(C')| > |V(C)|$ which is a contradiction since C is a circumference cycle. A similar argument holds if $|V(\overline{C}(u,v))| < |V(P)|$. □

The following lemma is a direct result of Lemma 2.5.

Lemma 2.6 *Suppose $c(G) = k$. Let C be a circumference cycle of G. Suppose $u, v \in V(G - V(C))$ are endvertices of a path in $G - V(C)$ of order at least $\lfloor \frac{k}{2} \rfloor$. If both u and v are attachment vertices of $G - V(C)$, then $|N_C(\{u,v\})| = 1$.*

We also require the following lemma in order to prove $g(10) = 15$.

Lemma 2.7 *Suppose G is a graph of order 10 as described in Case 3 of Theorem 2.3, with $H \cong K_1$ and $d_G(v) = 3$ for all $v \in V(G) - S$. Then $G - S$ is 2–connected.*

Proof: Suppose $G - S$ has a cut vertex y and that G has the structure as depicted in Figure 2.

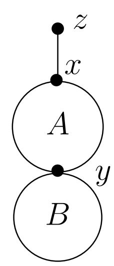


Fig. 2: Sketch for Lemma 2.7

Consider $G + zy$. Since $G + zy$ is traceable it follows that B is traceable from y. If B is not complete, then A is traceable between x and y making G traceable. Hence B is complete.

If $B = K_3$, then G has two vertices of degree 2, and if $B = K_4$, then $d_G(y) > 3$, both of which lead to contradictions.

Hence $G - S$ is 2–connected. □

3 xy–MNT and z–MNT graphs

In this section we investigate some properties of xy–MNT and z–MNT graphs and show how these graphs can be used in constructing MNT graphs. We also present some xy–MNT graphs of small order and smallest size, as well as some cubic xy–MNT graphs of larger orders.

Lemma 3.1 *Suppose F is an xy–MNT graph.*

(a) *Then $xy \in E(F)$.*

(b) *If x is not a universal vertex of F, then $F - x$ is traceable from y.*

(c) *If $N(x) \cap N(y) = \emptyset$, then $F - \{x,y\}$ is traceable from every vertex in $V(F) - \{x,y\}$.*

Proof:

(a) If $xy \notin E(F)$, then $F + xy$ is not xy–traceable.

(b) Suppose $z \notin N(x)$. Then $F+xz$ has a Hamilton path $xzQy$ and consequently $F-x$ has a Hamilton path zQy. Hence $F - x$ is traceable from y.

(c) Let $u \in V(F) - \{x, y\}$. Since $N(x) \cap N(y) = \emptyset$ it follows that $ux \notin E(F)$ or $uy \notin E(F)$. Suppose $ux \notin E(F)$. Then $F + ux$ has a Hamilton path $xuPy$, and hence uP is a Hamilton path of $F - \{x, y\}$, i. e. $F - \{x, y\}$ is traceable from u. A similar proof holds if $uy \notin E(F)$.

□

Lemma 3.2 *Suppose F is an xy–MNT graph with $n \geq 6$, where neither x nor y is a universal vertex and $w \in V(F) - \{x, y\}$ with $d(w) = 2$.*

(a) *Then w is nonadjacent to x and y.*

(b) *If $N(w) = \{v_1, v_2\}$ and $d(v_i) = 4$ for $i = 1, 2$, then $\langle v_1, v_2, x, y\rangle$ is not complete.*

Proof:

(a) Suppose w is adjacent to x and $u \notin N(x)$. Then $F + xu$ is not xy–traceable, a contradiction. A similar argument holds if w is adjacent to y.

(b) Suppose $d(v_i) = 4$ for $i = 1, 2$ and $\langle v_1, v_2, x, y\rangle$ is complete. Then $F + xw$ is not xy–traceable, a contradiction.

□

Lemma 3.3 *Suppose F is a z–MNT graph with $n \geq 5$, where z is not a universal vertex and $w \in V(F) - \{z\}$ with $d(w) = 2$.*

(a) *Then w is nonadjacent to z.*

(b) *If $N(w) = \{v_1, v_2\}$ and $d(v_i) = 4$ for $i = 1, 2$, then $\langle v_1, v_2, z\rangle$ is not complete.*

Proof:

(a) Suppose w is adjacent to z and v, and $u \notin N(z)$. Then $F + zu$ has a Hamilton path of the form $zuPvw$. But then $uPvwz$ is a Hamilton path in F, a contradiction.

(b) Suppose $d(v_i) = 4$ for $i = 1, 2$ and $\langle v_1, v_2, z\rangle$ is complete. Then $F+zw$ has a Hamilton path of the form zwv_iv_jP or zwv_iQv_j, where $\{i, j\} = \{1, 2\}$. But then zv_iwv_jP or zv_iQv_jw is a Hamilton path in F, a contradiction.

□

We consider the following construction.

Construction 1: Suppose F is an xy–MNT graph and $H_x \cong K_n$ and $H_y \cong K_m$, for positive integers n and m. Suppose the vertex sets $V(F)$, $V(H_x)$ and $V(H_y)$ are disjoint. Then $[F, H_x, H_y]$ denotes the graph obtained by joining each vertex of H_x to x and each vertex of H_y to y.

Theorem 3.4 *Suppose F is an xy–MNT graph and neither x nor y is a universal vertex of F. If $N_F(x) \cap N_F(y) = \emptyset$, then $G = [F, H_x, H_y]$ is MNT.*

Proof: Since F is not xy–traceable it follows that G is not traceable. We now prove that $G + uv$ is traceable for all $u, v \in V(G)$, where $uv \notin E(G)$.

Suppose that $u, v \in V(F)$. Since $F + uv$ has a Hamilton path between x and y it follows that $G + uv$ is traceable.

Suppose $u \in V(H_x) \cup \{x\}$ and $v \in V(H_y)$. Then, since x is not a universal vertex it follows from Lemma 3.1(b) that there is a Hamilton path in $G + uv$.

Suppose $u \in V(H_x)$ and $v \in V(F) - \{x, y\}$. Then, from Lemma 3.1(c), there is a Hamilton path in $G + uv$.

All other cases for u and v follow from symmetry. □

If $G = [F, H_x, H_y]$, where F is xy–MNT and $N_F(x) \cap N_F(y) \neq \emptyset$, then in order to show that G is MNT one needs to check if $G + uv$ is traceable for (i) $u \in H_x$ and $v \in H_y$, (ii) $u \in H_x$ and $v = y$, (iii) $u = x$ and $v \in H_y$, and (iv) $u \in H_x \cup H_y$ and $v \in N(x) \cap N(y)$.

We now consider z–MNT graphs. It follows from the definition that if F^z is a z–MNT graph, then $F^z - z$ is traceable from each $v \notin N_{F^z}(z)$, but not traceable from any $u \in N_{F^z}(z)$.

Construction 2: Suppose F^z is a z–MNT graph and $H_z \cong K_n$. Suppose the vertex sets $V(F^z)$ and $V(H_z)$ are disjoint. Then $[F^z, H_z]$ denotes the graph obtained by joining each vertex of H_z to z.

It is easy to see that the graph $G = [F^z, H_z]$, where F^z is z–MNT, is MNT if $G + uv$ is traceable for $u \in F^z - z$ and $v \in H_z$.

Suppose F is an xy–MNT graph and F^z is a z–MNT graph, where x and z are universal vertices and $[F, H_x, H_y]$ and $[F^z, H_z]$ are MNT graphs of order $n \geq 10$. Then $\langle V(F) \cup V(H_y) \rangle - x$ and $F^z - z$ are themselves MNT. If each of H_x, H_y and H_z is either a K_1 or K_2, then by using Theorem 2.3 and results for the minimum size of MNT graphs of orders less than 10 given in [8] it is not difficult to check that $[F, H_x, H_y]$ and $[F^z, H_z]$ have size greater than $\lceil \frac{3n}{2} \rceil$. Thus, since smallest MNT graphs of a specific order may arise from xy–MNT or z–MNT graphs, we searched, with the aid of a computer, for xy–MNT and z–MNT graphs of given order and smallest size in which neither x nor y nor z is a universal vertex. The algorithm for xy–MNT graphs F of order n starts with an initial graph of order n that is necessarily a subgraph of any xy–MNT graph of order n. From Lemmas 3.1(a) and (b) we choose an n–path of the form xyQ as the initial graph. The algorithm then performs a depth–first tree search, systematically adding all possible combinations of edges, but keeping the graph xy–nontraceable. When a predetermined upper bound on the number of edges is reached, that branch is terminated.

We could find all xy–MNT graphs of minimum size of order 5 up to 13 in reasonable computing time. These graphs are depicted in the Figures 3–6.

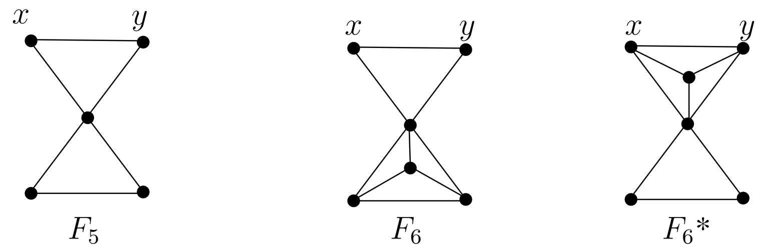


Fig. 3: Smallest xy–MNT graphs of order 5 (6 edges) and order 6 (9 edges)

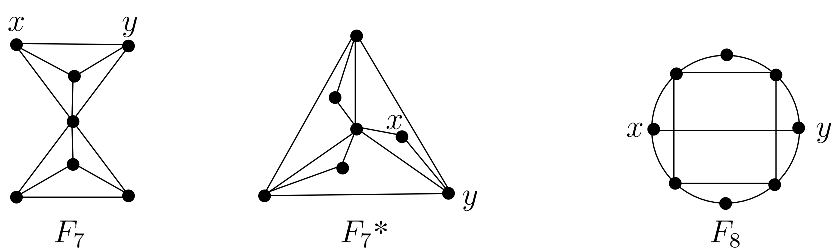


Fig. 4: Smallest xy–MNT graphs of order 7 (12 edges) and of order 8 (13 edges)

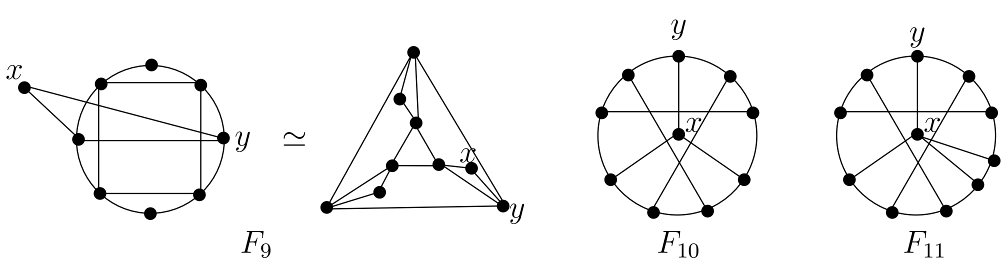


Fig. 5: Smallest xy–MNT graphs of order 9 (15 edges), order 10 (15 edges) and order 11 (17 edges)

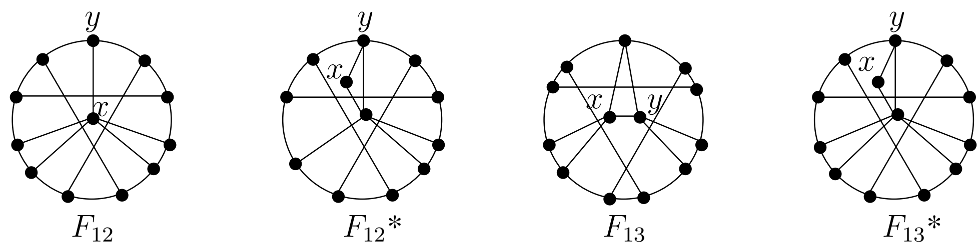


Fig. 6: Smallest xy–MNT graphs of order 12 (19 edges) and order 13 (21 edges)

We also established, by computer, that the smallest MNH graphs G_{14} of order 14 and size 22, and G_{15} of order 15 and size 24 (see Figure 7) given in [9] are in fact xy–MNT.

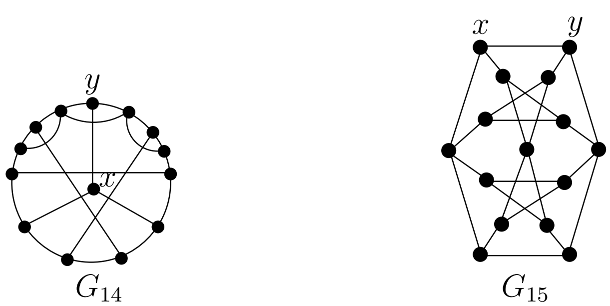


Fig. 7: Smallest maximal nonhamiltonian graphs G_{14} and G_{15}

A similar search for z–MNT graphs of minimum size was undertaken for orders up to 12. We noted that for $7 \leq n \leq 12$ all z–MNT graphs of minimum size can be constructed from xy–MNT graphs of minimum size in one or both of the following ways:
(1) Attach a K_1 to x (or y) of a smallest xy–MNT graph of order $n-1$.
(2) Attach a K_2 to x (or y) of a smallest xy–MNT graph of order $n-2$.
If K_i, $i = 1, 2$ is attached to x (or y), then z is y (or x).

We therefore do not include sketches of these graphs. Table 1 gives the minimum sizes of xy–MNT graphs and z–MNT graphs of orders up to 15 and 13 respectively.

n	5	6	7	8	9	10	11	12	13	14	15
Smallest xy–MNT	6	9	12	13	15	15	17	19	21	≤ 22	≤ 24
Smallest z–MNT	5	7	9	12	14	16	16	18	≤ 20		

Tab. 1: Sizes of smallest xy–MNT and z–MNT graphs

Remark 3.5 *We note the following:*

(a) *The xy–MNT graphs F_8, F_{11}, F_{12} and F_{13} are hamiltonian whereas the others in Figures 3–6 are MNH. The MNH ones for orders $5, 6, 7, 9$ and 10 are those listed as the MNH graphs of smallest size in [9]. Each of the MNH graphs of orders 12 and 13 have one more edge than the corresponding MNH graphs of smallest size listed in [9]*

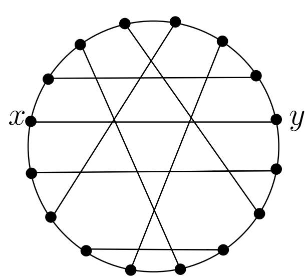
(b) $N_{F_j}(x) \cap N_{F_j}(y) \neq \emptyset$ *for* $j \in \{5, 6, 6^*, 7, 7^*, 9, 12^*, 13, 13^*\}$. *In these cases it is easy to check that [F_j, H_x, H_y] is MNT.*

(c) *The graph F_7^* is also a z–MNT graph (but not of smallest size), where z is the universal vertex.*

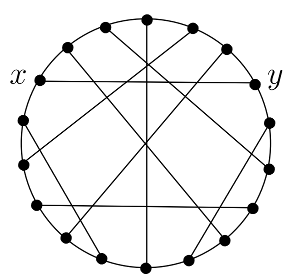
(d) *The xy–MNT graphs F_5, F_6, F_8 and F_{10} were used in [3] and [8] to construct smallest size MNT graphs of orders 7, 8, 9, 10, 12 and 13.*

(e) *In F_{10} and in G_{15}, x and y can be any two adjacent vertices of the graphs.*

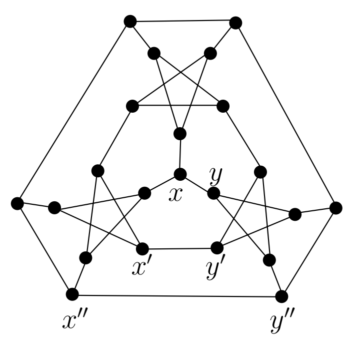
A cubic xy–MNT graph with $N(x) \cap N(y) = \emptyset$ of order n can be used to construct an MNT graph of order $n' = n + 2$ or $n' = n + 3$ by attaching a K_1 to x and a K_1 or K_2 to y. These MNT graphs realise the lower bound $\lceil \frac{3n'-2}{2} \rceil$. This is the construction used by Dudek, Katona and Wojda in [7] to produce MNT graphs of order n and size $\lceil \frac{3n-2}{2} \rceil$ for $n \geq 54$ as well as for $n \in \{22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$. This fact motivated us to search for cubic xy–MNT graphs of relevant orders. We used cubic graphs from Gordon Royle's Collection of small graphs (cf. [11]) and found xy–MNT graphs for orders 16, 18 and 22 (see Figure 8), but none of order 14. We also found xy–MNT graphs of orders 24, 26 and 30 (see Figure 9) from Brendan McKay's collection of cubic hypohamiltonian graphs (cf. [10]). We label these graphs Q_i, where i is the order of the graph.



(a) Cubic graph Q_{16}

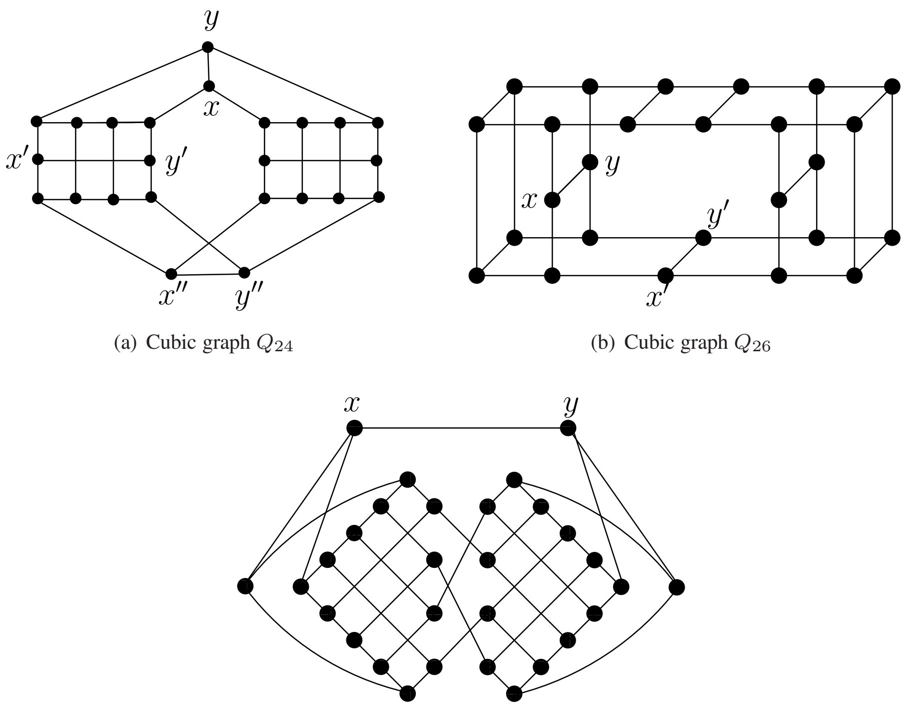


(b) Cubic graph Q_{18}



(c) Cubic graph Q_{22}

Fig. 8: Cubic xy–MNT graphs of order 16, 18 and 22



(a) Cubic graph Q_{24}

(b) Cubic graph Q_{26}

(c) Cubic graph Q_{30}

Fig. 9: Cubic hypohamiltonian xy–MNT graphs of order 24, 26 and 30

Remark 3.6 *We note the following:*

(a) *The graphs Q_i, $i = 16, 18, 22, 24, 26, 30$ all have $N(x) \cap N(y) = \emptyset$ and neither x nor y is a universal vertex of Q_i.*

(b) *The graphs Q_{16}, Q_{18} are hamiltonian. This is to be expected since the smallest MNH graphs of orders 16 and 18 listed in [9] have 25 and 28 edges respectively.*

(c) *In Q_{30}, x and y can be any two adjacent vertices of the graph and in Q_{22}, Q_{24} and Q_{26}, where applicable, x, y can be x', y' or x'', y''.*

(d) *There are no cubic xy–MNT graphs of order 14.*

4 Main Results

In this section we determine $g(n)$ for $n = 10, 11, 14-21, 24-29, 32, 33$.

Theorem 4.1 *For $n \in \{18, 19, 20, 21, 24, 25, 26, 27, 28, 29, 32, 33\}$, $g(n) = \lceil \frac{3n-2}{2} \rceil$.*

Proof: Let $F = Q_i$ for $i = 16, 18, 22, 24, 26, 30$, $H_x = K_1$ and $H_y = K_1$ or $H_y = K_2$. The result then follows from Theorems 2.3 and 3.4 and Remark 3.6(a). $\square$

We now prove that $g(n) = \lceil \frac{3n}{2} \rceil$ for $n = 10, 11, 14, 15, 16, 17$.

We prove $g(n) = \lceil \frac{3n}{2} \rceil$ for $n = 10$ as a separate theorem using "analytic" arguments. Alternatively, this result can be obtained by using computer results. The cases $n = 11, 14, 15, 16$ and 17 are proved using computer results.

Theorem 4.2 *The minimum size of an MNT graph of order 10 is 15.*

Proof: According to Theorem 2.3, $g(10) \geq 14$. Suppose G is an MNT graph of order 10 and size 14. We define M, S, m and s as in Theorem 2.3. It follows from Theorem 2.2 that $s \neq 0$. We consider the various cases given in the proof of Theorem 2.3.

Case 1 ($\langle S \rangle$ has 3 components)

The smallest graph of order 10 for this case is the Zelinka type II graph depicted in Figure 10 which has 15 edges.

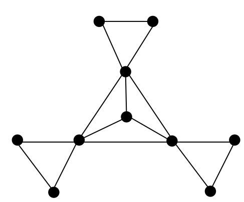


Fig. 10: Zelinka type II graph of order 10

Case 2 ($\langle S \rangle$ has 2 components, H_1 and H_2)

Case 2.1 $N_G(H_1) = N_G(H_2)$

According to Remark 2.4(b) we need not consider this case.

Case 2.2 $N_G(H_1) \neq N_G(H_2)$

Case 2.2.1 $G - S$ is hamiltonian

From inequality (1) of Section 2 it follows that the only values of m and s that may produce an MNT graph G of order 10 and 14 edges are $m = 0$ and $s = 2$, i. e. $G - S$ has no vertices of degree 2. Hence $G - S$ is a cubic xy–MNT graph of order 8. We show that no such graph $G - S$ exists.

The subgraphs G_s of G can have one of the three structures depicted in Figure 11.

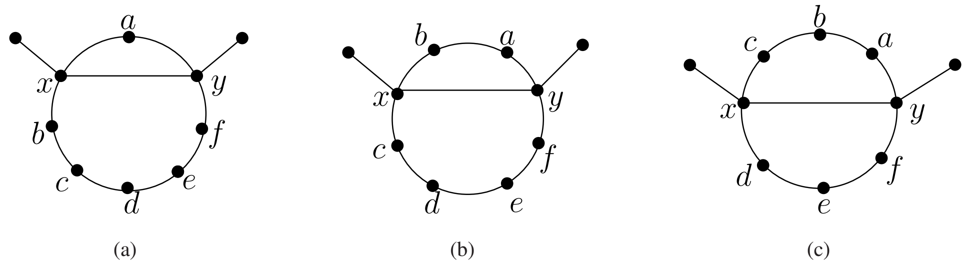


Fig. 11: Structures of G_s

We prove the subcase for G_s shown in Figure 11(b). The proofs of the other subcases are similar.

Firstly $ac, bf \notin E(G)$, otherwise $G - S$ is xy–traceable. Due to symmetry all subcases follow from the following three cases.

(i) Suppose $ad \in E(G)$. Then $bc, be \notin E(G)$. Hence $G - S$ is not cubic.
(ii) Suppose $ae \in E(G)$. Then $bd \notin E(G)$. If $bc, df \in E(G)$, then $G - S$ is xy–traceable.
(iii) Suppose $af \in E(G)$. Then $bc, be \notin E(G)$. If $bd, ce \in E(G)$, then $G - S$ is xy–traceable.

Thus there is no MNT graph G with 14 edges having G_s as subgraph.

Case 2.2.2 $G - S$ is MNH

From inequality (2) of Section 2 it follows that the only value of s that may produce an MNT graph G of order 10 and 14 edges is $s = 2$.

According to [9] the smallest MNH graph of order 8 has 15 edges and thus $e(G) > 14$.

Case 3 ($\langle S\rangle$ has 1 component, H)

According to inequality (3) of Section 2 the only possibility for a graph of order 10 to have 14 edges is when $m = 0$, i. e. when $d_G(v) \neq 2$ for all $v \in V(G) - S$.
For $s = 1$, $e(G) = 14 \iff d_G(v) = 3$ for all $v \in V(G) - S$.
For $s = 2$, taking Lemma 2.1 into account, we have

$$\begin{aligned} e(G) &= \frac{1}{2}\left(\sum_{v\in S} d_G(v) + \sum_{v\in V(G)-S} d_G(v)\right) \\ &\geq \frac{1}{2}\left(4 + 4 + 6(3) + 4\right) = 15. \end{aligned}$$

Thus we only consider $s = 1$, i. e. $H \cong K_1$. Then, according to Lemma 2.7, $G - S$ is 2–connected.

Let z denote the vertex adjacent to H in G, and $F = G - S$. Then $d_F(z) = 2$ and the other 8 vertices of F must have degree 3 each. Note that F is not traceable from z, otherwise G will be traceable. We show that such a graph G does not exist by attempting to construct F by considering circumference cycles of F.

Let C be a circumference cycle of F. We define, respectively, the degree deficiency, $dd(C)$, of C, and the degree deficiency, $dd(F - V(C))$, of $F - V(C)$ in F as

$$dd(C) = \sum_{v\in V(C)} d_F(v) - 2|V(C)|$$

and

$$dd(F - V(C)) = \sum_{v \in V(F-V(C))} d_F(v) - \sum_{v \in V(F-V(C))} d_{F-V(C)}(v).$$

Since F is connected $dd(F - V(C)) > 0$ and thus

$$dd(C) \geq dd(F - V(C)) > 0.$$

Since F is not traceable from z it follows that $c(F) \leq 8$. Suppose C is a circumference cycle of F of order $k \leq 8$. We denote the vertices on C in an anti-clockwise direction by $x_0, x_1, ..., x_{k-1}$, where $x_0 = z$ when $z \in V(C)$. We denote the vertices of $F - V(C)$ by $v_0, v_1, ..., v_{8-k}$, where $v_0 = z$ when $z \in V(F - V(C))$.

Since similar techniques are used in proving the cases $c(F) = 8$ and $c(F) = 7$ as in the case $c(F) = 6$, we only prove the following:

***c*(*F*) = 6**

(a) $z \in V(C)$

Now $dd(C) = 5$ and the only graphs $\langle v_0, v_1, v_2 \rangle$ with $0 < dd(\langle v_0, v_1, v_2 \rangle) \leq 5$ are P_3 and K_3. Each of P_3 and K_3 have vertices $v_i, v_j; i \neq j; i, j = 0, 1, 2$ such that $d_{F-V(C)}(v_i) < 3$ and $d_{F-V(C)}(v_j) < 3$ and v_i and v_j are endvertices of a path of order 3. Since, according to Lemma 2.7, F is 2–connected it follows that v_i and v_j have distinct neighbours on C which contradicts Lemma 2.6. Hence $d_F(v_i) < 3$ and $d_F(v_j) < 3$, a contradiction.

(b) $z \in F - V(C)$

Now $dd(C) = 6$ and the only graphs $\langle z, v_1, v_2 \rangle$ with $0 < dd(\langle z, v_1, v_2 \rangle) \leq 6$ are $K_1 \cup K_2$, P_3 and K_3. Using reasoning similar to that in (a) it can be seen that $\langle z, v_1, v_2 \rangle$ cannot be isomorphic to P_3 or K_3. Now consider $K_1 \cup K_2$. Let $V(K_1) = w_1$ and $V(K_2) = \{w_2, w_3\}$, where $z = w_1$ or $z = w_2$. Then $d_F(w_3) = 3$. Then, according to Lemma 2.5 we may assume $w_2x_0, w_3x_3 \in E(F)$, but then $w_3x_i \notin E(F)$ for $i = 1, 2, 4, 5$ and hence $d_F(w_3) < 3$.

***c*(*F*) at most 5**

Clearly at least one component of $F - V(C)$, say A, has at least two vertices. Then there are two distinct vertices in A which, since F is 2–connected, have distinct neighbours on C. This contradicts Lemma 2.6.

Thus $g(10) \geq 15$ and since there exist MNT graphs of order 10 and size 15 (for example, the graph in Figure 10 and $[F_8, K_1, K_1]$ which was first reported in [3]) it follows that $g(10) = 15$. □

In the proof of Theorem 4.5 we will consider degree sequences of graphs. Since all values in the degree sequences that we consider are less than 10, we will use, for example, the notation 123^44 for a graph having degrees $1, 2, 3, 3, 3, 3, 4$. We use a computer search using the same algorithm as described in Section 3, but with a restriction on the degree sequence of a specific path, where we define the degree sequence of a path $P = v_1v_2 \ldots v_n$ in G as $d_G(v_1), d_G(v_2), \ldots, d_G(v_n)$. These extra restrictions enabled the relevant searches to be completed in a reasonable time. The following two lemmas describe the paths used for different cases in the computer algorithms.

Lemma 4.3 *Suppose F is an xy–MNT graph with $n \geq 6$ and neither x nor y is a universal vertex of F. Then the following hold.*

(a) *There is a Hamilton path xyP in F.*

(b) *Let $u \in V(F) - \{x, y\}$. If $x \notin N(u)$, then F has a Hamilton path $xyPu$. If $x \in N(u)$, then F has a Hamilton path $xyPuQ$.*

(c) *If $w \in V(F) - \{x, y\}$ has degree 2 and $N(w) = \{v_1, v_2\}$ with $d(v_i) = 4$ for $i = 1, 2$, then F has a Hamilton path $xyPv_iwv_j$ or $yxPv_iwv_j$, where $\{i, j\} = \{1, 2\}$.*

Proof:

(a) and (b) follow directly from Lemma 3.1.

(c) From Lemmas 3.2(a) and (b) it follows that v_1 and v_2 are distinct from x and y, and that not both are adjacent to x or y. If x and v_j, $j \in \{1, 2\}$ are nonadjacent, then F has a Hamilton path $xyPv_iwv_j$, where $\{i, j\} = \{1, 2\}$. Similarly, if y and v_j, $j \in \{1, 2\}$ are nonadjacent, then F has a Hamilton path $yxPv_iwv_j$, where $\{i, j\} = \{1, 2\}$.

□

Lemma 4.4 *Suppose F is a z–MNT graph with $n \geq 5$ and z is not a universal vertex of F.*

(a) *There is a Hamilton path P in $F - z$.*

(b) *Let $u \in V(F) - \{z\}$. If $z \notin N(u)$, then $F - z$ has a Hamilton path uP. If $z \in N(u)$, then $F - z$ has a Hamilton path PuQ.*

(c) *If $w \in V(F) - \{z\}$ has degree 2 and $N(w) = \{v_1, v_2\}$ with $d(v_i) = 4$ for $i = 1, 2$, then $F - z$ has a Hamilton path Pv_iwv_j, where $\{i, j\} = \{1, 2\}$.*

Proof:

(a) and (b) are obvious.

(c) From Lemma 3.3(b) v_1 and v_2 are not both adjacent to z. Say v_j is not adjacent to z for $j \in \{1, 2\}$. Then $F + zv_j$ must be z–traceable. Therefore there is a Hamilton path zv_jwv_iP in F. (Note the path cannot end in w, otherwise $F - z$ is hamiltonian, making F z–traceable.) Therefore $F - z$ has a Hamilton path Pv_iwv_j, where $\{i, j\} = \{1, 2\}$.

□

Theorem 4.5 *For $n = 11, 14, 15, 16$ and 17, $g(n) = \lceil \frac{3n}{2} \rceil$.*

Proof: According to Theorem 2.3, $g(n) \geq \lceil \frac{3n-2}{2} \rceil$ for $n \geq 10$. In order to show that there are no MNT graphs of order n and size $\lceil \frac{3n-2}{2} \rceil$ for $n = 11, 14, 15, 16$ and 17, it follows from Theorem 2.2 and Remarks 2.4(a) and (b), that we need only consider Cases 2.2 and 3 of Theorem 2.3. This implies from Remark 2.4(c) that we need only consider MNT graphs constructed from xy–MNT and z–MNT graphs. In some cases we are able to use Table 1 to determine the minimum size of an MNT graph of a specific order constructed from an xy–MNT graph and/or a z–MNT graph of minimum size. If not, we consider values of m and s that could possibly produce MNT graphs G of order n with $\lceil \frac{3n-2}{2} \rceil$ edges and determine possible degree sequences for such graphs. In most cases, for the sake of brevity, we will only state the degree sequence(s) of G and the relevant lemma(s) used for the computation. When determining the degree sequences for G, we use Lemmas 2.1, 3.2(a) and 3.3(a), and the fact that $d(v) \geq 3$ for $v \in V(G) - S - M$. In the case where $G - S$ is a hamiltonian xy–MNT graph we use the fact that $d_{G-S}(x) \geq 3$ and $d_{G-S}(y) \geq 3$ (see Case 2.2.1 of Theorem 2.3). For all cases, no graphs with the relevant degree sequences were found when doing a computer search and hence we do not state this in every case.

Case I: $n = 11$

According to Table 1 the smallest MNT graphs of order 11 which can be constructed from xy–MNT or z–MNT graphs have 17 edges. Hence $g(11) = 17$. The graphs $[F_8, K_1, K_2]$ and $[F_9, K_1, K_1]$ are examples of such graphs.

Case II: $n = 14$

From Table 1 it follows that the smallest MNT graph of order 14 that can be constructed from an xy–MNT graph has 21 edges.

Suppose $G = [F, H_z]$ is an MNT graph of order 14 and size 20. Then from inequality (3) of Section 2 we can only have $m = 0$. We consider subcases according to possible values of s.

(i) $m = 0$ and $s = 1$: The only possible degree sequence is 13^{13} and we use Lemma 4.4(a) in our computation.

(ii) $m = 0$ and $s = 2$: It follows from Table 1 that the minimum number of edges is 21.

We have shown that $g(14) = 21$. The graphs $[F_{12}, K_1, K_1]$, $[F_{12}^*, K_1, K_1]$, $[F_{11}, K_1, K_2]$ and $[F_{10}, K_2, K_2]$ are examples of MNT graphs of order 14 and size 21.

Case III: $n = 15$

From Table 1 it follows that the smallest MNT graph of order 15 that can be constructed from an xy–MNT graph has 23 edges.

Suppose $G = [F, H_z]$ is an MNT graph of order 15 and size 22, where F is a z–MNT graph. Then from inequality (3) of Section 2 we have $m \leq 1$. Table 2 summarizes the possible subcases.

m	s	degree sequence	Lemma used for computer search
0	1	$13^{13}4$	4.4(b) if $d_G(z) = 3$ with $d(u) = 4$; 4.4(a) if $d_G(z) = 4$
0	2	$223^{12}4$	4.4(a) [Same as Case II(i)]
1	1	$123^{11}44$	4.4(c)
1	2	None	

Tab. 2: Subcases for $n = 15$, where $G - S$ is z–MNT

We have thus shown that $g(15) = 23$. The graphs $[F_{13}, K_1, K_1]$, $[F_{13}*, K_1, K_1]$, $[F_{12}, K_1, K_2]$, $[F_{12}*, K_1, K_2]$ and $[F_{11}, K_2, K_2]$ are examples of MNT graphs of order 15 and size 23.

Case IV: $n = 16$

We show that no MNT graph G of order 16 and size 23 exists.

We first consider the case where $G - S$ is an xy–MNT graph and $G - S$ is MNH. The smallest such MNT graph of order 16 has G_{14} as subgraph and has 24 edges. We now assume $G - S$ is hamiltonian. It follows from inequality (1) of Section 2 that $m + s \leq 2$. Thus $m = 0$ and $s = 2$, and the only possible degree sequence is $113^{12}44$ since $d_{G-S}(x) \geq 3$ and $d_{G-S}(y) \geq 3$. Hence $G-S$ must be cubic. Applying Lemma 4.3(a) and a computer search, no graph was found.

We next consider the case where $G-S$ is a z–MNT graph. If such a graph exists then from inequality (3) of Section 2 we have $m=0$. The possible subcases are:

(i) $m=0$ and $s=1$: The degree sequence is 13^{15} and we use Lemma 4.4(a).

(ii) $m=0$ and $s=2$: No degree sequence for G is possible.

We have thus shown that $g(16)=24$ and that $[G_{14}, K_1, K_1]$ is an MNT graph with order 16 and size 24 (see Figure 7).

Case V: $n=17$

We show that no MNT graph G of order 17 and size 25 exists. We first consider the case where $G-S$ is an xy–MNT graph and $G-S$ is MNH. The smallest such MNT graphs of order 17 have either G_{14} or G_{15} as subgraphs and have 26 edges. We thus need to consider the case where $G-S$ is a hamiltonian xy–MNT graph in which case $m+s \leq 3$ (inequality (1)); and the case where $G-S$ is a z–MNT graph in which case $m \leq 1$ (inequality (3)). Table 3 summarizes all possible subcases.

$G-S$	m	s	degree sequence	Lemma used for computation
hamiltonian	0	2	$113^{13}45$ or	4.3(a)
xy–MNT	0	2	$113^{12}444$	4.3(b) with $d(u)=4$
	0	3	$1223^{12}45$	4.3(a)
	1	2	$1123^{10}4444$	4.3(c)
z–MNT	0	1	$13^{15}4$	4.4(b) if $d_G(z)=3$ with $d(u)=4$; 4.4(a) if $d_G(z)=4$
	0	2	$223^{14}4$	4.4(a) [Same as Case IV(i)]
	1	1	$123^{13}44$	4.4(c)
	1	2	None	

Tab. 3: Subcases for $n=17$, where $G-S$ is hamiltonian xy–MNT or z–MNT

Hence $g(17)=26$ and $[G_{14}, K_1, K_2]$ and $[G_{15}, K_1, K_1]$ are examples of MNT graphs of order 17 and size 26. $\square$

Obviously, there can be some doubt about the correctness of an algorithm when no graphs are found. The fact that MNH graphs of certain orders that were found in [9] were also found with this algorithm is reassuring. Furthermore, since the algorithm does not take into account isomorphisms of graphs, it is unlikely that a graph would be missed since the algorithm has usually more than one chance to find a graph if it exists. The source code for the algorithm can be viewed at [4].

We summarize the results contained in [8] and in Theorems 4.1, 4.2 and 4.5 in the following theorem:

Theorem 4.6 *Let $g(n)$ denote the minimum number of edges of an MNT graph of order n. Then*

$$g(n) = \begin{cases} \lceil \frac{3n}{2} \rceil & \text{for } n \in J = \{10, 11, 14, 15, 16, 17\} \\ \lceil \frac{3n-2}{2} \rceil & \text{for } n \geq 10, n \notin J \cup K \text{ where } K = \{34, 35, 36, 37, 44, 45, 52, 53\}. \end{cases}$$

5 Conclusion

The smallest size of an MNT graph of order n has now been determined for all except 8 values of n, namely $n \in K = \{34, 35, 36, 37, 44, 45, 52, 53\}$. The minimum size of MNT graphs of order $n \geq 18$, where $n \notin K$, is $\lceil \frac{3n-2}{2} \rceil$ and graphs of this size are constructed from cubic xy–MNT graphs. We suspect that $g(n) = \lceil \frac{3n-2}{2} \rceil$ also for $n \in K$ and that these remaining cases could be resolved by finding cubic xy–MNT graphs of orders 32, 34, 42 and 50 either by construction or by the use of more sophisticated algorithms.

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