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# *The Erdős-Sós Conjecture for Geometric Graphs*[†]

Luis Barba[1,2] Ruy Fabila-Monroy[3‡] Dolores Lara[4†] Jesús Leaños[5]
Cynthia Rodríguez[6,7] Gelasio Salazar[8§] Francisco J. Zaragoza[6]

[1] *Boursier FRIA du FNRS, Département d'Informatique, ULB, Belgium*
[2] *School of Computer Science, Carleton University, Canada*
[3] *Departamento de Matemáticas, CINVESTAV, Mexico*
[4] *Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Barcelona, Spain.*
[5] *Unidad Académica de Matemáticas, UAZ. Mexico*
[6] *Departamento de Sistemas, UAM Azcapotzalco, Mexico*
[7] *Department of Combinatorics and Optimization, University of Waterloo, Canada*
[8] *Instituto de Física, UASLP, Mexico*

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Let $f(n,k)$ be the minimum number of edges that must be removed from some complete geometric graph $G$ on $n$ points, so that there exists a tree on $k$ vertices that is no longer a planar subgraph of $G$. In this paper we show that $\left(\frac{1}{2}\right)\frac{n^2}{k-1} - \frac{n}{2} \leq f(n,k) \leq 2\frac{n(n-2)}{k-2}$. For the case when $k=n$, we show that $2 \leq f(n,n) \leq 3$. For the case when $k=n$ and $G$ is a geometric graph on a set of points in convex position, we completely solve the problem and prove that at least three edges must be removed.

**Keywords:** extremal graph theory, geometric graph, spanning tree

## 1 Introduction

One of the most notorious problems in extremal graph theory is the Erdős-Sós Conjecture, which states that every simple graph with average degree greater than $k-2$ contains every tree on $k$ vertices as a subgraph. This conjecture was recently proved true for all sufficiently large $k$ (unpublished work of Ajtai, Komlós, Simonovits, and Szemerédi).

In this paper we investigate a variation of this conjecture in the setting of geometric graphs. Recall that a *geometric graph* $G$ consists of a set $S$ of points in the plane (these are the vertices of $G$), plus a set of straight line segments, each of which joins two points in $S$ (these are the edges of $G$). In particular, any set $S$ of points in the plane in *general position* (no three of its points are collinear) naturally induces a complete geometric graph. For brevity, we often refer to the edges of this graph simply as edges *of* $S$. If

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$S$ is in convex position then $G$ is a *convex geometric graph*. A geometric graph is *planar* if no two of its edges cross each other. An *embedding* of an abstract graph $H$ into a geometric graph $G$ is an isomorphism from $H$ to a planar geometric subgraph of $G$. For $r \geq 0$, an $r$*-edge* is an edge of $G$ such that in one of the two open semi-planes defined by the line containing it, there are exactly $r$ points of $G$. The convex hull of $S$ is the intersection of all convex sets containing $S$. We will frequently need to refer to the vertices and edges at the boundary of the convex hull of $S$, which for brevity we will denote simply as *convex hull vertices* and *convex hull edges* of $S$.

In this paper all point sets are in general position and $G$ is a complete geometric graph on $n$ points. It is well known that for every integer $1 \leq k \leq n$, $G$ contains every tree on $k$ vertices as a planar subgraph [3]. Even more, it is possible to embed any such tree into $G$, when the image of a given vertex is prespecified [5].

Let $F$ be a subset of edges of $G$, which we call *forbidden edges*. If $T$ is a tree for which every embedding into $G$ uses an edge of $F$, then we say that $F$ *forbids* $T$. In this paper we study the question of what is the minimum size of $F$ so that there is a tree on $k$ vertices that is forbidden by $F$. Let $f(n,k)$ be the minimum of this number taken over all complete geometric graphs on $n$ points. As $f(2,2)=1$, $f(3,3)=2$, $f(4,3)=3$, $f(4,4)=2$ and $f(n,2)=\binom{n}{2}$, we assume through out the paper that $n \geq 5$ and $k \geq 3$.

We show the following bounds on $f(n,k)$.

**Theorem 1.1**
$$\left(\frac{1}{2}\right)\frac{n^2}{k-1} - \frac{n}{2} \leq f(n,k) \leq 2\frac{n(n-2)}{k-2}$$

**Theorem 1.2**
$$2 \leq f(n,n) \leq 3$$

In the case when $G$ is a convex complete geometric graph, we show that the minimum number of edges needed to forbid a tree on $n$ vertices is three.

An equivalent formulation of the problem studied in this paper is to ask how many edges must be removed from $G$ so that it no longer contains *every* planar subtree on $k$ vertices. A related problem is to ask how many edges must be removed from $G$ so that it no longer contains *any* planar subtree on $k$ vertices. For the case of $k=n$, in [6], it is proved that if any $n-2$ edges are removed from $G$, it still contains a planar spanning subtree. Note that if the $n-1$ edges incident to any vertex of $G$ are removed, then $G$ no longer contains a spanning subtree. In general, for $2 \leq k \leq n-1$, in [1], it is proved that if any set of $\left\lceil \frac{n(n-k+1)}{2} \right\rceil - 1$ edges are removed from $G$, it still contains a planar subtree on $k$ vertices. In the same paper it is also shown that this bound is tight—a geometric graph on $n$ vertices and a subset of $\left\lceil \frac{n(n-k+1)}{2} \right\rceil$ of its edges are shown, so that when these edges are removed, every planar subtree has at most $k-1$ vertices. In [4] the authors study the seemingly unrelated (non-geometric) problem of packing two trees into planar(i) (abstract) graphs. That is, given two trees on $n$ vertices, the authors consider the question of when it is possible to find a planar graph having both of them as spanning trees and in which the trees are edge disjoint. However, although theirs is a combinatorial question rather than

(i) A *planar* (abstract) graph is an *abstract* graph that can be embedded in the plane; the embedding may not be unique.

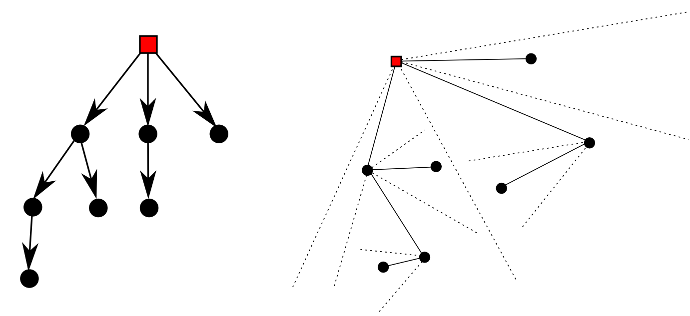

**Fig. 1:** An embedding of a tree using the algorithm.

geometric, their Theorem 2.1 implies our Lemma 2.2. We provide a self contained proof of Lemma 2.2 for completeness.

A previous version of this paper appeared in the conference proceedings of EUROCG'12 [2].

## 2 Spanning Trees

In this section we consider the case when $k = n$. Let $T$ be a tree on $n$ vertices. Consider the following algorithm to embed $T$ into $G$. Choose a vertex $v$ of $T$ and root $T$ at $v$. For every vertex of $T$ choose an arbitrary order of its children. Suppose that the neighbors of $v$ are $u_1, \dots, u_m$, and let $n_1, \dots, n_m$ be the number of nodes in their corresponding subtrees. Choose a convex hull point $p$ of $G$ and embed $v$ into $p$. Sort the remaining points of $G$ counter-clockwise by angle around $p$. A *wedge* is a region of the plane bounded by two infinite rays sharing a common apex. Choose $m+1$ rays centered at $p$ so that the wedge between two consecutive rays is convex and between the $i$-th ray and the $(i+1)$-th ray there are exactly $n_i$ points of $G$. Let $S_i$ be this set of points. A convex hull vertex $q$ of $S_i$ is *visible* from $p$ if the line segment with endpoints $p$ and $q$ intersects the convex hull of $S$ only at $q$. For each $u_i$ choose a convex hull vertex of $S_i$ visible from $p$ and embed $u_i$ into this point. Recursively embed the subtrees rooted at each $u_i$ into $S_i$. Note that this algorithm provides an embedding of $T$ into $G$. We will use this embedding frequently throughout the paper. See Figure 1.

For every integer $n \geq 2$ we define a tree $T_n$ as follows: If $n = 2$, then $T_n$ consists of only one edge; if $n$ is odd, then $T_n$ is constructed by subdividing once every edge of a star on $\frac{n+1}{2}$ vertices; if $n$ is even and greater than 2, then $T_n$ is constructed by subdividing an edge of $T_{n-1}$. See Figure 2.

We prove the lower bound of $f(n,n) \geq 2$ of Theorem 1.2.

**Theorem 2.1** *If $G$ has only one forbidden edge, then any tree on $n$ vertices can be embedded into $G$, without using the forbidden edge.*

**Proof:** Let $e$ be the forbidden edge of $G$. Let $T$ be a tree on $n$ vertices. Choose a root for $T$. Sort the children of each node of $T$, by increasing size of their corresponding subtrees. Embed $T$ into $G$ with the embedding algorithm, choosing at all times the *rightmost* point (the first point when sorting clockwise around the root) as the root of the next subtree. Suppose that $e$ is used in this embedding. Let $e := pq$ so that $u$ is embedded into $p$ and $v$ is embedded into $q$ (note that $u$ is the parent of $v$ in $T$).

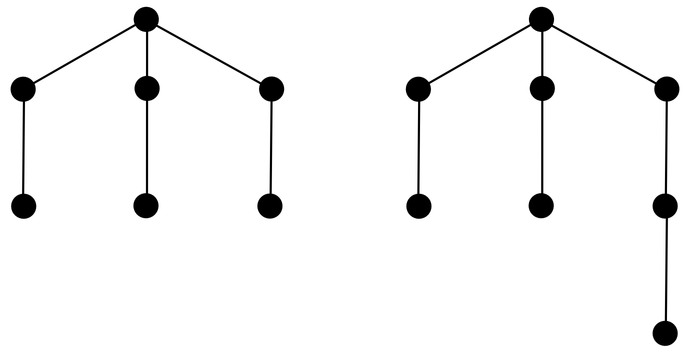

**Fig. 2:** $T_7$ and $T_8$.

Suppose that the subtree rooted at $v$ has $m \geq 2$ nodes. In the algorithm, we embedded this subtree into a set of exactly $m$ points. We chose a convex hull point $(q)$, of this set visible from $p$ to embed $v$. In this case we may choose another convex hull point visible from $p$ to embed $v$ and continue with the algorithm. Note that $pq$ is no longer used in the final embedding.

Suppose that $v$ is a leaf, and that $v$ has a sibling $v'$ whose subtree has at least two nodes. Then we may interchange $v$ and $v'$ in the order of the children of $u$, so that $e$ is no longer used in the embedding, or if it is, then $v'$ is embedded into $q$, but then we proceed as above.

Suppose that $v$ is a leaf, has at least one sibling and all its siblings are leaves. The subtree rooted at $u$ is a star. We choose a point distinct from $p$ and $q$ in the point set where this subtree is embedded, and embed $u$ into this point. Afterward we join it to the remaining points. This produces an embedding that avoids $e$.

Assume then, that $v$ is a leaf and that it has no siblings. We distinguish the following cases:

1. $u$ **has no siblings.** In this case, the subtree rooted at the parent of $u$ is a path of length two. If $u$ has no grandparent then $n = 3$ and $T$ can be trivially embedded into $G$ without using $e$. Suppose $u$ has a grandparent. In this case there are only four vertices to consider: $v$, $u$, the parent of $u$ and the grandparent of $u$. We keep the current location of the grandparent of $u$, and change the points into which the remaining vertices are embedded. This can always be done so that $e$ is not used in the embedding. All possible cases are shown in Figure 3.

2. $u$ **has a sibling** $u'$ **whose subtree is not an edge.** We may change the order of the siblings of $u$, with respect to their parent, so that the subtree rooted at $u'$ will be embedded into the point set containing $p$ and $q$. In the initial order—increasing by size of their corresponding subtrees—$u'$ is after $u$. We may assume that in the new ordering, the order of the siblings of $u$ before it, stays the same. Therefore $p$ is the rightmost point of the set into which the subtree rooted at $u'$ will be embedded. Embed $u'$ into $p$. Either we find an embedding not using $e$, or this embedding falls into one of the cases considered before.

3. $u$ **has at least one sibling, and the subtree at every sibling of** $u$ **is an edge.**

   Suppose that $u$ has no grandparent; then $T$ is equal to $T_n$ and $n$ is odd. Let $w$ be the parent of $u$. Embed $w$ into $p$. Let $p_1, \ldots, p_{n-1}$ be the points of $G$ different from $p$ sorted counter-clockwise

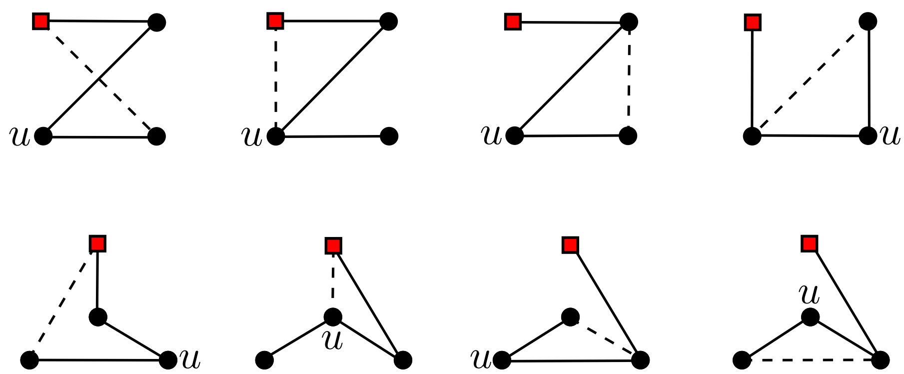

**Fig. 3:** The embedding of a path of length three. The grandparent of $u$ is highlighted and the forbidden edge is dashed.

by angle around $p$; choose $p_1$ so that the angle between two consecutive points is less than $\pi$. Let $u_1, \ldots, u_{(n-1)/2}$ be the neighbors of $w$. Embed each $u_i$ into $p_{2i-1}$ and its child into $p_{2i}$. If $q$ equals $p_{2j-1}$ for some $j$ then embed $u_j$ into $p_{2j}$ and its child into $p_{2j-1}$. This embedding avoids $e$.

Suppose that $w$ is the grandparent of $u$ and let $p'$ be the point into which $w$ is embedded. Let $S$ be the point set into which the subtree rooted at the parent of $u$ is embedded. Note that $S$ has an odd number of points. We replace the embedding as follows. Sort $S$ counter-clockwise by angle around $p'$. Call a point *even* if it has an even number of points before it in this ordering. Call a point *odd* if it has an odd number of points before it in this ordering. If $e$ is incident to an odd point, then we embed the parent of $u$ into this point. The remaining subtree rooted at $u$ can be embedded without using $e$. If the endpoints of $e$ are both even, between them there is an odd point. We embed the parent of $u$ into this point. The remaining vertices can be embedded without using $e$ (see Figure 4).

□

The upper bound of $f(n, n) \leq 3$ of Theorem 1.2 follows directly from Lemma 3.2. Now we prove in Lemma 2.2 and Theorem 2.3, that if $G$ is a convex geometric graph, at least three edges are needed to forbid some tree on $n$ vertices.

**Lemma 2.2** *Let $T$ be a tree on $n$ vertices. If $G$ is a convex geometric graph, then $T$ can be embedded into $G$ using less than $\frac{n}{2}$ convex hull edges of $G$.*

**Proof:** If $T$ is a star, then any embedding of $T$ into $G$ uses only two convex hull edges. If $T$ is a path then it can be embedded into $G$ using at most two convex hull edges. Therefore, we may assume that $T$ is neither a star nor a path.

Since $T$ is not a path, it has a vertex of degree at least three. Choose this vertex as the root. Since $T$ is not a star, the root has a child whose subtree has at least two nodes. Order the children of $T$ so that this node is first. Embed $T$ into $G$ with the embedding algorithm.

Let $u$ and $v$ be vertices of $T$, so that $u$ is the parent of $v$. Suppose that the subtree rooted at $v$ has at least two nodes. Then in the embedding algorithm we have at least two choices to embed $v$ once the ordering

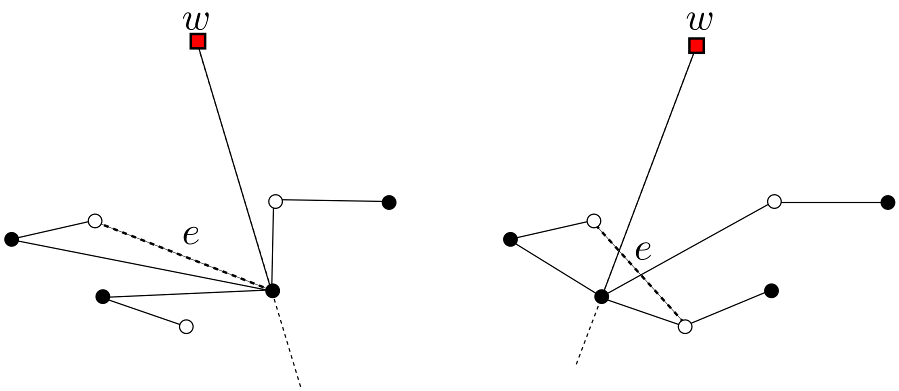


**Fig. 4:** The two sub-cases, when $u$ has a grandparent $w$, and all the subtrees of its children are edges. Odd points are painted in black and even points in white. The forbidden edges are dashed.

of the children of $u$ has been chosen. At least one of the choices is such that $uv$ is not embedded into a convex hull edge. Therefore, we may assume that the embedding is such that each convex hull edge used, is incident to a leaf.

Note that every vertex of $T$, distinct from the root, is incident to at most one convex hull edge in the embedding. Since the first child of the root is not a leaf, no convex hull edge is used to embed this child. Only in the embedding of the last child of the root a convex hull may have been used. Therefore every vertex of $T$ is incident to at most one convex hull edge. Thus the set of convex hull edges used in the embedding is a matching. Therefore at most $n/2$ convex hull edges are used in the embedding.

Suppose that exactly $n/2$ convex hull edges are used. One of these edges must be incident to the root. Since the root was chosen of degree at least three it has a child which is not a leaf nor the first child; we place this vertex last in the ordering of the children of the root. The leaf adjacent to the root can no longer be a convex hull edge and the embedding uses less than $n/2$ convex hull edges. □

**Theorem 2.3** *If $G$ is a convex geometric graph and has at most two forbidden edges, then any tree on $n$ vertices can be embedded into $G$, without using a forbidden edge.*

**Proof:** Let $f_0$ be an embedding given by Lemma 2.2, of $T$ into $G$. For $0 \leq i \leq n$, let $f_i$ be the embedding produced by rotating $f_0$, $i$ places to the right. Assume that in each of these rotations at least one forbidden edge is used, as otherwise we are done. Let $e_1, \ldots, e_m$ be the edges of $T$ that are mapped to a forbidden edge in some rotation. Assume that the two forbidden edges are an $l$-edge and an $r$-edge respectively.

Suppose that $l \neq r$. Then, each edge of $T$ can be embedded into a forbidden edge at most once in all of the $n$ rotations. Thus $m \geq n$. This is a contradiction, since $T$ has $n-1$ edges.

Suppose that $l = r$. Then, each of the $e_i$ is mapped twice to a forbidden edge. Thus $m \geq n/2$. By Lemma 2.2, $f_0$ uses less than $n/2$ convex hull edges. Therefore, $l = r > 0$. But a set of $n/2$ or more $r$-edges, with $r > 0$, must contain a pair of edges that cross. And we are done, since $f_0$ is an embedding. □

# 3 Bounds on $f(n,k)$

In this section we prove Theorem 1.1. First we show the upper bound.

**Lemma 3.1** *If $T_n$ is embedded into $G_n$ then every edge incident to a leaf of $T_n$ must be embedded into a convex hull edge.*

**Proof:** Let $e := uv$ be an edge of $T_n$ incident to leaf. Suppose that $u$ is the leaf vertex. Then $v$ is of degree two. Suppose that $e$ is not embedded into a convex hull edge of $G$. Then $e$ divides $S \setminus \{u,v\}$ into two non-empty subsets $S_1$ and $S_2$, so that $S_1$ lies on the opposite side of $S_2$ with respect to $e$. Assume that the parent of $v$ is embedded into $S_1$. Then no vertex of $T_n$ can be embedded into $S_2$ without crossing $e$. Therefore $e$ must be a convex hull edge of $G$. □

**Lemma 3.2** *If $G$ is a convex geometric graph, then forbidding three consecutive convex hull edges of $G$ forbids the embedding of $T_n$.*

**Proof:** Recall that $T_n$ comes from subdividing a star, let $v$ be the non leaf vertex of this star. Let $p_1p_2, p_2p_3, p_3p_4$ be the forbidden edges, in clockwise order around the convex hull of $G$. Note that by Lemma 3.1, in any embedding of $T_n$ into $G$, an edge incident to a leaf of $T_n$, must be embedded into a convex hull edge. Neither the leaves of $T_n$ nor its neighbors can be embedded into $p_2$ or $p_3$, without using a forbidden edge. Thus, $v$ must be embedded into $p_2$ or $p_3$. Without loss of generality assume that $v$ is embedded into $p_2$. But then, the embedding must use $p_2p_3$ or $p_3p_4$ □

**Lemma 3.3** *If $G$ is a convex geometric graph, then forbidding the convex hull edges incident to any three vertices $p_1, p_2$ and $p_3$ of $G$, forbids the embedding of $T_n$.*

**Proof:** Note that by Lemma 3.1, neither a leaf of $T_n$, nor its neighbor can be embedded into $p_1$, $p_2$ or $p_3$, without using a forbidden edge. But at most two points do not fall into this category. □

**Lemma 3.4**
$$f(n,k) \le 2\frac{n(n-2)}{k-2}$$

**Proof:** Let $G$ be a complete convex geometric graph. We forbid every $r$-edge of $G$ for $r = 0, \dots, \left\lceil 2\frac{n-2}{k-2} - 2 \right\rceil$. Note that, in total we are forbidding at most $n\left(\left\lceil 2\frac{n-2}{k-2} - 2\right\rceil + 1\right) \le 2\frac{n(n-2)}{k-2}$ edges. As every subset of points of $G$ is in convex position, it suffices to show that every induced subgraph $H$ of $G$ on $k$ vertices is in one of the two configurations of Lemmas 3.2 and 3.3.

Assume then, that $H$ does not contain three consecutive forbidden edges in its convex hull nor three vertices, each with its two convex hull edges forbidden. $H$ has at most two (non-adjacent) pairs of consecutive forbidden edges in its convex hull. Therefore every forbidden edge of $H$ in its convex hull—with the exception of at most two—must be preceded by an $\ell$-edge (of $G$), with $\ell > \left\lceil 2\frac{n-2}{k-2} - 2\right\rceil$. The number of these $\ell$-edges contained in $H$ is at least $\frac{k-2}{2}$. The points separated by these edges amount to more than $\frac{k-2}{2}\left\lceil 2\frac{n-2}{k-2} - 2\right\rceil \ge n-k$ points of $G$. This is a contradiction, since together with the $k$ points of $H$ this is strictly more than $n$. □

Now, we show the lower bound of Theorem 1.1.

**Lemma 3.5** $$f(n,k) \geq \left(\frac{1}{2}\right)\frac{n^2}{k-1} - \frac{n}{2}$$

**Proof:** Let $F$ be a set of edges whose removal from $G$ forbids some $k$-tree. Let $H := G \setminus F$. Note that $H$ contains no complete $K_k$ as a subgraph, otherwise any $k$-tree can be embedded into this subgraph. By Turán's Theorem [7], $H$ cannot contain more than $\left(\frac{k-2}{k-1}\right)\frac{n^2}{2}$ edges. Thus $F$ must have size at least

$$\binom{n}{2} - \left(\frac{k-2}{k-1}\right)\frac{n^2}{2} = \left(\frac{1}{2}\right)\frac{n^2}{k-1} - \frac{n}{2}$$

□

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