Discrete Mathematics and Theoretical Computer Science DMTCS vol. **17:1**, 2015, 339–356

Maximum difference about the size of optimal identifying codes in graphs differing by one vertex

Mikko Pelto*

Department of Mathematics and Statistics, University of Turku, Finland

received 25th June 2013, revised 9th Feb. 2015, accepted 12th Feb. 2015.

Let $G = (V, E)$ be a simple undirected graph. We call any subset $C \subseteq V$ an identifying code if the sets $I(v) = \{c \in C \mid \{v, c\} \in E$ or $v = c\}$ are distinct and non-empty for all vertices $v \in V$. A graph is called twin-free if there is an identifying code in the graph. The identifying code with minimum size in a twin-free graph G is called the optimal identifying code and the size of such a code is denoted by $\gamma(G)$. Let G_S denote the induced subgraph of G where the vertex set $S \subset V$ is deleted. We provide a tight upper bound for $\gamma(G_S) - \gamma(G)$ when both graphs are twin-free and $|V|$ is large enough with respect to $|S|$. Moreover, we prove tight upper bound when G is a bipartite graph and $|S| = 1$.

Keywords: identifying codes, graph theory, twin-free graphs

1 Introduction

Let $G = (V, E)$ be an undirected graph where V is the vertex set and E the edge set. The *order* of the graph G is the number of vertices $|V|$ and it is denoted by n or n_G. The *neighbourhood* of a vertex v is

$$N(v, G) = \{v\} \cup \{u \in V \mid \{u, v\} \in E\}.$$

If the graph is clear from the context, we usually denote $N(v)$ instead of $N(v, G)$. Two vertices u and v are called *twins* if their neighbourhoods are the same, i.e. $N(u) = N(v)$. A graph is called *twin-free* if there are no twins, i.e. all sets $N(v)$ are distinct.

We call any $C \subseteq V$ a *code*. The vertices of C are called *codewords*. A code is called an *identifying code* if the sets

$$I(v, C) = \{c \in C \mid \{v, c\} \in E \text{ or } c = v\}$$

are distinct and non-empty for all vertices $v \in V$. Usually we denote these sets by $I(v)$ instead of $I(v, C)$ if the code C is clear from the context. Furthermore, if $I(v, C) \neq I(u, C)$ and $w \in (I(v, C) \setminus I(u, C)) \cup (I(u, C) \setminus I(v, C))$ we say that C or w *separates* v from u. If $w \in I(v, C)$, we say that v is *dominated* by w or C. Otherwise stated, C is an identifying code if all vertices are dominated and separated from each other by C.

*Email: mikko.pelto@utu.fi. Research supported by Finnish Academy of Science and Letters.

Identifying codes were introduced in the late 1990s by Karpovsky, Chakrabarty and Levitin [9]. The purpose of such codes is to create a safeguard analysis of a facility [11, 12] or a fault diagnosis of multiprocessor systems [9]. More papers on identifying codes and related topics can be found in the web bibliography [10]. It has been shown that there is an identifying code in a given graph if and only if the graph is twin-free [3, Remark 1]. Therefore, in this paper we restrict to twin-free graphs. In this case, the code which consists of all the vertices of a graph is always an identifying code. Moreover, if we add a vertex to an identifying code, the extended code is also an identifying code.

An interesting question among identifying codes is what the minimum size of any identifying code in a given graph is. A minimum sized identifying code is called *optimal*. The size of an optimal identifying code of G is denoted by $\gamma(G)$.

This paper is about how much the size of an optimal identifying code can increase if some vertices are deleted from the graph. In other words, we study the difference of $\gamma(G_S) - \gamma(G)$ where $G = (V, E)$ is any graph and

$$G_S = (V \setminus S, \{\{u, v\} \in E \mid u \notin S \text{ and } v \notin S\}),$$

i.e. the induced subgraph of G where the subset S of vertices is deleted. If $S = \{x\}$ contains only one vertex we typically refer to G_x instead of G_S. Moreover, we denote the order of graphs G_x and G_S by n_x and n_S respectively. Foucaud et al [6] proved that $\gamma(G_S) - \gamma(G) \geq -|S|$. Moreover, many such graphs as G and G_S which satisfy the equation $\gamma(G_S) - \gamma(G) = -|S|$ are known. On the other hand, Charon et al [2] showed that there are twin-free graphs G and G_x such as $\gamma(G_x) - \gamma(G) \approx \frac{n}{2} - \frac{3}{2}\log_2 n$. This in particular means that $\gamma(G)$ is not a monotone parameter with respect to the subgraph inclusion order.

In this paper, we prove the following inequalities of the difference between the sizes of optimal identifying codes in G and G_S:

$$\begin{aligned}
\gamma(G_x) - \gamma(G) &\leq \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor - 2 & \text{if } n \in \{2, 4, 5, 6, 8\} \\ \left\lfloor \frac{n}{2} \right\rfloor - 1 & \text{otherwise} \end{cases} \\
\gamma(G_S) - \gamma(G) &\leq n - 2|S| - \left\lfloor \frac{n - |S|}{2^{|S|}} \right\rfloor \qquad \text{if } n \geq 2^{|S|-1}.
\end{aligned}$$

Besides, we show that the first inequality is tight for all n and the second inequality is tight when n is large enough with respect to S. We also show that there are connected graphs G and G_x which attain this maximal value if n is odd and greater than 3. Instead, if n is even, then there are no such connected graphs except if $n = 6$ or $n = 8$. Finally, we improve the previous upper bound for the case when G is bipartite. We prove the bound

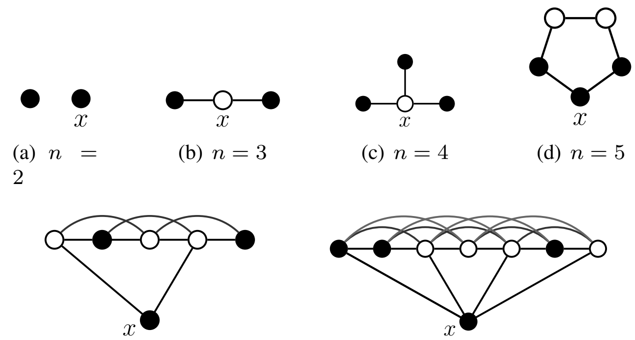
$$\gamma(G_x) - \gamma(G) \leq \left\lfloor \frac{n - \log_2(n - \log_2 n)}{2} \right\rfloor - 1$$

for such graphs and show that it is tight for all $n \geq 3$.

2 Constructions

Theorem 1. *For every $n \geq 2$ there exists a twin-free graph $G = (V, E)$ and a vertex $x \in V$ such that G_x is twin-free and*

$$\gamma(G_x) - \gamma(G) \geq \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor - 2 & \text{if } n \in \{2, 4, 5, 6, 8\} \\ \left\lfloor \frac{n}{2} \right\rfloor - 1 & \text{otherwise.} \end{cases}$$



(a) $n = 2$ (b) $n = 3$ (c) $n = 4$ (d) $n = 5$

(e) $n = 6$ (f) $n = 8$

Fig. 1: A graph G where $\gamma(G_x) - \gamma(G)$ is maximal for small orders n. Black dots are codewords of an optimal identifying code in G.

Proof: For $n \leq 6$ or $n = 8$, an example of a graph for which the inequality is tight is given in Figure 1.

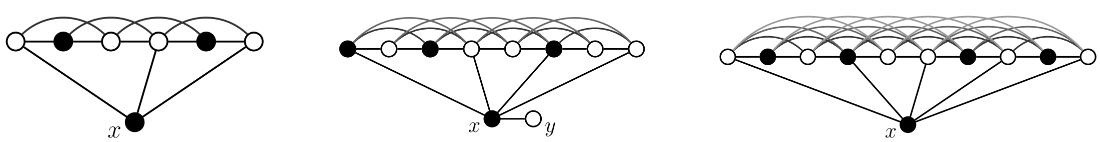
Assume that $n = 7$ or $n \geq 9$. First, we define $t = \left\lfloor \frac{n-1}{2} \right\rfloor$. We can now construct the following family of graphs. Let G be the graph with the vertex set

$$V = \begin{cases} \{v_1, v_2, \ldots, v_{n-1} = v_{2t}; x\} & \text{if } n \text{ is odd} \\ \{v_1, v_2, \ldots, v_{n-2} = v_{2t}; x, y\} & \text{if } n \text{ is even,} \end{cases}$$

and the edge set

$$E = \{\{v_i, v_j\} : |i - j| < t\} \cup \{\{v_i, x\} : i \geq 4 \text{ and even}\} \cup \{\{v_1, x\}\}.$$

Moreover, there is an edge $\{x, y\} \in E$ if n is even. Some examples of these constructions are shown in Figure 2.



(a) $n = 7$ (b) $n = 10$ (c) $n = 11$

Fig. 2: Three graphs G such that $\gamma(G_x) - \gamma(G)$ is maximal. The black dots are an identifying code of G.

Now, we show that

$$C = \begin{cases} \{x; v_2, v_4, \ldots, v_{t-1}; v_{t+2}, v_{t+4}, \ldots, v_{2t-1}\} & \text{if } t \text{ is odd} \\ \{x; v_1, v_3, \ldots, v_{t-1}; v_{t+2}, v_{t+4}, \ldots, v_{2t-2}\} & \text{if } t \text{ is even} \end{cases}$$

is an identifying code of G. The black dots in Figures 2(a), 2(b) and 2(c) illustrate the code C when $n = 7$, $n = 10$ and $n = 11$ respectively.

We first show that C separates x from all the other vertices. First, if $n = 7$, x is the only vertex whose neighbourhood does not contain any codeword other than x. If t is even, so x is the only vertex whose neighbourhood contains v_1 but not v_3. Otherwise t is odd and at least 5. Then $v_2 \notin I(x)$ and $v_{t+2} \notin I(x)$. Now, v_1 and v_2 are the only vertices in addition to x whose neighbourhoods do not contain v_{t+2}. However, the neighbourhoods of both v_1 and v_2 contain v_2, so x is also separated from these vertices by C.

Next, we prove that v_i and v_j are separated by C for all i and j when $i \neq j$. Without loss of generality, we assume $i < j$. If $j - i = 1$ then exactly one of $I(v_i)$ and $I(v_j)$ contains x except if $i = 2$ and $j = 3$. However, since $v_{t+2} \in I(v_3) \setminus I(v_2)$, then v_2 and v_3 are also separated. Now assume that $j - i \geq 2$. This claim essentially follows since the only couples of consecutive non-codewords are v_t and v_{t+1} and (for n even) v_{2t-1} and v_{2t}. Strictly speaking, if $1 \leq i \leq t-2$ and then $i+2 \leq j \leq 2t$, then $v_{i+t} \in N(v_j) \setminus N(v_i)$ and $v_{i+t+1} \in N(v_j) \setminus N(v_i)$ and one of v_{i+t} and v_{i+t+1} is a codeword and separates v_i from v_j. Analogously, if $t+2 \leq j \leq 2t$ and then $1 \leq i \leq j-2$, then either v_{j-t-1} or v_{j-t} separates v_i from v_j. Otherwise, $t-1 \leq i \leq j \leq t+1$ and since $j - i \geq 2$, then $i = t-1$ and $j = t+1$. Now, $v_1 \in N(v_{t-1}) \setminus N(v_{t+1})$ and $v_{2t-1} \in N(v_{t+1}) \setminus N(v_{t-1})$ and one of these two vertices is a codeword which separates v_{t+1} and v_{t-1}.

If n is even and $n \geq 10$, we must also separate y from the other vertices. Now, $I(y) = \{x\}$ and for all $v \in V \setminus \{y\}$, the set $I(v)$ contains either v_{t-1} or v_{t+2} (including when $v = x$ since $t = \left\lfloor \frac{n-1}{2} \right\rfloor \geq 4$). Therefore, y is separated from all other vertices by C. We can also see that the set $I(v)$ is non-empty for every $v \in V$.

Thus, C is an identifying code of size $t = \left\lfloor \frac{n-1}{2} \right\rfloor$, and so $\gamma(G) = \left\lfloor \frac{n-1}{2} \right\rfloor$.

Notice that G_x is twin-free since it is the disjoint union of a vertex and the power of a path. Besides, it has been shown in [6] that such power of a path is a tight example for Lemma 7 and hence, any minimum identifying code of G_x has size $n-2$.

Hence

$$\gamma(G_x) - \gamma(G) = n - 2 - \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

□

Theorem 2. *For every $s \geq 1$ and n such that $n \geq 4 \cdot 2^s + s$, there exists a twin-free graph $G = (V, E)$ and a set $S = \{x_1, \ldots, x_s\} \subseteq V$ such that G_S is twin-free and*

$$\gamma(G_S) - \gamma(G) \geq n - 2s - \left\lfloor \frac{n-s}{2^s} \right\rfloor.$$

Proof: Let $n = s + 2^s \cdot k + l$ such that $0 \leq l < 2^s$. From this follows that $k = \left\lfloor \frac{n-s}{2^s} \right\rfloor$. Moreover, we define $t = 2^s \cdot k$. First, we define the graph $G_S = (V_S, E_S)$ that consists on vertices $P = \{v_1, v_2, \ldots, v_t\}$ which form a power of a path such that there is an edge between v_i and v_j if $|i - j| < \frac{t}{2}$. Moreover, there are l isolated vertices $Y = \{y_1, y_2, \ldots, y_l\}$. Now we can show in the same way as in the proof of Theorem 1 that $\gamma(G_S) = n_S - 1$.

The graph G is constructed from G_S by adding the set of vertices $S = \{x_1, x_2, \ldots, x_s\}$. Every vertex of Y will have a unique set of vertices of S in its neighbourhood. Similarly, we can partition vertices of P into k sets which all contain 2^s consecutive vertices $(v_{i \cdot 2^s + 1}, \ldots, v_{(i+1)2^s})$. Every vertex of such a subset has a unique set of vertices of S in its neighbourhood. Thus, all the vertices in a given subset are separated from each other by S.

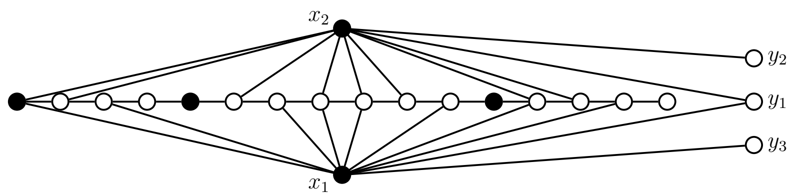


Fig. 3: A such graph G where $\gamma(G_S)-\gamma(G)$ is maximal when $|S|=2$ and $n=21$. The black dots are an identifying code of G. Notice that all edges between the vertices in the path are not drawn in the figure.

The graph $G=(V,E)$ can be defined as follows:

$$V=\{v_1,v_2,\dots,v_t;x_1,x_2,\dots,x_s;y_1,y_2,\dots,y_l\}$$

and

$$E=\begin{cases}E_v\cup E_y\cup E_o & \text{if } k \text{ is odd}\\ E_v\cup E_y\cup E_e & \text{if } k \text{ is even,}\end{cases}$$

where

$$E_v=\left\{\{v_i,v_j\}\,\middle|\,|i-j|<2^{s-1}k=2^{s-1}\cdot\frac{n-s-l}{2^s}\right\},$$

$$E_y=\left\{\{x_i,y_j\}\,\middle|\,j\equiv b \pmod{2^i} \text{ for some } b\in\{1,2,\dots,2^{i-1}\}\right\},$$

$$E_e=\left\{\{x_i,v_j\}\,\middle|\,i=1,2,\dots,s; j\neq\frac{t}{2}+1-2^s \text{ and } j\equiv b \pmod{2^i} \text{ for some } b=1,2,\dots,2^{i-1}\right\}$$
$$\cup\left\{\{x_i,v_{\frac{t}{2}}\}\,\middle|\,i=1,2,\dots,s\right\}$$

and

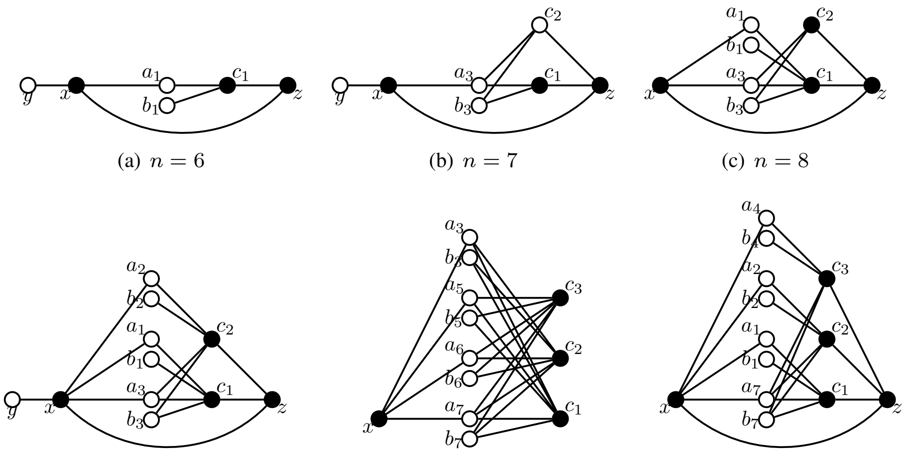
$$E_o=\left\{\{x_i,v_j\}\,\middle|\,i=1,2,\dots,s-1; j\neq\frac{t}{2}+1-2^s \text{ and } j\equiv b \pmod{2^i} \text{ for some } b=1,2,\dots,2^{i-1}\right\}$$
$$\cup\left\{\{x_s,v_j\}\,\middle|\,j\neq\frac{t}{2}+1-2^s \text{ and } j\equiv b+1 \pmod{2^i} \text{ for some } b=1,2,\dots,2^{s-1}\right\}$$
$$\cup\left\{\{x_i,v_{\frac{t}{2}-2^{s-1}}\}\,\middle|\,i=1,2,\dots,s\right\}.$$

We can now see that

$$C=S\cup\left\{v_i\,\middle|\,i=\frac{t}{2}+1-2^s\cdot j,\ j=1,2,\dots,\left\lfloor\frac{k}{2}\right\rfloor\right\}\cup\left\{v_i\,\middle|\,i=\frac{t}{2}+2^s\cdot j,\ j=1,2,\dots,\left\lfloor\frac{k-1}{2}\right\rfloor\right\}$$

is an optimal identifying code in G and it contains $\lfloor\frac{n-s}{2^s}\rfloor+s-1$ codewords. An example of such a graph and an optimal code is given in Figure 3, when $|S|=2$ and $n=21$. On the other hand, $V_S\setminus\{v_1\}$ is an optimal identifying code in G_S.

The details of the proof can be proved in a similar way as in the proof of Theorem 1 and the details of this proof are ignored. □



(a) $n = 6$ (b) $n = 7$ (c) $n = 8$

(d) $n = 11$ (e) $n = 12$ (f) $n = 13$

Fig. 4: Bipartite graphs G such that $\gamma(G_x) - \gamma(G)$ is maximal within these bipartite graphs. The maximality of Constructions 4(a), 4(d) and 4(f) follows from Theorem 4. The maximality of Constructions 4(c) and 4(b) and 4(e) follows from Remarks 5 and 6. The black dots are an identifying code of G. The vertices a_i, c_j, x and y are an optimal identifying code of G_x.

Remark 3. Theorem 2 is also valid if $3 \cdot 2^s + s \leq n < 4 \cdot 2^s$. The proof of Theorem 2 is valid if the set Y is chosen so that all these vertices have at least two neighbours which belong to the set S.

Theorem 4. *Let k be the smallest integer such that $n \leq 2^k + k$. There is a bipartite graph $G = (V, E)$ and a vertex $x \in V$ such that*

$$\gamma(G_x) - \gamma(G) \geq \left\lfloor \frac{n-k}{2} \right\rfloor - 1$$

if $n = 5, 6, 10, 11$ or $n \geq 13$.

Proof: Let $t = \left\lfloor \frac{n-k-1}{2} \right\rfloor$. Then $2^{k-2} \leq t \leq 2^{k-1} - 1$. We define $G = (V, E)$ as follows:

$$V = \{a_i, b_i \mid i = 1, 2, \ldots, t-1; 2^{k-1} - 1\} \cup \{c_j \mid j = 1, 2, \ldots, k-1\} \cup \{x, z\}$$

If $n - k$ is even, then a vertex y is added to V. If $t = 2^{k-2}$ and $n \geq 13$, then a_3 and b_3 are replaced by $a_{2^{k-2}}$ and $b_{2^{k-2}}$. Thus, $a_{2^i} \in V$ and $b_{2^i} \in V$ belong to V for all $i = 0, 1, \ldots, k-2$. The edge set is

$$\begin{aligned} E &= \{\{a_i, c_j\}, \{b_i, c_j\} \mid i \equiv -1, -2, \ldots, -2^{j-1} \pmod{2^j}\} \\ &\quad \cup \{\{a_i, x\} | \forall i\} \cup \{\{z, c_j\} | \forall j\} \cup \{z, x\} \cup \{y, x\} \end{aligned}$$

The edge $\{y, x\}$ naturally belongs to E only if $y \in V$. The graph G is bipartite since exactly one of the ends of each edge is x or c_i for some i. The graph G is given in Figures 4(a), 4(d) and 4(f) when n is 6, 11 and 13, respectively.

Now, we show that

$$C = \{x, z\} \cup \{c_i \mid \forall i\}$$

is an identifying code of G, and we see that $\gamma(G) \leq |C| = k+1$. First, z is the only vertex which is dominated by x, z and c_1. Second, x is the only vertex which is dominated by x and z but not c_1. Moreover, y is the only vertex in addition to x which is not dominated by c_i for any i. Then, x, y and z are separated from all other vertices. Now, c_i is the only vertex which is dominated by c_i and z. Therefore, every c_i is separated from all other vertices. Furthermore, x dominates a_i for all i, but not any b_j, i.e. a_i is separated from b_j for all i and j. Finally, every a_i is dominated by a unique set of vertices c_j. Thus, a_i is separated from a_j if $i \neq j$. In the same way we show that b_i and b_j are separated. Thus, we have shown that C is an identifying code.

Let

$$C_x = \{y, z\} \cup \{a_i \,|\, \forall i\} \cup \{c_i \,|\, \forall i\}$$

be a code of G_x. Again, y naturally belongs to C_x only if $y \in V$. We prove that there is not an identifying code of G_x with smaller cardinality than $|C_x|$. First, if $y \in V$, it is isolated and must be a codeword of every identifying code of G_x. Second, a_i and b_i are the only vertices which are able to separate a_i and b_i. Without loss of generality, we can assume that a_i belongs to the identifying code for all i. The vertices z and $b_{2^{k-1}-1}$ must still be separated and z and $b_{2^{k-1}-1}$ are the only vertices which can do it since every c_i belongs to the neighbourhood of both z and $b_{2^{k-1}-1}$. Again, we can assume without loss of generality that z belongs to C_x. Finally, the vertices b_i are not dominated by z and a_j. Moreover, the neighbourhood of the vertex b_{2^i} contains only c_i in addition to b_{2^i}. Then, either c_i or b_{2^i} must belong to any identifying code. Now we have seen that the cardinality of every identifying code of G_x is at least

$$|C_x| = \begin{cases} t+k+1 = \left\lfloor \frac{n-k-1}{2} \right\rfloor + k + 1 = \frac{n+k}{2} & \text{if y} \in V \text{, i.e. } n-k \text{ is even,} \\ t+k = \left\lfloor \frac{n-k-1}{2} \right\rfloor + k = \frac{n+k-1}{2} & \text{if y} \notin V \text{, i.e. } n-k \text{ is odd} \end{cases}$$

since C_x contains t vertices of a_i, $k-1$ vertices of c_j, and z and possibly y. Hence,

$$\gamma(G_x) - \gamma(G) \geq |C_x| - |C| = \left\lfloor \frac{n+k}{2} \right\rfloor - (k+1) = \left\lfloor \frac{n-k}{2} \right\rfloor - 1.$$

□

Remark 5. Theorem 4 and the construction of the proof of this theorem are also valid if $n = 8$ or $n = 9$. The reasons why $|C|$ and $|C_x|$ are optimal identifying codes of G and G_x respectively, are valid despite the fact that there is not a vertex b_2 in G. The vertex b_2 only has b_2 and c_2 in its neighbourhood.

We can assume as in the proof of Theorem 4 that a_1, a_3 and z are codewords of C_x. Now, b_1 and b_3 are not dominated. These two vertices can not both be dominated and separated by only one codeword. Therefore, the set $I(b_1) \cup I(b_3) \subseteq N(b_1) \cup N(b_3) = \{b_1, b_3, c_1, c_2\}$ must contain at least two vertices. Thus, C_x is optimal.

Remark 6. Theorem 4 is valid also if $n \in \{3, 4, 7, 12\}$ but either G is not defined or C_x is not an optimal identifying code of G_x. The claim follows from the constructions in Figures 1(b) and 1(c) in the cases $n = 3$ and $n = 4$ respectively. If $n = 7$, the same construction as the proof of Theorem 4 is a valid example, but the optimal codes of G and G_x are $\{z, c_1, x\}$ and $\{z, c_1, a, y\}$ respectively. The case $n = 12$ follows from the construction where $V = \{x, c_1, c_2, c_3\} \cup \{a_i, b_i : i = 3, 5, 6, 7\}$ and the edges define in a similar way as in the proof of Theorem 4. Now, it is possible to show that $C = \{x, c_1, c_2, c_3\}$ and $C_x = \{a_3, a_5, a_6, a_7, c_1, c_2, c_3\}$ are optimal identifying codes of G and G_x respectively.

3 Upper bounds for $\gamma(G_x) - \gamma(G)$

We first give some basic results which are needed in this section.

Lemma 7 ([7, 1])**.** *Let $G = (V, E)$ be a twin-free graph with n vertices, and at least one edge, then*

$$\gamma(G) \leq n - 1.$$

Lemma 8 ([9])**.** *Let $G = (V, E)$ be a twin-free graph with n vertices. Then,*

$$\gamma(G) \geq \lceil \log_2 (n + 1) \rceil .$$

Lemma 9 ([9])**.** *Let $G = (V, E)$ be a twin-free graph of order n and $E \neq \emptyset$. If (exactly) one end of each edge of E is $x \in V$, then*

$$\gamma(G) = n - 1.$$

We next show the upper bound for $\gamma(G_S) - \gamma(G)$ in general case.

Theorem 10. *Let G be a twin-free graph on n vertices and let $S \subseteq V$ such that satisfying $n \geq 2^{|S|-1}$. If G_S is twin-free, then*

$$\gamma(G_S) - \gamma(G) \quad \leq \quad n - 2|S| - \left\lfloor \frac{n - |S|}{2^{|S|}} \right\rfloor . \tag{1}$$

Proof: First, if $2^{|S|-1} \leq n \leq 2^{|S|} - 1$, then the claim follows since $|S| = \lceil \log_2 (n + 1) \rceil$ by Lemma 8 and since an optimal identifying code of G_S contains at most $n_S = n - |S|$ codewords.

Assume then that $n \geq 2^{|S|}$. Let C be an optimal identifying code of G. Now, we construct an identifying code C_S of G_S. Let $C_S = C \setminus S$ in the beginning. Next, we partition the vertices of G_S into equivalence classes T_R such that the vertex v belongs to T_R if $I(v, C \setminus S) = R$. We can also assume that all vertices of S belong to C. Otherwise we can reduce it to a smaller graph. Moreover, the right hand side of Equation (1) increases when $|S|$ increases and $n \geq 2^{|S|}$.

First, we can observe that every class T_R can contain at most $2^{|S|}$ vertices. However, there can be at most $2^{|S|} - 1$ vertices in $T_\emptyset$. Now we can separate vertices of T_R by adding at most $|T_R| - 1$ codewords to C_S if $R \neq \emptyset$. Instead we may need to add $|T_\emptyset|$ vertices to C_S so that all vertices of $T_\emptyset$ are dominated (and separated from each other).

There have to be at least $\frac{|G_S|}{2^{|S|}}$ non-empty equivalence classes if $T_\emptyset = \emptyset$. Otherwise, there are at least $\frac{|G_S| - (2^{|S|} - 1)}{2^{|S|}}$ equivalence classes in addition to $T_\emptyset$. In all cases, the number of the other than $T_\emptyset$ equivalence classes is at least

$$\min \left\{ \left\lceil \frac{|G_S|}{2^{|S|}} \right\rceil , \left\lceil \frac{|G_S| - (2^{|S|} - 1)}{2^{|S|}} \right\rceil \right\} = \left\lceil \frac{|G_S| - (2^{|S|} - 1)}{2^{|S|}} \right\rceil = \left\lfloor \frac{|G_S|}{2^{|S|}} \right\rfloor .$$

From this follows that

$$
\begin{aligned}
|C_S| &\le |C \setminus S| + \left(|T_\emptyset| + \sum_{R\neq\emptyset, T_R\neq\emptyset} (|T_R| - 1) \right) \\
&= (|C| - |S|) + \sum_{T_R\neq\emptyset} |T_R| - \sum_{R\neq\emptyset, T_R\neq\emptyset} 1 \le |C| - |S| + |G_S| - \left\lfloor \frac{|G_S|}{2^{|S|}} \right\rfloor \\
&= |C| - |S| + (n - |S|) - \left\lfloor \frac{n-|S|}{2^{|S|}} \right\rfloor = |C| + n - 2|S| - \left\lfloor \frac{n-|S|}{2^{|S|}} \right\rfloor .
\end{aligned}
$$

□

Theorem 11. *Let G be a twin-free graph on $n \in \{2,4,5,6,8\}$ vertices. For every $x \in V$, if G_x is twin-free, then*

$$\gamma(G_x) - \gamma(G) \le \left\lfloor \frac{n}{2} \right\rfloor - 2.$$

Proof: The case $n = 2$ is straightforward to check.

Assume now that $n \ge 4$. If an optimal identifying code of G_x contains n_x codewords, there are no edges in G_x. Then one end of all edges in G is x and $\gamma(G) \ge n - 1 = n_x = \gamma(G_x)$ by Lemma 9, i.e. $\gamma(G_x) - \gamma(G) \le 0 \le \left\lfloor \frac{n}{2} \right\rfloor - 2$. Otherwise, $\gamma(G_x) \le n_x - 1$ and now the claim follows again by Lemma 8. □

Corollary 12. *Let $G = (V, E)$ be a twin-free graph with n vertices. Then,*

$$\gamma(G_x) - \gamma(G) \le \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor - 2 & \text{if } n \in \{2,4,5,6,8\} \\ \left\lfloor \frac{n}{2} \right\rfloor - 1 & \text{in other cases} \end{cases}$$

and the inequality is tight.

Proof: The claim follows by Theorems 1, 10 and 11. □

Corollary 13.

$$\frac{\gamma(G_x)}{\gamma(G)} \le \frac{\lceil \log_2(n+1) \rceil + \left\lfloor \frac{n}{2} \right\rfloor - 1}{\lceil \log_2(n+1) \rceil} \approx \frac{n}{2\log_2 n}.$$

Proof: The claim follows by Theorem 10 and Lemma 8. □

The inequality of Corollary 13 is almost tight. Indeed, Charon et al [2] proved that there are graphs G and G_x such that $\frac{\gamma(G_x)}{\gamma(G)} \approx \frac{n}{2\log_2 n}$ (Conclusion 6 [2]). There is nevertheless a small gap between the construction of Conclusion 6 [2] and Corollary 13. In fact, the construction of the proof in Theorem 4 is very similar to the construction of Conclusion 6 [2] and it also satisfies the approximate equation $\frac{\gamma(G_x)}{\gamma(G)} \approx \frac{n}{2\log_2 n}$.

Next, we show that if n is even and large enough and $\gamma(G_x) - \gamma(G)$ reaches the upper bound $\gamma(G_x) - \gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor - 1$, then G_x must be disconnected.

Proposition 14. *There does not exist graphs G and G_x such that G_x is connected and $\gamma(G_x) - \gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor - 1$ if n is even and at least 10.*

Proof: Assume the contrary that G and G_x would be such graphs. Let C be an optimal identifying code of G. The vertices of G_x can be divided into equivalence classes T_R as the proof of Theorem 10. In particular, every equivalence class contains at most 2 vertices. Therefore, two vertices are called a *pair* if the vertices belong to the same equivalence class. Since n is even, then n_x is odd. From this follows that there are at most $\frac{n_x-1}{2}$ pairs and at least one equivalence class which contains only one vertex. Now, if $\gamma(G_x) - \gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor - 1$, then every vertex must belong to a pair except one vertex y which satisfies the condition $I(y, C) = \{x\}$. Moreover, any two pairs are not separated by the same vertex of G_x and any vertex which dominates y does not separate any of these pairs.

Since G_x is connected, y has at least one neighbour in G_x. Assume that v_1 and y are connected by an edge. Let u_1 be the pair of v_1 that is $C \setminus \{x\}$ does not separate v_1 and u_1. Now, if u_1 and y were not neighbours then y would separate v_1 and u_1 in addition to dominating y. Thus, all the vertices y, v_1 and u_1 can dominate y. Therefore any of these three vertices can not separate any of these $\frac{n_x-1}{2}$ pairs.

Now, we show the claim by induction. Assume that y, $v_1, \ldots, v_k$ and $u_1, \ldots, u_k$ can separate only the pairs $\{v_i, u_i\}$, where $i = 1, 2, \ldots, k-1$. Let $\{v_{k+1}, u_{k+1}\}$ be the pair such that at least one of these vertices separate the pair $\{v_k, u_k\}$. Without loss of generality, we can assume that there is an edge between v_k and v_{k+1} but not between u_k and v_{k+1}. Now, if there is not an edge between v_k and u_{k+1}, then v_k separates the pair $\{v_{k+1}, u_{k+1}\}$ against the assumption of induction. Similarly, if there is an edge between u_k and u_{k+1}, then u_k separates $\{v_{k+1}, u_{k+1}\}$. Again, this is against the assumption of induction. Therefore both u_{k+1} and v_{k+1} can separate the pair $\{v_k, u_k\}$. Thus, none of these two vertices can separate any other pair. Hence, any of the vertices y, $v_1, \ldots, v_{k+1}$ and $u_1, \ldots u_{k+1}$ can not separate any other pairs except $\{v_i, u_i\}$, when $i = 1, 2, \ldots, k$. In other words, y and $k+1$ pairs can separate only k pairs and this holds for all k. This is nevertheless impossible since the graph is finite. Therefore it is impossible that G_x is connected if $n \geq 10$, n is even and $\gamma(G_x) - \gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor - 1$. □

Next we prove an upper bound for $\gamma(G_x) - \gamma(G)$ among the bipartite graphs.

Theorem 15. *For bipartite twin-free graphs G and G_x, the maximum difference between $\gamma(G_x)$ and $\gamma(G)$ is*

$$\gamma(G_x)-\gamma(G) \leq \frac{n-3}{2}-\frac{1}{3}\left\lfloor \log_2\left(n-3-\left\lfloor \log_2\left(n-\left\lfloor \log_2(4n)\right\rfloor\right)\right\rfloor\right)\right\rfloor-\frac{1}{6}\left\lfloor \log_2\left(n-\left\lfloor \log_2(4n)\right\rfloor\right)\right\rfloor$$

if $n \geq 7$.

Proof: Let C be an optimal identifying code in G and $C \setminus \{x\}$ be a code in G_x. Let A and B be a partition of all vertices V such that if there is an edge between vertices u and v, then one of these two vertices belongs to A and the other belongs to B. Assume also that x belongs to B. Moreover, if there is a vertex v such that $I(v, C) = \{x\}$ and $v \neq x$, the vertex is denoted by y. Observe that y is not dominated by $C \setminus \{x\}$.

Step 1. The partition of pairs

If $C \setminus \{x\}$ does not separate u and v, then u and v is said to be a *pair*. A pair of u and v is called *AA-pair* if both u and v belong to A. Also, the set AA denotes the set of vertices which belong to some AA-pair. Furthermore, $|AA|$ denotes the number of vertices of AA-pairs. Thus, the number of AA-pairs is $\frac{1}{2}|AA|$. Similarly, a pair is called AB-*pair* if exactly one of the vertices of the pair belongs to A and the other one belongs to B. Notice that there can not be a pair between two vertices of the set B since x can not belong to the neighbourhood of any vertex of B except the neighbourhood of x.

We still divide AB-pairs into two disjoint sets AB^*- and AB'-pairs. Let $a \in A$ and $b \in B$ be an AB-pair.

- If there is an AA-pair of a' and a'' such that both a' and a'' are the neighbours of b, then a and b are an AB^*-pair.
- Otherwise, the pair of a and b is an AB'-pair.

We furthermore partition AB'-pairs into four disjoint sets $AB'_{\alpha 1}$, $AB'_{\alpha 2}$, $AB'_{\alpha 3}$ and AB'_{β} or two sets $AB'_{\alpha} = AB'_{\alpha 1} \cup AB'_{\alpha 2} \cup AB'_{\alpha 3}$ and AB'_{β}. We first define an induced subgraph $G' = (AB', E')$ of G, where AB' is the set of vertices in AB'-pairs and $E' = \{\{u, v\} \in E \mid u \in AB', v \in AB'\}$. Let $a \in AB'$ and $b \in AB'$ be an AB'-pair and a' and b' be another AB'-pair. Notice that a and b belong to the same connected component in G' since there must be an edge between a and b or else the neighbourhoods of a and b would be totally distinct in G.

- If the pair of a and b belongs to the connected component whose cardinality is at least six vertices in G', the pair belongs to $AB'_{\alpha 3}$.
- If a, b, a' and b' form a connected component of four vertices and if at least one of the pairs has no neighbours in G except x, a, a', b and b', then all the vertices a, b, a' and b' belong to $AB'_{\alpha 2}$.
- If there is an AB'-pair of $a \in A$ and $b \in B$ such that the vertices do not belong to the set $AB'_{\alpha 2} \cup AB'_{\alpha 3}$ and b has a neighbour which belongs to the set AA, then a and b belong to $AB'_{\alpha 1}$.
- Otherwise, an AB'-pair is an AB'_{β}-pair.

Notice that at least one of the vertices of AB'_{β}-pair must have such a non-codeword neighbour that does not belong to $AA \cup AB'$. Otherwise, the pair would be an AB'_{α}-pair or G_x would not be twin-free.

Let us have a look some basic properties of AA and AB-pairs. First, no vertex of AA belongs to the code C since there are no edges between two vertices of AA. Secondly, if there is an AB-pair of $a \in A$ and $b \in B$, then a and b are the only vertices which can belong to the neighbourhoods of both a and b. Then a, b and x are the only possible codewords in their neighbourhoods. On the other hand, a or b belongs to C, since b has to be dominated by C. Especially, if a and b belong to AB'_{α}, then a or b is able to separate another pair, and from this follows that this vertex can not be a codeword. It also means that either a or b is a leaf of G' and a codeword of C.

Step 2. The set B^*

We first prove that the cardinality of $B^* = (B \cap C \setminus \{x\} \setminus AB')$ is $|B^*| \geq \lfloor \log_2 |AA| \rfloor$ where $|AA|$ is the number of vertices in AA-pairs. We see that all codeword neighbours of $a \in AA$ belong to the set $B^* \cup \{x\}$ since a and its pair a' must have the same codeword neighbours apart from x, and if $b \in AB$ is a codeword neighbour of both a and a', then b belongs to AB^*. Moreover, every vertex in AA has to be dominated by at least one codeword of B^* since vertices of AA have to be non-codewords and only one vertex of each pair is dominated by x. Furthermore, codewords in B^* are the only codewords which can separate AA-pairs from other AA-pairs. Since we can form at most $2^{|B^*|}-1$ non-empty subsets from $|B^*|$ codewords, then the number of AA-pairs is $\frac{1}{2}|AA| \le 2^{|B^*|}-1$, i.e.

$$|B^*| \geq \left\lceil \log_2\left(\frac{1}{2}|AA|+1\right)\right\rceil = 1+\left\lfloor \log_2\left(\frac{1}{2}|AA|\right)\right\rfloor = \lfloor \log_2 |AA| \rfloor . \tag{2}$$

Step 3. The construction of C_x

Next, we construct an identifying code C_x of G_x and show that $|C_x|-|C| \le \frac{1}{2}|AA|+\frac{1}{3}|AB'_\alpha|+|S|+|y|-1$ where $|S| \le \frac{1}{2}|AB'_\beta|$ and $|y| \in \{0,1\}$ is the number of non-codewords which are dominated only by x.

Let $C_x = C \setminus \{x\}$ or $C_x = C \cup \{y\} \setminus \{x\}$ in the beginning, depending on whether there exists a vertex y in G. Then, we add and delete vertices to/from C_x in the following way:

Step 3.1. Separation of $AB'_{\alpha 3}$-pairs: Firstly, we study an $AB'_{\alpha 3}$-pair of $a \in A$ and $b \in B$. Let T be a connected component of G' that contains a and b. (Since ab is an $AB'_{\alpha 3}$-pair, the size of the connected component is at least 6.) Let T' be an induced subgraph of T which contains non-codewords of C, i.e. T' contains one vertex of each pair of T and the vertices of T' are exactly all non-leaves of T. In particular, T' is connected and its order is at least three. Let T_D be a minimum total dominating set of T', i.e. such a set that every vertex of T' has a neighbour which belongs to T_D. We now note that $C_x \cup T_D$ separates a and b and also all the other pairs of T. Since either a or b belongs to T', this vertex must have a neighbour which belongs to T_D. Moreover, it is known that every connected graph T' of order at least three has a total dominating set with at most $\frac{2}{3} \cdot |T'|$ codewords [5, 8]. Moreover, T contains $|T'|$ $AB'_{\alpha 3}$-pairs. Therefore all $AB'_{\alpha 3}$-pairs can be separated adding $\frac{1}{3}|AB'_{\alpha 3}|$ codewords to C_x. (Notice that there are $\frac{1}{2}|AB'_{\alpha 3}|$ different $AB'_{\alpha 3}$-pairs.)

Step 3.2. Separation of $AB'_{\alpha 2}$-pairs: Let a, b, a' and b' be the vertices of a connected component of G' such that a and b is an $AB'_{\alpha 2}$-pair and a' and b' is another $AB'_{\alpha 2}$-pair. As we formerly mentioned in the end of Step 1 there has to be exactly two leaves of the connected component and the leaves are codewords of C, but the other vertices are not. Therefore, the component has to be a path (either $a-b-a'-b'$ or $a'-b'-a-b$). Without loss of generality, we can assume that the path is $a-b-a'-b'$ and a and b have no neighbours apart from x, a, b and a' in G. Now, a and b' belong to C. We show that $C \cup \{b, a'\} \setminus \{a, x\}$ separates any vertices $u \in V \setminus \{x\}$ and $v \in V \setminus \{x, u\}$ if $C \setminus \{x\}$ also separates them or u or v belongs to the set $\{a, b, a', b'\}$. First, a and b are the only vertices of G_x such that a vertex has been deleted from their identifying set. Then it is enough to show that a and b are separated from all other vertices of G_x and a' is separated from b'. This is valid, since a, b and a' are the only vertices which are dominated by b and the identifying sets $I(a)=\{b\}$, $I(b)=\{b,a'\}$ and $I(a') \supseteq \{b, a', b'\}$ are different. All in all, we need to add two vertices to C_x and we can delete a vertex from C_x. Therefore, all $AB'_{\alpha 2}$-pairs can be separated by increasing the number of codewords of C_x by $\frac{1}{4}|AB'_{\alpha 2}|$.

Step 3.3. Separation of AB'_β-pairs: If a and b belong to an AB'_β-pair, then a or b must have a non-codeword neighbour v such that $v \notin AA \cup AB'$. We add v to C_x. Notice that a and b may have many such neighbours v but we add only one of such vertices to C_x. Moreover, it is possible that we add the same vertex v because of more than one AB'_β-pair. All in all, we need to add at most $\frac{1}{2}|AB'_\beta|$ codewords to C_x. The set of the vertices which add to C_x by AB'_β-pairs is called the set S. Thus, $|S| \le \frac{1}{2}|AB'_\beta|$ and we also know that $S \cap (AA \cup AB' \cup B^*) = \emptyset$ since $B^* \subseteq C$ and $S \cap C = \emptyset$.

Step 3.4. Separation of AA-pairs and $AB'_{\alpha 1}$-pairs: Finally, we add the following non-codewords of AA to the code C_x. First, every $AB'_{\alpha 1}$-pair has at least one neighbour which belongs to AA. We add one of these neighbours to C_x for every $AB'_{\alpha 1}$-pair. Second, if none of the vertices of an AA-pair is not yet added to C_x, then we add one of these two vertices to C_x. Now we have added at least one of the vertices of each AA-pair to C_x. However, if there is an AA-pair of a and a' such that both a and a' belong to C_x now, this means that there have to be two $AB'_{\alpha 1}$-pairs such that a has been added by another $AB'_{\alpha 1}$-pair and a' by the other $AB'_{\alpha 1}$-pair. Therefore, a and a' are able to separate three pairs: an AA-pair and two $AB'_{\alpha 1}$-pairs. In particular, we add at most $\frac{1}{2}|AA| + \frac{1}{4}|AB'_{\alpha 1}|$ vertices of the set AA to C_x in the final state.

Step 3.5. The Conclusion of Step 3 and the separation of AB^*-pairs: Now, we have seen that C_x separates AA-pairs and AB'-pairs. Moreover, it separates AB^*-pairs since at least one of the vertices of every AA-pair is a codeword of C_x. Therefore, C_x is an identifying code of G_x. Moreover,

$$\begin{aligned} |C_x| - |C| &\leq \frac{1}{2}|AA| + \frac{1}{4}|AB'_{\alpha 1}| + \frac{1}{4}|AB'_{\alpha 2}| + \frac{1}{3}|AB'_{\alpha 3}| + |S| + |y| - 1 \\ &\leq \frac{1}{2}|AA| + \frac{1}{3}|AB'_{\alpha}| + |S| + |y| - 1, \end{aligned} \tag{3}$$

where the term -1 follows since x belongs to C but not C_x.

Step 4. An upper bound for $|C_x| - |C|$

We recall that the sets AA, AB'_{α}, AB'_{β}, S and B^* are disjoint and none of these contain x or y. If we denote the other vertices of G by $S' = V \setminus (AA \cup AB' \cup S \cup B^* \cup \{x, y\})$, then we know that

$$\begin{aligned} n - 1 \; = \; n_x \; &= \; |AA| + |AB'_{\alpha}| + |AB'_{\beta}| + |S| + |B^*| + |y| + |S'| \\ &\geq \; |AA| + |AB'_{\alpha}| + 3|S| + |B^*| + |y| + |S'|. \end{aligned} \tag{4}$$

Now, the next inequality follows from Equations (3) and (4):

$$\begin{aligned} |C_x| - |C| &\leq \frac{1}{2}|AA| + \frac{1}{3}|AB'_{\alpha}| + |S| + |y| - 1 \\ &= \frac{1}{3}\Big(|AA| + |AB'_{\alpha}| + 3|S| + |y| + |B^*| + |S'|\Big) + \frac{1}{6}|AA| + \frac{2}{3}|y| - \frac{1}{3}|B^*| - \frac{1}{3}|S'| - 1 \\ &\leq \frac{1}{3}(n-1) + \frac{1}{6}|AA| - \frac{1}{3}|B^*| + \frac{2}{3}|y| - \frac{1}{3}|S'| - 1 \\ &= \frac{1}{3}\left(n - 4 + \frac{|AA|}{2} - |B^*| + 2|y| - |S'|\right). \end{aligned}$$

Next, we denote $S'' = AB' \cup S$. Then, the first equality of Equation 4 can be written as follows:

$$n - 1 = |AA| + |B^*| + |S'| + |S''| + |y| \tag{5}$$

Step 4.1. Case $|AA| \geq 2$ and $|S'| + |S''| + |y| \leq n - \lfloor \log_2 n \rfloor - 2$:

We now make two assumptions. We first assume that $|AA| \geq 2$, i.e. there is at least one AA-pair. Secondly, we assume that $|S'| + |S''| + |y| \leq n - \lfloor \log_2 n \rfloor - 2$. Steps 4.2 and 4.3 handle the cases when these assumptions are not valid.

Next, we show that

$$|AA| \leq n - 1 - |S'| - |S''| - |y| - \lfloor \log_2 (n - |S'| - |S''| - |y| - \lfloor \log_2 n \rfloor) \rfloor .$$

Let us assume to the contrary, that

$$|AA| \geq n - |S'| - |S''| - |y| - \left\lfloor \log_2 \left(n - |S'| - |S''| - |y| - \lfloor \log_2 n \rfloor \right) \right\rfloor .$$

Now, we can use Equation (2) and write

$$|AA| + |B^*| + |S'| + |S''| + |y| \geq |AA| + \lfloor \log_2 |AA| \rfloor + |S'| + |S''| + |y| \geq n$$

which is a contradiction according to Equation (5). We still notice that the right hand side of the inequality

$$\frac{|AA|}{2} - |B^*| \leq \frac{|AA|}{2} - \lfloor \log_2 |AA| \rfloor$$

is increasing when $|AA|$ increases since $|AA|$ has to be even and $|AA| \geq 2$. Now,

$$\begin{aligned}
|C_x| - |C| &\leq \frac{1}{3}\Big(n - 4 + \frac{|AA|}{2} - |B^*| + 2|y| - |S'| \Big) \\
&\leq \frac{1}{3}\Big(n - 4 + 2|y| - |S'| + \frac{|AA|}{2} - \lfloor \log_2 |AA| \rfloor \Big) \\
&\leq \frac{1}{3}\Bigg(n - 4 + 2|y| - |S'| \\
&\quad + \frac{1}{2}\Big(n - 1 - |S'| - |S''| - |y| - \left\lfloor \log_2 \left(n - |S'| - |S''| - |y| - \lfloor \log_2 n \rfloor \right) \right\rfloor \Big) \\
&\quad - \Big\lfloor \log_2 \Big(n - 1 - |S'| - |S''| - |y| - \left\lfloor \log_2 \left(n - |S'| - |S''| - |y| - \lfloor \log_2 n \rfloor \right) \right\rfloor \Big) \Big\rfloor \Bigg) \\
&= \frac{n - 3 - |S'| - \frac{1}{3}|S''| + |y|}{2} - \frac{1}{6} \Big\lfloor \log_2 \Big(n - |S'| - |S''| - |y| - \lfloor \log_2 n \rfloor \Big) \Big\rfloor \\
&\quad - \frac{1}{3} \Big\lfloor \log_2 \Big(n - 1 - |S'| - |S''| - |y| - \left\lfloor \log_2 \left(n - |S'| - |S''| - |y| - \lfloor \log_2 n \rfloor \right) \right\rfloor \Big) \Big\rfloor \\
&:= f(n, |S'|, |S''|, |y|),
\end{aligned}$$

Step 4.1.1. The analysis of $f(n, |S'|, |S''|, |y|)$: Now, we show that

$$\gamma(G_x) - \gamma(G) \leq \max(f(n,0,0,0), f(n,1,0,1), f(n,0,3,1) - 1, f(n,0,4,1)).$$

First, $|S'|$, $|S''|$ and $|y|$ are non-negative integers. If y is constant, then f is decreasing when $|S'|$ or $|S''|$ increases. Therefore, $f(n,0,0,0)$ is an upper bound for $|C_x| - |C|$ if $|y| = 0$. Otherwise, $|y| = 1$. Again, if $|S'| \geq 1$, then $f(n,1,0,1)$ is an upper bound for $|C_x| - |C|$. Assume then that $|S'| = 0$ and $|y| = 1$. Now x must have a codeword neighbour $c \in C$ in G, since $I(y, C) = \{x\}$. Moreover, since no codeword belongs to $S \cup AA$, then c has to belong to $AB' = S'' \setminus S$. Furthermore, at least either c or x must have another neighbour $c' \in C$ or else $I(x, C) = \{x, c\} = I(c, C)$. If c' is a neighbour of x, then there are at least two AB'-pairs and then $|S''| \geq |AB'| \geq 4$. If there are at least 4 vertices in S', then $|C_x| - |C| \leq f(n,0,4,1)$. Otherwise, there are at most 3 vertices in S', and c' is a neighbour of c. Then, $c \in A$ and $c' \in B$ must be the only AB'-pair. Now c or c' has to be a neighbour v in addition to x, c and c', since G_x is twin-free. On the other hand, v does not belong to any pair. Indeed, c or c' would separate this pair since c and c' do not belong to AB^*. Then, v has to belong to $S \subseteq S''$ (or S' which is a contradiction according to our assumption) and v is the only neighbour for c and c' in addition to x, c and c'. Now, C_x contains all the vertices v, c and c' following from Step 3.3. However, $C_x \setminus \{c\}$ or $C_x \setminus \{c'\}$ is also an identifying code depending on which are the neighbours of v. Thus, $\gamma(G_x) - \gamma(G) \leq f(n,0,3,1) - 1$ in this case.

Now, $f(n,1,0,1) \geq f(n,0,0,0) \geq f(n,0,3,1) - 1$ for all $n \geq 7$. Moreover, $f(n,0,4,1) \leq f(n,1,0,1) + \frac{1}{3}$ for all $n \geq 10$. However, $f(n,1,0,1)$ is a multiple of $\frac{1}{2}$ except if $n = 2^t + t + 2$, where t is an integer. Furthermore, $f(n,1,0,1) = f(n,0,4,1)$, if $n = 2^t + t + 2$. Therefore,

$$\gamma(G_x) - \gamma(G) \leq \lfloor f(n,1,0,1) \rfloor = \left\lfloor f(n,1,0,1) + \frac{1}{3} \right\rfloor \geq \lfloor f(n,0,4,1) \rfloor$$

for all $n \geq 10$, since $\gamma(G_x) - \gamma(G)$ is an integer. If $n \leq 9$, then the case $f(n,0,4,1)$ is impossible, since $|S'| + |S''| + |y|$ would be $5 > n - \lfloor \log_2 n \rfloor - 2$. All in all, $\gamma(G_x) - \gamma(G) \leq \lfloor f(n,1,0,1) \rfloor$ if $|S'| + |S''| + |y| \leq n - \lfloor \log_2 n \rfloor - 2$ and $AA \neq \emptyset$.

Step 4.2. Case $|AA| \geq 4$ and $|S'| + |S''| + |y| \geq n - \lfloor \log_2 n \rfloor - 1$:
If $|S'| + |S''| + |y| \geq n - \lfloor \log_2 n \rfloor - 1$, then $|AA| \leq \lfloor \log_2 n \rfloor - |B^*|$ according to Equation (5). If $|AA| \geq 4$, then $|B^*| \geq \lfloor \log_2 |AA| \rfloor \geq 2$ and then $|C_x| - |C| \leq \frac{n-4}{3} + \frac{\lfloor \log_2 n \rfloor - 2}{6} \leq f(n,1,0,1)$, when $n \geq 7$.

Step 4.3. Case $|AA| \leq 2$: If $|AA| \leq 2$, then $\frac{\lfloor AA \rfloor}{2} - |B^*| \leq 0$ and $|C_x| - |C| \leq \frac{n-2}{3} < f(n,1,0,1)$ when $n \geq 11$. In fact, $|C_x| - |C|$ is an integer, then $|C_x| - |C| \leq \left\lfloor \frac{n-2}{3} \right\rfloor \leq \lfloor f(n,1,0,1) \rfloor$, when $n = 7$ or $n \geq 9$.

Finally, we have proved the claim which is equivalent with $\gamma(G_x) - \gamma(G) \leq \lfloor f(n,1,0,1) \rfloor$ except if $n = 8$ and $|AA| \leq 2$. The case $n = 8$ follows from Lemma 16. □

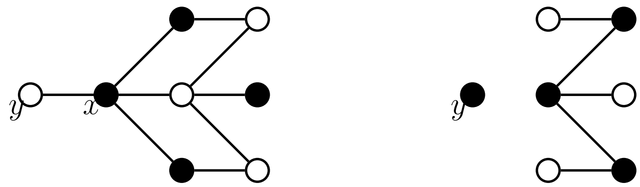
Lemma 16. *If G is a bipartite graph of order 8, then*

$$\gamma(G_x) - \gamma(G) \leq 1.$$

Proof: We use the same markings as the proof of Theorem 15. Assume the contrary that the claim is invalid, i.e. there is a graph G of order 8 such that $\gamma(G_x) - \gamma(G) \geq 2$. This is possible only if $|C| = 4$ and $|C_x| = \gamma(G_x) = 6$ according to Lemma 8 and the proof of Theorem 11. Now, G has to contain at least three pairs or two pairs *and* the vertex y. These pairs and the vertex y should provide three codewords to C_x.

Assume first that there is an AA-pair of a' and a''. Without loss of generality, we can assume that $\{a', x\} \in E$ and $\{a'', x\} \notin E$. Now, a' and a'' must have a common codeword neighbour $b^* \in B^*$. This AA-pair can provide at most one codeword to C_x in addition to $C \setminus \{x\}$. Moreover, at least one codeword must be provided to C_x by another pair, i.e. by $AB' \cup S$. However, vertices of $AB' \cup S$ can provide at most $\frac{1}{3}|AB' \cup S|$ codewords to C_x. Thus, there must be an AB'-pair and a vertex y. Moreover, the AB'-pair has to be an AB'_β-pair since a single AB'_α-pair does not provide any codeword to C_x. Thus, the eightest vertex belongs to S and it is denoted by s. Now, $\{s, y, a', a''\} \cap C = \emptyset$. Moreover, b^* or a'' can not have a or b in its neighbourhood or else a and b would not be an AB'_β-pair. Therefore, $I(a'') = \{b^*\} = I(b^*)$ which is a contradiction.

Assume second that there is not an AA-pair. Now, at least two codewords must be provided to C_x by AB'-pairs and this is possible only if there are three $AB'_{\alpha 3}$-pairs or two AB'_β-pairs and 2 vertices of $|S|$.



(a) G (b) G_x

Fig. 5: The bipartite graph of order 8 with three $AB'_{\alpha 3}$-pairs. It is possible that there are more edges between y and the other vertices. The black dots represent an identifying code.

The third codeword must be provided by y. First, assume that $s \in S$ separates an AB'_{β}-pair of a and b and s' separates another AB'_{β}-pair of a' and b'. Moreover, there can not be edges $\{s', a\}$, $\{s', b\}$, $\{s, a'\}$ and $\{s, b'\}$ since s or s' can not separate both of the pairs. Thus, $I(a) \cup I(b) \cup I(s) \subseteq \{a, b, x\}$. Moreover, since none of $I(a)$, $I(b)$ and $I(s)$ can be empty or $\{x\} = I(y)$, both a and b in addition to x have to be codewords. Similarly, a' and b' are codewords and $|C| = 5$, which is a contradiction.

Finally, let us assume that G contains three $AB'_{\alpha 3}$-pairs and x and y. Now, x must have at least two codeword neighbours, since $I(x) \neq I(y) = \{x\}$ and $|I(v)| \leq 2$ for all $v \in AB'_{\alpha 3}$. Then, G is uniquely defined apart from the edges between y and vertices of AB'. See Figure 5. Now, $\gamma(G_x) \leq 4$ regardless of the edges from y. $\square$

Corollary 17. *Let $G = (V, E)$ be a twin-free bipartite graph with n vertices. Then,*

$$\gamma(G_x) - \gamma(G) \leq \left\lfloor \frac{n - \log_2(n - \log_2 n)}{2} \right\rfloor - 1$$

and the inequality is tight for all $n = 3, 4, 5, \ldots$.

Proof: The claim follows by Theorems 1, 4, 10, 11 and 15 and Remarks 5 and 6. Notice that the values of the equations in Theorems 4 and 15 and this corollary are same for all integers $n \geq 3$ when the equations are defined. $\square$

4 Acknowledgements

I would like to thank anonymous referees for numerously linguistic and mathematical comments. Also I am grateful to referees for suggestions to extend this paper to bipartite graphs and to cases where $|S|$ is larger than 1.

References

[1] Bertrand, N.: Codes identifiants et codes localisateurs-dominateurs sur certains graphes, Mémoire de stage de maîtrise, ENST, Paris, France, 28 pages (2001).

[2] Charon, I.; Honkala, I.; Hudry, O.; Lobstein A.: Minimum sizes of identifying codes in graphs differing by one vertex, Cryptography and Communications, Vol. 5, 119–136 (2013).

[3] Charon, I.; Honkala, I.; Hudry, O.; Lobstein A.: Structural Properties of Twin-Free Graphs, Electronic Journal of Combinatorics, Vol. 14(1), R16 (2007).

[4] Charon, I.; Hudry, O.; Lobstein A.: Extremal cardinalities for identifying and locating-dominating codes in graphs, Discrete Mathematics, Vol. 307, 356–366 (2007).

[5] Cockayne, E. J.; Dawes, R. M.; Hedetniemi, S. T.: Total domination in graphs, Networks, Vol. 10, 211–219 (1980).

[6] Foucaud, F.; Guerrini, E.; Kovse, M.; Naserasr, R.; Parreau, A.; Valicov, P.: Extremal graphs for the identifying code problem, European Journal of Combinatorics, Vol. 32, 628–638 (2011).

[7] Gravier, S.; Moncel, J.: On graphs having a $V \setminus \{x\}$ set as an identifying code, Discrete Mathematics, Vol. 307, 432–434 (2007).

[8] Haynes, T. W.; Hedetniemi, S. T., Slater, P. J.: Fundamentals of Domination in graphs, Marcel Dekker, Inc., New York (1998).

[9] Karpovsky, M. G.; Chakrabarty, K.; Levitin, L. B.: On a new class of codes for identifying vertices in graphs, IEEE Transactions on Information Theory, Vol. IT-44, 599–611 (1998).

[10] Lobstein, A.: Watching systems, identifying, locating-dominating and discriminating codes in graphs, Bibliography.
http://www.infres.enst.fr/~lobstein/bibLOCDOMetID.html

[11] Slater, P. J.: Domination and location in acyclic graphs, Networks, Vol. 17, 55–64 (1987).

[12] Slater, P. J.: Dominating and reference sets in a graph, Journal of Mathematical and Physical Sciences, Vol. 22, 445–455 (1988).