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Arithmetic completely regular codes

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In this paper, we explore completely regular codes in the Hamming graphs and related graphs. Experimental evidence suggests that many completely regular codes have the property that the eigenvalues of the code are in arithmetic progression. In order to better understand these "arithmetic completely regular codes", we focus on cartesian products of completely regular codes and products of their corresponding coset graphs in the additive case. Employing earlier results, we are then able to prove a theorem which nearly classifies these codes in the case where the graph admits a completely regular partition into such codes (e.g, the cosets of some additive completely regular code). Connections to the theory of distance-regular graphs are explored and several open questions are posed.

Keywords: completely regular code, distance-regular graph, Hamming graph, coset graph, Leonard's Theorem

1 Introduction

In this paper, we present the theory of completely regular codes in the Hamming graph enjoying the property that the eigenvalues of the code are in arithmetic progression. We call these codes *arithmetic completely regular codes* and we classify them under some additional conditions. Our results are strongest when the Hamming graph admits a completely regular partition into such codes (e.g., the partition into cosets of some additive completely regular code), since it is known that the quotient graph obtained from any such partition is distance-regular. Using Leonard's Theorem, the list of possible quotients is determined, with a few special cases left as open problems. In the case of linear arithmetic completely regular codes, more can be said.

Aside from the application of Leonard's Theorem, the techniques employed are mainly combinatorial and products of codes as well as decompositions of codes into "reduced" codes play a fundamental role.

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Indeed, whenever a product of two completely regular codes is completely regular, all three codes are necessarily arithmetic and taking cartesian products with entire Hamming graphs does not affect either the completely regular or the arithmetic property.

These results are not so relevant to classical coding theory; they belong more to the theory of distance-regular graphs. Coset graphs of additive completely regular codes provide an important family of distance-regular graphs (and dual pairs of polynomial association schemes). While the local structure of an arbitrary distance-regular graph is hard to recover from its spectrum alone, these coset graphs naturally inherit much of their combinatorial structure from the Hamming graphs from which they arise. What is interesting here is how this additional information gives leverage in the combinatorial analysis of these particular distance-regular coset graphs, a tool not available in the unrestricted case. Recent related work appears in [RZ15].

It is important here to draw a connection between the present paper and a companion paper [KLM10]. In [KLM10], the first three authors introduced the concept of *Leonard completely regular codes* and developed the basic theory for them. We also gave several important families of Leonard completely regular codes in the classical distance-regular graphs, demonstrating their fundamental role in structural questions about these graphs. In fact, we wonder if there exist any completely regular codes in the Hamming graphs with large covering radius that are neither Leonard nor closely related to Leonard codes. The class of arithmetic codes we introduce in this paper is perhaps the most important subclass of the Leonard completely regular codes in the Hamming graphs and something similar is likely true for the other classical families, but this investigation is left as an open problem.

The layout of the paper is as follows. After an introductory section outlining the required background, we explore products of completely regular codes in Hamming graphs. First noting (Proposition 3.1) that a completely regular product must arise from completely regular constituents, we determine in Proposition 3.4 exactly when the product of two completely regular codes in two Hamming graphs is completely regular. At this point, the role of the arithmetic property becomes clear and we understand that Lemma 3.3 gives a generic form for the quotient matrix of such a code.

With this preparatory material out of the way, we are then ready to present the main results in Section 3.2. Applying the celebrated theorem of Leonard, Theorem 3.7 imposes powerful limitations on the combinatorial structure of a quotient of a Hamming graph when the underlying completely regular partition is composed of arithmetic codes. Stronger results are obtained in Proposition 3.9, Theorem 3.12 and Proposition 3.13 when one makes additional assumptions about the minimum distance of the codes or the specific structure of the quotient. When C is a linear completely regular code with the arithmetic property and C has minimum distance at least three and covering radius at most two, we show that C is closely related to some Hamming code. These results are summarized in Theorem 3.16, which gives a full classification of possible codes and quotients in the linear case (always assuming the arithmetic and completely regular properties) and Corollary 3.17 which characterizes Hamming quotients of Hamming graphs.

2 Preliminaries and definitions

In this section, we summarize the background material necessary to understand our results. Most of what is covered here is based on Chapter 11 in the monograph [BCN89] by Brouwer, Cohen and Neumaier. The theory of codes in distance-regular graphs began with Delsarte [Del73]. The theory of association schemes is also introduced in the book [BI84] of Bannai and Ito while connections between these and related material (especially equitable partitions) can be found in Godsil [God93]. See also the survey [DKT14] by Van Dam, Koolen and Tanaka for recent updates on the theory of distance-regular graphs.

2.1 Distance-regular graphs

Suppose that Γ is a finite, undirected, connected graph with vertex set $V\Gamma$. For vertices x and y in $V\Gamma$, let $d(x,y)$ denote the distance between x and y, i.e., the length of a shortest path connecting x and y in Γ. Let D denote the diameter of Γ; i.e., the maximal distance between any two vertices in $V\Gamma$. For $0 \le i \le D$ and $x \in V\Gamma$, let $\Gamma_i(x) := \{y \in V\Gamma \mid d(x,y) = i\}$ and put $\Gamma_{-1}(x) := \emptyset$, $\Gamma_{D+1}(x) := \emptyset$. The graph Γ is called *distance-regular* whenever it is regular of valency k, and there are integers b_i, c_i $(0 \le i \le D)$ so that for any two vertices x and y in $V\Gamma$ at distance i, there are precisely c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$. It follows that there are exactly $a_i := k - b_i - c_i$ neighbors of y in $\Gamma_i(x)$. The numbers c_i, b_i and a_i are called the *intersection numbers* of Γ and we observe that $c_0 = 0$, $b_D = 0$, $a_0 = 0$, $c_1 = 1$ and $b_0 = k$. The array $\iota(\Gamma) := \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$ is called the *intersection array* of Γ.

From now on, assume Γ is a distance-regular graph of valency $k \ge 2$ and diameter $D \ge 2$. Define A_i to be the square matrix of size $|V\Gamma|$ whose rows and columns are indexed by $V\Gamma$ with entries

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } d(x,y) = i \\ 0 & \text{otherwise} \end{cases} \qquad (0 \le i \le D,\ x, y \in V\Gamma).$$

We refer to A_i as the i^{th} *distance matrix* of Γ. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. Since Γ is distance-regular, we have

$$AA_{i-1} = b_{i-2}A_{i-2} + a_{i-1}A_{i-1} + c_iA_i \quad (2 \le i \le D)$$

so that $A_i = p_i(A)$ for some polynomial $p_i(t)$ of degree i. Let $\mathbb{A}$ be the *Bose-Mesner algebra*, the matrix algebra over $\mathbb{C}$ generated by A. Then $\dim \mathbb{A} = D + 1$ and $\{A_i \mid 0 \le i \le D\}$ is a basis for $\mathbb{A}$. As $\mathbb{A}$ is semi-simple and commutative, $\mathbb{A}$ has also a basis of pairwise orthogonal idempotents $\left\{E_0 = \frac{1}{|V\Gamma|}J, E_1, \ldots, E_D\right\}$, where J denotes the all ones matrix. We call these matrices the *primitive idempotents* of Γ. As $\mathbb{A}$ is closed under the entrywise (or Hadamard or Schur) product $\circ$, there exist real numbers q_{ij}^{ℓ}, called the *Krein parameters*, such that

$$E_i \circ E_j = \frac{1}{|V\Gamma|} \sum_{\ell=0}^{D} q_{ij}^{\ell} E_\ell \quad (0 \le i, j \le D).$$

The graph Γ is called Q-*polynomial* if there exists an ordering $E_0, \ldots, E_D$ of the primitive idempotents and there exist polynomials q_i of degree i such that $E_i = q_i(E_1)$, where the polynomial q_i is applied entrywise to E_1. We recall that the distance-regular graph Γ is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ provided its Krein parameters satisfy

- $q_{ij}^{\ell} = 0$ unless $|j - i| \le \ell \le i + j$;
- $q_{ij}^{\ell} \ne 0$ whenever $\ell = |j - i|$ or $\ell = i + j \le D$.

By an *eigenvalue* of Γ, we mean an eigenvalue of $A = A_1$. Since Γ has diameter D, it has at least $D+1$ distinct eigenvalues; but since Γ is distance-regular, it has exactly $D+1$ distinct eigenvalues and $E_0, \ldots, E_D$ are the orthogonal projections onto the eigenspaces. We denote by θ_i the eigenvalue associated with E_i and, aside from the convention that $\theta_0 = k$, the valency of Γ, we make no further assumptions at this point about the eigenvalues except that they are distinct.

2.2 Codes in distance-regular graphs

Let Γ be a distance-regular graph with distinct eigenvalues $\theta_0 = k, \theta_1, \ldots, \theta_D$. By a *code* in Γ, we simply mean any nonempty subset C of $V\Gamma$. We call C *trivial* if $|C| = 1$ or $C = V\Gamma$ and *non-trivial* otherwise. For $|C| > 1$, the *minimum distance* of C, $\delta(C)$, is defined as

$$\delta(C) := \min\{\, d(x,y) \,|\, x, y \in C, x \neq y \,\}$$

and for any $x \in V\Gamma$ the distance $d(x, C)$ from x to C is defined as

$$d(x,C) := \min\{\, d(x,y) \,|\, y \in C \,\}.$$

The number

$$\rho(C) := \max\{\, d(x,C) \,|\, x \in V\Gamma \,\}$$

is called the *covering radius* of C.

For a code C and for $0 \le i \le \rho := \rho(C)$, define

$$C_i = \{\, x \in V\Gamma \,|\, d(x,C) = i \,\}.$$

Then $\Pi(C) := \{C_0 = C, C_1, \ldots, C_\rho\}$ is the *distance partition* of $V\Gamma$ with respect to code C.

A partition $\Pi = \{P_0, P_1, \ldots, P_\ell\}$ of $V\Gamma$ is called *equitable* if, for all i and j, the number of neighbors a vertex in P_i has in P_j is independent of the choice of vertex in P_i. Following Neumaier [Neu92], we say a code C in Γ is *completely regular* if this distance partition $\Pi(C)$ is equitable(i). In this case the following quantities are well-defined:

$$\begin{aligned} \gamma_i &= \left|\{y \in C_{i-1} \,|\, d(x,y) = 1\}\right|, \\ \alpha_i &= \left|\{y \in C_i \,|\, d(x,y) = 1\}\right|, \\ \beta_i &= \left|\{y \in C_{i+1} \,|\, d(x,y) = 1\}\right| \end{aligned}$$

where x is chosen from C_i. The numbers $\gamma_i, \alpha_i, \beta_i$ are called the *intersection numbers* of code C. Observe that a graph Γ is distance-regular if and only if each vertex is a completely regular code and these $|V\Gamma|$ codes all have the same intersection numbers. Set the tridiagonal matrix

$$U = U(C) := \begin{pmatrix} \alpha_0 & \beta_0 & & & \\ \gamma_1 & \alpha_1 & \beta_1 & & \\ & \gamma_2 & \alpha_2 & \beta_2 & \\ & & \ddots & \ddots & \ddots \\ & & & \gamma_\rho & \alpha_\rho \end{pmatrix}.$$

For C a completely regular code in Γ, we say that η is an *eigenvalue of C* if η is an eigenvalue of the *quotient matrix* U defined above. By $\mathrm{Spec}(C)$, we denote the set of eigenvalues of C. Note that, since $\gamma_i + \alpha_i + \beta_i = k$ for all i, $\theta_0 = k$ belongs to $\mathrm{Spec}(C)$.

(i) This definition of a completely regular code is equivalent to the original definition, due to Delsarte [Del73].

2.3 Completely regular partitions

Given a partition Π of the vertex set of a graph Γ (into nonempty sets), we define the *quotient graph* Γ/Π on the cells of Π by calling two cells $C, C' \in \Pi$ adjacent if $C \neq C'$ and Γ contains an edge joining some vertex of C to some vertex of C'. A partition Π of $V\Gamma$ is *completely regular* if it is an equitable partition of Γ and all $C \in \Pi$ are completely regular codes with the same intersection numbers. If Π is completely regular we write $\mathrm{Spec}(\Pi) = \mathrm{Spec}(C)$ for $C \in \Pi$ and we say $\rho := \rho(C)$ is the covering radius of Π.

Proposition 2.1 (cf. [BCN89, p352-3])**.** *Let Π be a completely regular partition of any distance-regular graph Γ such that each $C \in \Pi$ has intersection numbers γ_i, α_i and β_i $(0 \leq i \leq \rho)$. Then Γ/Π is a distance-regular graph with intersection array*

$$\iota(\Gamma/\Pi) = \left\{ \frac{\beta_0}{\gamma_1}, \frac{\beta_1}{\gamma_1}, \ldots, \frac{\beta_{\rho-1}}{\gamma_1}; 1, \frac{\gamma_2}{\gamma_1}, \ldots, \frac{\gamma_\rho}{\gamma_1} \right\},$$

remaining intersection numbers $a_i = \frac{\alpha_i - \alpha_0}{\gamma_1}$, and eigenvalues $\frac{\theta_j - \alpha_0}{\gamma_1}$ for $\theta_j \in \mathrm{Spec}(C)$. All of these lie among the eigenvalues of the matrix $\frac{1}{\gamma_1}(A - \alpha_0 I)$.

Proposition 2.2. *Let Π be a non-trivial completely regular partition of a distance-regular graph Γ and assume that $\mathrm{Spec}(\Pi) = \{\eta_0 \geq \eta_1 \geq \cdots \geq \eta_\rho\}$. Then $\eta_\rho \leq \alpha_0 - \gamma_1$.*

Proof: By Proposition 2.1, the eigenvalues of Γ/Π are $\frac{\eta_0 - \alpha_0}{\gamma_1}, \frac{\eta_1 - \alpha_0}{\gamma_1}, \ldots, \frac{\eta_\rho - \alpha_0}{\gamma_1}$. As Γ/Π has at least one edge, it follows that its smallest eigenvalue is at most -1. Hence $\frac{\eta_\rho - \alpha_0}{\gamma_1} \leq -1$. $\square$

Question. Let C be a non-trivial completely regular code in a distance-regular graph Γ. Let $\theta := \min\{\eta \,|\, \eta \in \mathrm{Spec}(C)\}$. Is it true that $\theta \leq \alpha_0 - \gamma_1$?

3 Codes in the Hamming graph

Let X be a finite abelian group. A *translation distance-regular graph* on X is a distance-regular graph Γ with vertex set X such that if x and y are adjacent then $x + z$ and $y + z$ are adjacent for all $x, y, z \in X$. A code $C \subseteq X$ is called *additive* if for all $x, y \in C$, also $x - y \in C$; i.e., C is a subgroup of X. If C is an additive code in a translation distance-regular graph Γ on X, then we obtain the usual coset partition $\Delta(C) := \{C + x \,|\, x \in X\}$ of X; whenever C is a completely regular code, it is easy to see that $\Delta(C)$ is a completely regular partition. The quotient graph $\Gamma/\Delta(C)$ is usually referred to as the *coset graph* of C in Γ. An important special class of additive codes are the linear codes. Here, assume X is a vector space over some finite field $GF(q)$; a code $C \subseteq X$ is a *linear code* if C is a vector subspace of X. In our results, we are careful to assume only additivity when possible, but some results only apply in the linear case.

An important result of Brouwer, Cohen and Neumaier [BCN89, p353] states that every translation distance-regular graph of diameter at least three defined on an elementary abelian group X is necessarily a coset graph of some additive completely regular code in some Hamming graph. Of course, the Hamming graph itself is a translation graph, usually in a variety of ways.

Let $\mathcal{Q}$ be an abelian group with $|\mathcal{Q}| = q$. Then we may identify the vertex set of the Hamming graph $H(n, q)$ with the group $X = \mathcal{Q}^n$, two vertices being adjacent if they agree in all but one coordinate position. It is well known (e.g., [BCN89, Theorem 9.2.1]) that $H(n, q)$ has eigenvalues $\theta_j = n(q-1) - qj$,

$0 \le j \le n$. The corresponding "natural" ordering $E_0, E_1, \ldots, E_n$ is a Q-polynomial ordering. When $q > 2$ and $n > 2$, this is the unique Q-polynomial ordering for $H(n,q)$ but when $q = 2$ and n is even, $H(n,2)$ is also Q-polynomial with respect to the ordering $E_0, E_{n-1}, E_2, E_{n-3}, \ldots, E_1, E_n$ [BI84, p305]. (This second ordering plays a role in the addendum.)

For the remainder of this paper, we will consider q-ary codes of length n as subsets of the vertex set of the Hamming graph $H(n,q)$. In this section we will focus on q-ary completely regular codes, i.e., q-ary codes of length n which are completely regular in $H(n,q)$.

3.1 Products of completely regular codes

We now recall the cartesian product of two graphs Γ and Σ. This new graph $\Gamma \times \Sigma$ has vertex set $V\Gamma \times V\Sigma$ and adjacency $\sim$ defined by $(x,y) \sim (u,v)$ precisely when either $x = u$ and $y \sim v$ in Σ or $x \sim u$ in Γ and $y = v$. Now let C be a code in Γ and let C' be a code in Σ. The cartesian product of C and C' is the code defined by

$$C \times C' := \{(x, x') \in V\Gamma \times V\Sigma \,|\, x \in C,\, x' \in C'\}\,.$$

We are interested in the cartesian product of codes in the Hamming graphs $H(n,q)$ and $H(n',q')$. Note that if C and C' are additive codes then $C \times C'$ is also additive. Also, if C is a (not necessarily additive) completely regular code in $H(n,q)$ with vertex set $\mathcal{Q}^n$, then $\mathcal{Q} \times C$ is a completely regular code in $H(n+1,q)$. Also if Π is a completely regular partition of $H(n,q)$ then $\mathcal{Q} \times \Pi := \{\mathcal{Q} \times P \,|\, P \in \Pi\}$ is a completely regular partition of $H(n+1,q)$. We say a completely regular code C in $H(n,q)$ is *non-reduced* if $C \cong \mathcal{Q} \times C'$ for some C' in $H(n-1,q)$, and *reduced* otherwise. In similar fashion we say that a completely regular partition is *non-reduced* or *reduced*.

The next three results will determine exactly when the cartesian product of two arbitrary codes in two Hamming graphs is completely regular.

Proposition 3.1. *Let C and C' be non-trivial codes in the Hamming graphs $H(n,q)$ and $H(n',q')$ respectively. If the cartesian product $C \times C'$ is completely regular in $H(n,q) \times H(n',q')$, then both C and C' themselves must be completely regular codes in their respective graphs.*

Proof: Let $\{C_0 = C, C_1, \ldots, C_\rho\}$ be the distance partition of $H(n,q)$ with respect to C. Then we easily see that every vertex (x,y) of $C_i \times C'$ is at distance i from $C \times C'$ in the product graph. Moreover, the neighbors of this vertex which lie at distance $i-1$ from the product code are precisely $\{(u,y) \,|\, u \sim x,\, u \in C_{i-1}\}$; so the size of the set $\{u \in C_{i-1} \,|\, u \sim x\}$ must be independent of the choice of $x \in C_i$. This shows that the intersection numbers γ_i are well-defined for C. An almost identical argument gives us the intersection numbers β_i. This shows that C is a completely regular code and, swapping the roles of C and C', we show the same is true for C'. □

It is well known that equitable partitions are preserved under products. If Π is an equitable partition in any graph Γ and Δ is an equitable partition in another graph Σ, then $\{P \times P' \,|\, P \in \Pi,\; P' \in \Delta\}$ is an equitable partition in the cartesian product graph $\Gamma \times \Sigma$. The following special case will prove useful in our next proposition.

Lemma 3.2. *Let C be a completely regular code in $H(n,q)$ with distance partition $\Pi = \{C_0 = C, \ldots, C_\rho\}$, and let C' be a completely regular code in $H(n',q')$ with distance partition $\Pi' = \{C'_0 = C', \ldots, C'_{\rho'}\}$.*

(a) The partition

$$\left\{C_i \times C'_j \,|\, 0 \le i \le \rho,\, 0 \le j \le \rho'\right\}$$

is an equitable partition in the product graph $H(n,q) \times H(n',q')$.

(b) A vertex in $C_i \times C'_j$ has all its neighbors in

$$\left[(C_{i-1} \cup C_i \cup C_{i+1}) \times C'_j\right] \cup \left[C_i \times \left(C'_{j-1} \cup C'_j \cup C'_{j+1}\right)\right].$$

(c) If C has intersection numbers $\gamma_i, \alpha_i, \beta_i$ and C' has intersection numbers $\gamma'_i, \alpha'_i, \beta'_i$, then in the product graph, a vertex in $C_i \times C'_j$ has: γ_i neighbors in $C_{i-1} \times C'_j$; β_i neighbors in $C_{i+1} \times C'_j$; γ'_j neighbors in $C_i \times C'_{j-1}$; β'_j neighbors in $C_i \times C'_{j+1}$; and $\alpha_i + \alpha'_j$ neighbors in $C_i \times C'_j$.

Proof: Straightforward. □

Lemma 3.3. *Let k, γ, β and ρ be positive integers. The tridiagonal matrix*

$$L = \begin{pmatrix} \alpha_0 & \rho\beta & & & & \\ \gamma & \alpha_1 & (\rho-1)\beta & & & \\ & 2\gamma & \alpha_2 & (\rho-2)\beta & & \\ & & \ddots & \ddots & \ddots & \\ & & & (\rho-1)\gamma & \alpha_{\rho-1} & \beta \\ & & & & \rho\gamma & \alpha_\rho \end{pmatrix}$$

where $\alpha_i = k - i\gamma - (\rho - i)\beta$ $(0 \le i \le \rho)$, has eigenvalues $\mathrm{Spec}(L) = \{k - ti \,|\, 0 \le i \le \rho\}$ *where $t = \gamma + \beta$.*

Proof: By direct verification. (This is not new. See, for example, type (IIC) in Terwilliger [Ter92, Theorem 2.1].) □

Proposition 3.4. *Let C be a non-trivial completely regular code in $H(n,q)$ with $\rho := \rho(C) \ge 1$ and intersection numbers α_i, β_i and γ_i $(0 \le i \le \rho)$. Let C' be a non-trivial completely regular code in $H(n',q')$ with $\rho' := \rho(C') \ge 1$ and intersection numbers α'_i, β'_i and γ'_i $(0 \le i \le \rho')$. Then $C \times C'$ is a completely regular code in $H(n,q) \times H(n',q')$ if and only if there exist integers γ and β satisfying (a) and (b):*

(a) $\gamma_i = \gamma i$ for $0 \le i \le \rho$ and $\gamma'_i = \gamma i$ for $0 \le i \le \rho'$;

(b) $\beta_{\rho-i} = \beta i$ for $0 \le i \le \rho$ and $\beta'_{\rho'-i} = \beta i$ for $0 \le i \le \rho'$.

In this case, $C \times C'$ has covering radius $\bar\rho := \rho + \rho'$ and intersection numbers $\bar\gamma_i = \gamma i$ and $\bar\beta_i = \beta(\bar\rho - i)$ for $0 \le i \le \bar\rho$, and all three codes — C, C' and $C \times C'$ — are arithmetic completely regular codes.

Proof: Assume, without loss, that $\rho \le \rho'$. ($\Rightarrow$) Suppose first that $C \times C'$ is a completely regular code. From Lemma 3.2, we see that $C \times C'$ has covering radius $\bar\rho = \rho + \rho'$. For $0 \le j \le \bar\rho$, let

$$S_j = \{(x,y) \,|\, d((x,y), C \times C') = j\}.$$

Then it follows easily from Lemma 3.2(b) that

$$S_j = \bigcup_{h+i=j} C_h \times C'_i.$$

Moreover, by Lemma 3.2(c), a vertex in $C_h \times C'_i$ has $\gamma_h + \gamma'_i$ neighbors in S_{j-1} and $\beta_h + \beta'_i$ neighbors in S_{j+1}. For $j = 1$, this forces $\gamma_1 = \gamma'_1 =: \gamma$. Assume inductively that $\gamma_i = \gamma i$ and $\gamma'_i = \gamma i$ for $i < j$. Then, considering a vertex in

$$S_j = C_r \times C'_{j-r} \cup \cdots \cup C_{j-s} \times C'_s$$

(where $r = \max\{0, j - \rho'\}$ and $s = \max\{0, j - \rho\}$), we find

$$\gamma_r + \gamma'_{j-r} = \gamma_{r+1} + \gamma'_{j-r-1} = \cdots = \gamma_{j-s} + \gamma'_s.$$

For $j \leq \rho$, this gives $\gamma_j = \gamma j = \gamma'_j$. For $\rho < j \leq \rho'$, we deduce, $\gamma'_j = \gamma j$. So we have (a) by induction. A symmetrical argument establishes part (b).

($\Leftarrow$) Considering the same partition of S_j into cells of the form $C_i \times C'_{j-i}$, we obtain the converse in a straightforward manner: if C and C' have intersection numbers given by (a) and (b), then their cartesian product $C \times C'$ is completely regular in the product graph.

The fact that all three codes are arithmetic completely regular codes when (a) and (b) hold now follows directly from Lemma 3.3. □

Remark 3.5. Note that, if $q \neq q'$, then the product graph is not distance-regular. Although this case is not of primary interest, there are examples where $C \times C'$ can still be a completely regular code (in the sense of Neumaier) in such a graph. For instance, suppose that C is a perfect code with covering radius one in $H(n, q)$ and C' is a perfect code with covering radius one in $H(n', q')$. If we happen to have $n(q-1) = n'(q'-1)$, then, by the above proposition, $C \times C'$ is a completely regular code in $H(n, q) \times H(n', q')$ with intersection numbers $\gamma_1 = 1$, $\gamma_2 = 2$, $\beta_0 = 2n(q-1)$ and $\beta_1 = n(q-1)$.

From now on we will look at a completely regular partition Π of $H(n, q)$ into arithmetic completely regular codes with $\mathrm{Spec}(\Pi) = \{n(q-1), n(q-1) - qt, \ldots, n(q-1) - q\rho t\}$. As a direct consequence of Proposition 2.2, we have the following proposition which says that α_0 must be quite large unless at least one of ρ (the covering radius) or t (the spectral gap) is large.

Proposition 3.6. *Let Π be a non-trivial completely regular partition of the Hamming graph $H(n, q)$ such that Π has covering radius ρ and $\mathrm{Spec}(\Pi) = \{n(q-1), n(q-1) - qt, \ldots, n(q-1) - q\rho t\}$ for some t. Then $n(q-1) - \alpha_0 \leq q\rho t$.*

3.2 Classification

In this section, we will employ results from [KLM10], in which *Leonard completely regular codes* are defined and studied. But we must first address an error in that paper. Lemma 6.3 in [KLM10] claims that every harmonic completely regular code is Leonard. This fails to hold in some cases where the Q-polynomial ordering does not place the eigenvalues of Γ in decreasing order. See the addendum below for a discussion of these counterexamples as well as a corrected statement of that lemma along with a full proof. Since arithmetic completely regular codes are harmonic completely regular codes in the Hamming graphs (with respect to the natural ordering), Lemma 6.3 in the addendum applies and all arithmetic completely regular codes are indeed Leonard codes.

Now we will classify the possible quotients Γ/Π where Π is a completely regular partition of the Hamming graph $\Gamma = H(n,q)$ into arithmetic completely regular codes. Recall that a *Doob graph* [BCN89, p27] is any graph formed as a cartesian product of $s \geq 1$ copies of the Shrikhande graph [BCN89, p104-5] and $t \geq 0$ copies of K_4. A graph so constructed has the same parameters as the Hamming graph $H(2s+t, 4)$ but quite different local structure; e.g., the graph contains a four-clique if and only if $t \neq 0$ but always contains maximal cliques of size three. The Doob graphs are therefore important in the theory of distance-regular graphs [BCN89, Section 9.2B].

Theorem 3.7. *Let Γ be the Hamming graph $H(n,q)$, and let Π be a completely regular partition of $V\Gamma$. Assume that Π has covering radius $\rho \geq 3$ and eigenvalues $\mathrm{Spec}(\Pi) = \{n(q-1), n(q-1)-qt, \ldots, n(q-1) - q\rho t\}$ for some t. Then Γ/Π has diameter ρ and is isomorphic to one of the following:*

(a) a folded cube;

(b) a Hamming graph;

(c) a Doob graph;

(d) an incidence graph of some 2-$(16,6,2)$ design (i.e., with intersection array $\{6,5,4;1,2,6\}$).

Proof: Denote $\eta_i := k - i\tau$ $(0 \leq i \leq \rho)$, where $k := \frac{n(q-1)-\alpha_0}{\gamma_1}$ and $\tau := \frac{qt}{\gamma_1}$. Then by Proposition 2.1, Γ/Π is a distance-regular graph with eigenvalues $\{\eta_i \,|\, 0 \leq i \leq \rho\}$ and diameter ρ. As stated above, each code in partition Π is a Leonard completely regular code. So we can apply the proof of [KLM10, Proposition 4.5] to conclude that the quotient graph Γ/Π is Q-polynomial with respect to $\overline{E}_0, \overline{E}_1, \ldots, \overline{E}_\rho$ where $\overline{E}_i$ is the primitive idempotent of Γ/Π corresponding to η_i. (Note that the statement of Proposition 4.5 in [KLM10] assumes that Π is a coset partition, but the proof applies in the more general situation we face here.)

By Leonard's Theorem [BI84, p263] (cf. [Leo82]), Q-polynomial distance-regular graphs with diameter $D \geq 3$ fall into seven types, namely, (I), (IA), (II), (IIA), (IIB), (IIC), and (III). See also [Ter92, Section 2]. As the eigenvalues of Γ/Π are in arithmetic progression and $\rho \geq 3$, neither type (I), (IA), (II), (IIA) nor (III) can occur. The only remaining possibilities are types (IIB) and (IIC). Terwilliger [Ter88] showed that a Q-polynomial distance-regular graph of type (IIB) with diameter $D \geq 3$ is either the folded $(2D+1)$-cube, or has the same intersection numbers as the folded $2D$-cube. By [BCN89, Theorem 9.2.7], it follows that Γ/Π is either a folded n-cube for $n \geq 7$ or has intersection array $\{6,5,4;1,2,6\}$. For type (IIC), we can easily check (using $c_1 = 1$) that Γ/Π has the same parameters as a Hamming graph (cf. [Tan11, Proposition 6.2]). But Egawa [Ega81] showed that a distance-regular graph with the same parameters as a Hamming graph must be a Hamming graph or a Doob graph. Hence the theorem is proved. □

Remark 3.8. **A.** Theorem 3.7 also holds when Γ is a Doob graph. There are completely regular partitions of Doob graphs whose corresponding quotient graphs are Hamming graphs with $q \neq 4$. For example, let $s \geq 2$ be an integer. In any Doob graph Γ of diameter at least five containing at least one four-clique, there exists an additive completely regular code, say C, with covering radius one and sixteen cosets [KM00]. Let $n \geq 1$ be an integer. Then C^n is an additive completely regular code with covering radius n in Γ^n, and the coset graph $\Gamma^n/\Delta(C^n)$ is isomorphic to $H(n, 16)$. Moreover, when Γ contains a subgraph isomorphic to $H((4^s - 1)/3, 4)$, we may also obtain $H(n, 4^s)$ as a quotient.

B. There are three nonisomorphic graphs with the same intersection array, $\{6, 5, 4; 1, 2, 6\}$, as the folded 6-cube. They are the point-block incidence graphs of the 2-$(16, 6, 2)$ designs. We do not know any example in which the quotient graph has intersection array $\{6, 5, 4; 1, 2, 6\}$, but is not the folded 6-cube.

C. If Γ is a Hamming graph and C is an additive completely regular code in Γ satisfying the eigenvalue conditions of Theorem 3.7, then $\Gamma/\Delta(C)$ is always a Hamming graph, a Doob graph or a folded cube. We do not know any example where Γ is a Hamming graph and Π is a completely regular partition such that Γ/Π is a Doob graph. But we will prove below in Proposition 3.13 that this cannot occur when $\Pi = \Delta(C)$ for some linear completely regular code C.

D. We wonder whether it is true that if the quotient graph Γ/Π in Theorem 3.7 is isomorphic to a Hamming graph or a Doob graph, then each cell in Π can be expressed as a cartesian product of completely regular codes with covering radius at most two. We will show in Theorem 3.16 below that this holds when Π is the coset partition of a linear completely regular code satisfying the eigenvalue conditions of Theorem 3.7.

In the special case when no two adjacent vertices are in the same cell of partition Π, we can strengthen Theorem 3.7 by looking at the images of the cliques in Γ.

Proposition 3.9. *Let Γ be the Hamming graph $H(n, q)$. Let Π be any completely regular partition of Γ where each code C in Π has minimum distance $\delta(C) \geq 2$.*

(a) If $\Gamma/\Pi \cong H(m, q')$ then $q' \geq q$.

(b) If $q \geq 4$ then Γ/Π is not isomorphic to any Doob graph.

(c) If $q \geq 3$ then Γ/Π can not have the same intersection array as any folded cube of diameter at least two (including $\{6, 5, 4; 1, 2, 6\}$).

(d) Suppose further that $\delta(C) \geq 3$, and that Γ/Π has intersection array $\{6, 5, 4; 1, 2, 6\}$. Then $q = 2$ and Γ/Π is the folded 6-cube.

Proof: Since each C in Π satisfies $\delta(C) \geq 2$, the vertices in any clique in Γ belong to pairwise distinct cells in Π. So, in the quotient Γ/Π, every edge lies in a clique of size at least q, the clique number of $H(n, q)$. This immediately implies (a)–(c) as: $H(m, q')$ has clique number q'; any Doob graph has maximal cliques of size three; a folded cube other than K_4 has clique number two, as does any graph with intersection array $\{6, 5, 4; 1, 2, 6\}$.

For part (d), we clearly need only consider the case where $q = 2$. As $\delta(C) \geq 3$, we use [BCN89, Theorem 11.1.8] to determine $n = \delta(C) = 6$ from the intersection array of Γ/Π, and this means that Γ/Π is the folded 6-cube. $\square$

As a side remark, we note that every perfect 1-code in $H(n, q)$, additive or not, gives us an equitable partition — the partition into the code and its translates by the various vectors of Hamming weight one — with complete quotient graph. (This was a small oversight in [BCN89], near the bottom of p355.)

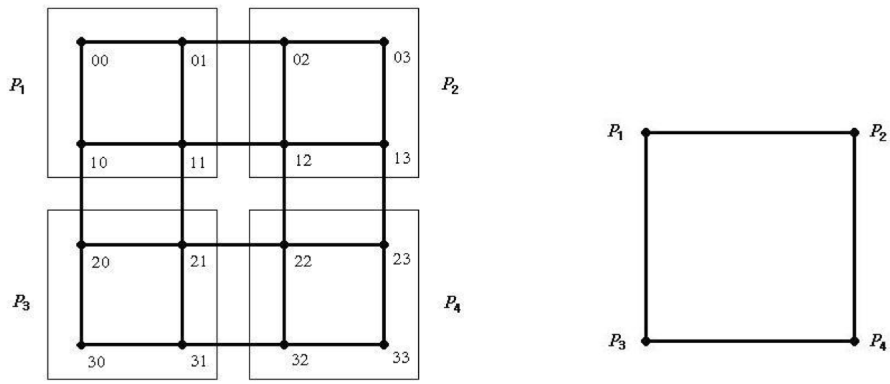

Fig. 1: Hamming graph $H(2,4)$ and its quotient graph $H(2,2)$

Example 3.10. Let Γ be the Hamming graph $H(2,4)$ and let $\Pi = \Pi(\Gamma) = \{P_1, P_2, P_3, P_4\}$, where $P_1 = \{00, 01, 10, 11\}$, $P_2 = \{02, 03, 12, 13\}$, $P_3 = \{20, 21, 30, 31\}$, and $P_4 = \{22, 23, 32, 33\}$. Then the quotient graph Γ/Π is the Hamming graph $H(2,2)$ as depicted in Figure 1. Observe that each cell in partition Π is expressible as a cartesian product:

$$P_1 = \{0,1\} \times \{0,1\}, \quad P_2 = \{0,1\} \times \{2,3\},$$
$$P_3 = \{2,3\} \times \{0,1\}, \quad P_4 = \{2,3\} \times \{2,3\}.$$

Finally note that this example can be easily extended to give $H(n,q)$ as a quotient of $H(n,sq)$, $n,s \geq 1$.

From Example 3.10, we see that the n-cube and the folded n-cube can be quotients of $H(n,4)$. So Proposition 3.9(a) does not hold when $\delta(C) = 1$, even for additive codes.

From now on, we will focus on linear completely regular codes over $GF(q)$. We begin with a comment. Recall that C reduced means that it is impossible to express C in the form $\mathcal{Q} \times C'$, where $\mathcal{Q} = GF(q)$ in this case. Let $\mathbf{e}_i$ denote the vertex of $H(n,q)$ with i^{th} position 1 and all other positions 0.

Lemma 3.11. *Let C be a non-trivial linear completely regular code over $GF(q)$ of length n. Then C is reduced if and only if $\delta(C) \geq 2$.*

Proof: If C contains a vector of weight one, say $\zeta\mathbf{e}_i$, then by linearity $\mathbf{e}_i$ is in C and hence C is not reduced. The converse is also obvious. $\square$

We next show that, if Π is the coset partition of $H(n,q)$ with respect to a linear code with a quotient of Hamming type, then each cell in Π may be expressed as a cartesian product of completely regular codes with covering radius one.

Theorem 3.12. *Let Γ be the Hamming graph $H(n,q)$. Let C be a linear completely regular code with minimum distance $\delta(C) \geq 2$ in Γ and let $\Delta(C)$ be the coset partition of Γ with respect to C.*

(a) Suppose $\Gamma/\Delta(C) \cong H(m,q')$. Then m divides n and $C \cong \prod_{i=1}^{m} C^{(i)}$, where each $C^{(i)}$ is a linear q-ary completely regular code with covering radius one and length n/m;

(b) If $\Gamma/\Delta(C)$ is isomorphic to a folded m-cube, $m \geq 4$, then $q = 2$ and

$$C \cong \mathrm{nullsp}\left[\begin{array}{c|c|c|c|c} H_1 & \cdots & H_{m-1} & J \end{array}\right],$$

where H_i is the $(m-1)\times\gamma_1$ matrix with row i all ones and all other rows all zero and J is the $(m-1)\times\gamma_1$ all ones matrix, where $\gamma_1 = \gamma_1(C)$.

Proof: Note that $\ell_i := \{\zeta\mathbf{e}_i \mid \zeta \in GF(q)\}$ $(1 \leq i \leq n)$ are the singular lines (cliques of size q) in Γ containing $\mathbf{0}$. Since $\delta(C) \geq 2$, the quotient map $\pi : V\Gamma \to V(\Gamma/\Delta(C))$ sending $\mathbf{x} \in V\Gamma$ to $\mathbf{x} + C$, maps each singular line in Γ into some singular line in $\Gamma/\Delta(C)$. Hence we can consider the equivalence relation $\equiv$ on $\{1,\ldots,n\}$ given by $i \equiv j$ if $\pi(\ell_i)$ and $\pi(\ell_j)$ belong to the same singular line in $\Gamma/\Delta(C)$.

First we prove (a). Since C is linear, we have either $\pi(\ell_i) = \pi(\ell_j)$ or $\pi(\ell_i) \cap \pi(\ell_j) = \{\mathbf{0}\}$ for $1 \leq i, j \leq n$. Since there are precisely $(q'-1)\gamma_1$ neighbors of $\mathbf{0}$ which are mapped to a fixed singular line in $\Gamma/\Delta(C)$ containing $\pi(\mathbf{0}) = C$, each equivalence class has size $(q'-1)\gamma_1/(q-1) = n/m$, showing that m divides n.

Let $R_1,\ldots,R_m$ denote the equivalence classes of $\equiv$. For $1 \leq i \leq m$, let $D_i := \{\mathbf{x} \in V\Gamma \mid \mathrm{supp}(\mathbf{x}) \subseteq R_i\}$.

Claim 1. *For any $\mathbf{d} \in D_i$, $\pi(\mathbf{d})$ belongs to the singular line of $\Gamma/\Delta(C)$ corresponding to R_i.*

Proof of Claim 1: Write $\mathbf{d} = \sum_{j\in\mathrm{supp}(\mathbf{d})} \zeta_j\mathbf{e}_j$. If $\pi(\zeta_j\mathbf{e}_j) = \pi(\zeta_h\mathbf{e}_h)$ for some distinct $j, h \in \mathrm{supp}(\mathbf{d})$, then $\zeta_j\mathbf{e}_j - \zeta_h\mathbf{e}_h \in C$ and we may replace $\mathbf{d}$ by $\tilde{\mathbf{d}} := \mathbf{d} - (\zeta_j\mathbf{e}_j - \zeta_h\mathbf{e}_h)$. Note that $\mathrm{wt}(\tilde{\mathbf{d}}) < \mathrm{wt}(\mathbf{d})$. Hence we may assume from the first that $\pi(\zeta_j\mathbf{e}_j) \neq \pi(\zeta_h\mathbf{e}_h)$ for distinct $j, h \in \mathrm{supp}(\mathbf{d})$. We now prove the result by induction on $\mathrm{wt}(\mathbf{d})$. The result is obvious when $\mathrm{wt}(\mathbf{d}) = 1$, so assume $\mathrm{wt}(\mathbf{d}) \geq 2$. Take distinct $j, h \in \mathrm{supp}(\mathbf{d})$, and let $\mathbf{d}' := \mathbf{d} - \zeta_j\mathbf{e}_j$ and $\mathbf{d}'' := \mathbf{d} - \zeta_h\mathbf{e}_h$. Then we have $\pi(\mathbf{d}') \neq \pi(\mathbf{d}'')$; for otherwise $\zeta_j\mathbf{e}_j - \zeta_h\mathbf{e}_h = \mathbf{d}'' - \mathbf{d}' \in C$, so $\pi(\zeta_j\mathbf{e}_j) = \pi(\zeta_h\mathbf{e}_h)$, a contradiction. Now $\pi(\mathbf{d})$ is adjacent to both $\pi(\mathbf{d}')$ and $\pi(\mathbf{d}'')$ which, by induction, belong to the above singular line of $\Gamma/\Delta(C)$, hence $\pi(\mathbf{d})$ must also belong to the same singular line. □

For $1 \leq i \leq m$, let Σ_i be the subgraph of Γ induced on D_i, so $\Sigma_i \cong H(n/m, q)$. Let $C^{(i)} := D_i \cap C$. Note that $C^{(i)} \neq D_i$ since C is reduced.

Claim 2. *$C^{(i)}$ is a linear q-ary completely regular code with covering radius one in Σ_i with $U(C^{(i)}) = \left(\begin{smallmatrix} 0 & \beta \\ \gamma_1 & \beta-\gamma_1 \end{smallmatrix}\right)$ where $\beta := n(q-1)/m$.*

Proof of Claim 2: Let $\mathbf{d} \in D_i$ with $\mathbf{d} \notin C^{(i)}$. By Claim 1, $\pi(\mathbf{d})$ is adjacent to $\pi(\mathbf{0})$; i.e., $d(\mathbf{d}, C) = 1$. Let $\mathbf{x} \in \Gamma(\mathbf{d}) \cap C$, so $\mathbf{x} = \mathbf{d} + \zeta\mathbf{e}_j$ for some j and non-zero $\zeta \in GF(q)$. Then we have $\pi(-\zeta\mathbf{e}_j) = \pi(\mathbf{d})$ and hence $\pi(\ell_j)$ must be contained in the singular line of $\Gamma/\Delta(C)$ corresponding to R_i. This shows $j \in R_i$ and we have $\Gamma(\mathbf{d}) \cap C = \Gamma(\mathbf{d}) \cap C^{(i)}$. This gives us the result. □

Since C is additive, we have $\prod_{i=1}^m C^{(i)} \subseteq C$. Moreover, by Proposition 3.4, $\prod_{i=1}^m C^{(i)}$ is a completely regular code with the same intersection numbers as C (i.e., the intersection numbers of $H(m, q')$ scaled by γ_1). So $C = \prod_{i=1}^m C^{(i)}$.

The proof of (b) is almost identical with that of (a) above. First note that, as the folded m-cube has singular line size two, it follows that $q = 2$ and $|R_i| = \gamma_1 = n/m$ $(1 \leq i \leq m)$. Moreover, each $C^{(i)}$ is a binary linear completely regular code with covering radius one, length γ_1 and $U(C^{(i)}) = \left(\begin{smallmatrix} 0 & \gamma_1 \\ \gamma_1 & 0 \end{smallmatrix}\right)$; i.e.,

$C^{(i)}$ is the binary even weight code. Let $C' := \prod_{i=1}^{m} C^{(i)} \subseteq C$. Then we have $\dim C = n - m + 1$, $\dim C' = m(\gamma_1 - 1) = n - m$, and $\Gamma/\Delta(C') \cong H(m,2)$. It follows that C/C' is the repetition code in $H(m,2)$ and hence $C = C' \cup (\mathbf{x} + C')$ for any word $\mathbf{x}$ with odd weight inside each equivalence class R_i. □

The next result rules out Doob graphs as quotients.

Proposition 3.13. *If C is a linear q-ary completely regular code of length n, then $H(n,q)/\Delta(C)$ is not isomorphic to a Doob graph.*

Proof: Since the Doob graph has 2^{2m} vertices for some m, we restrict to the case where q is a power of 2. Without loss of generality, we may assume C is non-trivial and reduced, so $\delta(C) \geq 2$ by Lemma 3.11. Now if $H(n,q)/\Delta(C)$ is a Doob graph, then by Proposition 3.9(b), we have $q \leq 3$, i.e., $q = 2$. Moreover, every vertex in $H(n,q)/\Delta(C)$ is contained in a maximal clique of size three. Let $\{C, C_1, C_2\}$ be such a clique. We may assume $\mathbf{e}_1 \in C_1$ and $\mathbf{e}_2 \in C_2$. Then some neighbor of $\mathbf{e}_1$, say $\mathbf{e}_1 + \mathbf{e}_i$, belongs to C_2. By linearity, $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_i \in C$, so we have $i \neq 1, 2$, and $\mathbf{e}_i + C$ is adjacent in $H(n,q)/\Delta(C)$ to all of C, C_1, C_2, contradicting our choice of a maximal clique. □

The following result was also shown by Borges et al. [BRZ10]. We give a proof for the convenience of the reader.

Theorem 3.14. *Let C be a linear completely regular code over $GF(q)$ with minimum distance $\delta(C) \geq 3$ in $H(n,q)$.*

(a) If $\mathrm{Spec}(C) = \{n(q-1), n(q-1) - qt\}$ for some $t \geq 1$, then C is a Hamming code.

(b) If $\mathrm{Spec}(C) = \{n(q-1), n(q-1) - qt, n(q-1) - 2qt\}$ for some $t \geq 1$, then $C = D \times D$ where D is a Hamming code, or $q = 2$ and C is an extended Hamming code, or $q = 2$, $n = 5$ and C is the repetition code.

Proof: (a) Since C has covering radius one and minimum distance three, it follows immediately that C has to be perfect.

(b) Assume first that $t = 1$. Then, by a result of Meyerowitz [BGKM03, Theorem 7], C takes the form $\{0\}^{\ell} \times \mathcal{Q}^{n-\ell}$ for some $0 \leq \ell \leq n$ where $\mathcal{Q} = GF(q)$; so C is either trivial or reduced, a contradiction. Hence we have $t \geq 2$.

The coset graph $\Sigma := H(n,q)/\Delta(C)$ is a connected strongly regular graph (i.e., a distance-regular graph with diameter two); consider the associated parameters k, λ, μ in the standard notation so that $\iota(\Sigma) = \{k, k-1-\lambda; 1, \mu\}$. Note that $\mu > 0$ since Σ is connected. Using [BCN89, Theorem 1.3.1], we obtain

$$\begin{aligned} k &= n(q-1), \\ \lambda &= (n(q-1) - qt)(n(q-1) - 2qt + 3), \\ \mu &= n(q-1) + (n(q-1) - qt)(n(q-1) - 2qt). \end{aligned}$$

Since $\mu \leq k$, we have $\frac{qt}{q-1} \leq n \leq \frac{2qt}{q-1}$. This, combined with the constraint $0 \leq \lambda \leq k-2$, gives us three possibilities: (i) $n = \frac{qt}{q-1}$; (ii) $n = \frac{2qt-3}{q-1}$; (iii) $n = \frac{2qt-2}{q-1}$.

For (i), we have $k = qt$, $\lambda = 0$, $\mu = qt$, and hence Σ is the complete bipartite graph $K_{qt,qt}$. Note that $2qt$ must divide q^n, so we have $q-1 = 1$, i.e., $q = 2$, since n is an integer. Moreover, C is an even weight code; for otherwise Σ would have a cycle of odd length. In particular, we have $\delta(C) \geq 4$. By a result of Brouwer [Bro90], it follows that a truncation of C is a linear completely regular code with covering radius one, so it is a perfect code with minimum distance three. Hence C is an extended Hamming code.

For (ii), we have $0 < \mu = 6 - qt$. Since $t \geq 2$, we have $q = t = 2$. So $n = 5$, $k = 5$, $\lambda = 0$, $\mu = 2$, and hence Σ has the intersection array $\{5, 4; 1, 2\}$ of the folded 5-cube. By [BCN89, Theorem 9.2.7], Σ is isomorphic to the folded 5-cube. By Theorem 3.12(b) and since $\gamma_1 = 1$, it follows that C is the repetition code with length five.

For (iii), we have $k = 2qt-2$, $\lambda = qt-2$, $\mu = 2$, so Σ has the intersection array $\{2qt-2, qt-1; 1, 2\}$ of the Hamming graph $H(2, qt-1)$. By [Ega81], Σ is isomorphic to $H(2, qt-1)$. (Note that $qt - 1 \neq 4$ as $q \geq 2$ and $t \geq 2$.) It follows from Theorem 3.12(a) that C is the cartesian product of two copies of the Hamming code with length $n/2$. □

Let C be a non-trivial, reduced linear completely regular code over $GF(q)$ in $\Gamma = H(n, q)$ with intersection numbers $\gamma_i, \alpha_i, \beta_i$ and parity check matrix H. Note that the columns of H are all non-zero since C is reduced. Consider the equivalence relation $\simeq$ defined on $\{1, \ldots, n\}$ by[(ii)] $i \simeq j$ if the i^{th} and j^{th} columns of H are linearly dependent, or equivalently, $i = j$ or $\{i, j\} = \operatorname{supp}(\mathbf{x})$ for some $\mathbf{x} \in C$. Note that such $\mathbf{x}$ is unique up to scalar multiplication; for otherwise we would have $\mathbf{e}_i, \mathbf{e}_j \in C$ and hence C is non-reduced, a contradiction. One also easily checks that all equivalence classes have size γ_1.

Lemma 3.15. *With the above notation, let $P \subseteq \{1, \ldots, n\}$ be a complete set of representatives for the equivalence classes of $\simeq$. Let $D := \{\mathbf{x} \in V\Gamma \mid \operatorname{supp}(\mathbf{x}) \subseteq P\}$ and let $\Sigma \cong H(n/\gamma_1, q)$ be the subgraph of Γ induced on D. Then $C_P := C \cap D$ is a linear completely regular code in Σ with $\delta(C_P) \geq 3$, $\rho(C_P) = \rho(C)$, and intersection numbers $\gamma_i/\gamma_1, \alpha_i/\gamma_1, \beta_i/\gamma_1$. Moreover, $\Gamma/\Delta(C)$ and $\Sigma/\Delta(C_P)$ are isomorphic.*

Proof: Let $\Pi(C) = \{C_0 = C, C_1, \ldots, C_\rho\}$ be the distance partition of $V\Gamma$ with respect to C, where $\rho := \rho(C)$. We claim that $\{C_0 \cap D = C_P, C_1 \cap D, \ldots, C_\rho \cap D\}$ is the distance partition of $V\Sigma$ with respect to C_P. Indeed, let $\mathbf{x} \in C_i \cap D$, and suppose $\mathbf{x} + \zeta_i \mathbf{e}_i \in C_{i-1}$ for some non-zero $\zeta_i \in GF(q)$. Then for any j with $i \simeq j$, there is a unique non-zero $\zeta_j \in GF(q)$ such that $\zeta_i \mathbf{e}_i - \zeta_j \mathbf{e}_j \in C$, so $\mathbf{x} + \zeta_j \mathbf{e}_j = \mathbf{x} + \zeta_i \mathbf{e}_j - (\zeta_i \mathbf{e}_i - \zeta_j \mathbf{e}_j) \in C_{i-1}$. Hence it follows that $\mathbf{x}$ has precisely γ_i/γ_1 neighbors in $C_{i-1} \cap D$. By the same reasoning, $\mathbf{x}$ has α_i/γ_1 neighbors in $C_i \cap D$ and β_i/γ_1 neighbors in $C_{i+1} \cap D$. In particular, C_P is a completely regular code in Σ with $\rho(C_P) = \rho$. Since any word in $V\Gamma$ is congruent modulo C to some word in D, it also follows that $\Gamma/\Delta(C)$ and $\Sigma/\Delta(C_P)$ are (canonically) isomorphic. Finally, since C_P can not contain words with weight two, we have $\delta(C_P) \geq 3$. □

Theorem 3.16. *Let C be a non-trivial, reduced, linear completely regular code over $GF(q)$ in $H(n, q)$ with intersection number $\gamma_1 = \gamma_1(C)$. Suppose that* $\operatorname{Spec}(C) = \{n(q-1), n(q-1) - qt, \ldots, n(q-1) - q\rho t\}$ *for some t. Then one of the following holds:*

(a) $q = 2$ *and*

$$C \cong \operatorname{nullsp} \underbrace{\left[\begin{array}{c|c|c} M & \cdots & M \end{array}\right]}_{\gamma_1 \text{ copies}},$$

(ii) With the notation in the proof of Theorem 3.12, we have $i \simeq j$ if and only if $\pi(\ell_i) = \pi(\ell_j)$.

where $M = [I \mid \mathbf{1}]$ is a parity check matrix for a binary repetition code, and the quotient is a folded cube;

(b) $\rho = 1$ *and*

$$C \cong \operatorname{nullsp} \underbrace{\begin{bmatrix} H & | & \cdots & | & H \end{bmatrix}}_{\gamma_1 \text{ copies}},$$

where H is a parity check matrix for some Hamming code, and the quotient is a complete graph;

(c) $\rho = 2$, $q = 2$ *and*

$$C \cong \operatorname{nullsp} \underbrace{\begin{bmatrix} E & | & \cdots & | & E \end{bmatrix}}_{\gamma_1 \text{ copies}},$$

where E is a parity check matrix for a fixed extended Hamming code, and the quotient is a regular complete bipartite graph;

(d) $\rho \geq 2$ *and*

$$C \cong \underbrace{C_1 \times \cdots \times C_1}_{\rho \text{ copies}},$$

where C_1 is a completely regular code with covering radius one, and the quotient is a Hamming graph.

Proof: The reader can easily check that examples (a)–(d) are all completely regular. (See also Bier [Bie87].)

Recall the code C_P defined in Lemma 3.15, which is a completely regular code in $\Sigma \cong H(n/\gamma_1, q)$ with $\delta(C_P) \geq 3$ and still satisfies our arithmetic hypothesis. Note that if $\delta(C) \geq 3$ then we have $\gamma_1 = 1$ so $C = C_P$. Note also that if K is a parity check matrix for C_P, then we have $C \cong \operatorname{nullsp} \begin{bmatrix} K \mid \cdots \mid K \end{bmatrix}$ (γ_1 copies). Hence we may restrict to the case where $\delta(C) \geq 3$.

The result follows from Theorem 3.14 when $\rho \in \{1, 2\}$, so we assume $\rho \geq 3$. Then Theorem 3.7 tells us the possible quotients, i.e., a folded cube, a Hamming graph, a Doob graph, or the incidence graph of a symmetric 2-$(16, 6, 2)$ design. If the coset graph is a Hamming graph, then (d) applies by Theorem 3.12(a). If it is a folded m-cube, then Theorem 3.12(b) gives us a parity check matrix for C and one easily sees that rearranging columns of this matrix, we obtain the parity check matrix given in (a). By Proposition 3.13, a Doob graph cannot arise as coset graph. Finally, if the coset graph has intersection array $\{6, 5, 4; 1, 2, 6\}$, then it is indeed the folded 6-cube by Proposition 3.9(d) and we are done. □

We remark that the above result includes the following generalization of the result of Bier [Bie87] concerning coset graphs which are isomorphic to Hamming graphs (cf. [BCN89, p354]):

Corollary 3.17. *Let C be a reduced linear completely regular code in $H(n, q)$ with intersection number $\gamma_1 = \gamma_1(C)$, whose coset graph is a Hamming graph $H(m, q')$. Then one of the following holds:*

(a) $\rho = 1$ *and*

$$C \cong \operatorname{nullsp} \underbrace{\begin{bmatrix} H & | & \cdots & | & H \end{bmatrix}}_{\gamma_1 \text{ copies}},$$

where H is a parity check matrix for some Hamming code;

(b) $\rho \geq 2$ *and*

$$C \cong \underbrace{C_1 \times \cdots \times C_1}_{\rho \text{ copies}},$$

where C_1 *is a completely regular code with covering radius one.*

Proof: The eigenvalues of $H(m, q')$ are in arithmetic progression, and hence so are the eigenvalues of C in view of Proposition 2.1. Now the result follows from Theorem 3.16. □

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Addendum: Correcting an error in Lemma 6.3 of [KLM10]

We now point out an error in the statement of Lemma 6.3 in our companion paper [KLM10]; a key hypothesis is missing from the statement of the lemma.

We first recall several definitions from [KLM10].

Let Γ be a distance-regular graph. We say that a subspace W of $V := \mathbb{C}^{V\Gamma}$ is *Schur closed* if $\mathbf{u} \circ \mathbf{v} \in W$ whenever $\mathbf{u}$ and $\mathbf{v}$ themselves belong to W. The *Schur closure* of a subspace (or set) is the smallest Schur closed subspace containing it. A subspace is easily seen to be Schur closed if and only if it contains some basis of pairwise orthogonal 01-vectors.

Let $E_0, E_1, \ldots, E_D$ be any ordering of the primitive idempotents of Γ. Let $V_j \subseteq V$ denote the eigenspace corresponding to E_j. For a completely regular code $C \subseteq V\Gamma$ with $\operatorname{Spec}(C) = \{\theta_{i_0}, \ldots, \theta_{i_\rho}\}$, consider the characteristic vector $\mathbf{x} = \mathbf{x}_C \in V$ of C and the *outer distribution module* $\mathbb{A}\mathbf{x}$ which has $\left\{E_{i_0}\mathbf{x}, \ldots, E_{i_\rho}\mathbf{x}\right\}$ as basis. It is well known that $\mathbb{A}\mathbf{x}$ is Schur closed whenever C is completely regular. We say [KLM10, Definition 4.1] C is *Leonard* (with respect to this ordering) if

$$\mathbf{u}^{(\ell)} := \underbrace{\mathbf{u} \circ \cdots \circ \mathbf{u}}_{\ell \text{ times}} \in (V_{i_0} + V_{i_1} + \cdots + V_{i_\ell}) \setminus (V_{i_0} + V_{i_1} + \cdots + V_{i_{\ell-1}}),$$

for $1 \leq \ell \leq \rho$, where $\mathbf{u} := E_{i_1}\mathbf{x}$. On the other hand, when $E_0, E_1, \ldots, E_D$ is a Q-polynomial ordering, we say that completely regular code C is *harmonic* (with respect to this ordering) if $\operatorname{Spec}(C) = \{\theta_{ti} \mid 0 \leq i \leq \rho\}$ for some t [KLM10, Definition 6.1].

We now give the corrected statement of Lemma 6.3 in [KLM10] together with a full proof[iii].

Lemma 6.3. *Let Γ be a distance-regular graph which is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of its primitive idempotents. Let C be a completely regular code which is harmonic with respect to this ordering with $\operatorname{Spec}(C) = \{\theta_{ti} \mid 0 \leq i \leq \rho\}$. If $E_t\mathbf{x}$ has $\rho + 1$ distinct entries, then C is a Leonard completely regular code. In particular, whenever the Q-polynomial ordering satisfies $\theta_0 > \theta_1 > \cdots > \theta_D$, every completely regular code which is harmonic with respect to this ordering is also Leonard with respect to this ordering.*

Proof: The outer distribution module $\mathbb{A}\mathbf{x}$ has $\{E_{ti}\mathbf{x} \mid 0 \leq i \leq \rho\}$ as a basis. Since $\mathbb{A}\mathbf{x}$ is Schur closed, there exist numbers $\omega_{\ell,j}$ such that $E_t\mathbf{x} \circ E_{tj}\mathbf{x} = \sum_{\ell=0}^{\rho} \omega_{\ell,j} E_{t\ell}\mathbf{x}$. Using a fundamental result of Cameron, Goethals and Seidel (see, e.g., [KLM10, Theorem 2.2]), we see that, since Γ is Q-polynomial, we must have $\omega_{\ell,j} = 0$ whenever $|\ell - j| > 1$.

(iii) We found a typographical error in the proof of Lemma 6.3 in [KLM10]: where we write "$\leq t$" and "≤ 1" in the displayed equations, it should instead read "$= t$" and "$= 1$", respectively.

Now we employ the assumption (missing in the statement of Lemma 6.3 in [KLM10]) that $E_t\mathbf{x}$ has $\rho+1$ distinct entries to show that $\omega_{\ell,j} \neq 0$ when $\ell = j+1$ for $0 \leq j < \rho$. Under this assumption, the Schur closure of $\{E_t\mathbf{x}\}$ has dimension $\rho+1$. If there were some $j < \rho$ with $\omega_{\ell,j} = 0$ for all $\ell > j$, this would give us a proper Schur closed $\mathbb{A}$-submodule $\{E_{ti}\mathbf{x} \,|\, 0 \leq i \leq j\}$ which is impossible. Now let $\mathbf{u} := E_t\mathbf{x}$. Then we use $\omega_{\ell,j} = 0$ for $\ell > j+1$ to see that

$$\mathbf{u}^{(j+1)} \in V_0 + V_t + \cdots + V_{t(j+1)}$$

while

$$\mathbf{u}^{(j+1)} \notin V_0 + V_t + \cdots + V_{tj}$$

follows since $\omega_{j+1,j} \neq 0$ and, by induction, $E_{tj}\mathbf{u}^{(j)} \neq 0$. Thus the code C is Leonard with respect to this ordering.

The most common Q-polynomial orderings known have $\theta_0 > \theta_1 > \cdots > \theta_D$. In this case θ_t is the second largest eigenvalue of the tridiagonal matrix $U = U(C)$ and standard results about Sturm sequences (see, e.g., [Mar94, Mar00]) guarantee that $E_t\mathbf{x}$ has $\rho+1$ distinct entries in all such cases. □

Of particular relevance here is the observation that the lemma applies to all arithmetic completely regular codes in the Hamming graphs. For if C is arithmetic, then it is harmonic, and thus Leonard, with respect to the natural Q-polynomial ordering for $H(n,q)$. But what of the second Q-polynomial ordering in the case $q=2$?

In [Dic95], Dickie showed that the only distance-regular graphs with diameter $D \geq 5$ which admit more than one Q-polynomial ordering are dual polar graph $[{}^2A_{2D-1}(r)]$ ($r \geq 2$, a prime power), the D-cube $H(D,2)$ (when D is even), the halved $(2D+1)$-cube, the folded $(2D+1)$-cube, and the regular n-gons $n = 2D, 2D+1$. In all these cases, one must be careful in applying Lemma 6.3 in [KLM10] when the Q-polynomial ordering does not correspond to the natural ordering of graph eigenvalues from largest to smallest.

We end with an example which underscores the necessity of the condition that $E_t\mathbf{x}$ have $\rho+1$ distinct entries.

The trivial completely regular codes $C = \{0,1\}^{n-\rho} \times \{0\}^{\rho}$ in the D-cube $H(n,2)$ have $\mathrm{Spec}(C) = \{n, n-2, n-4, \ldots, n-2\rho\}$ so, with respect to the standard ordering, these codes are arithmetic (with $t=1$), hence both harmonic and Leonard. But when n is even, the n-cube admits a second Q-polynomial structure, with eigenvalue ordering

$$\theta_j = (-1)^j(n-2j) \quad (0 \leq j \leq n).$$

With respect to this ordering, $E_1\mathbf{x}$ has only $1 + \lfloor \rho/2 \rfloor$ distinct entries and covering radius ρ. Under this ordering,

$$\mathrm{Spec}(C) = \{\theta_0, \theta_2, \ldots, \theta_{2\lfloor \rho/2 \rfloor},\ \theta_{n+1-2\lceil \rho/2 \rceil}, \ldots, \theta_{n-3}, \theta_{n-1}\}$$

so that, in the particular case $\rho = n-1$, C is arithmetic. Varying n, we find an infinite family of examples (codes of size two) which are harmonic but not Leonard.