Discrete Mathematics and Theoretical Computer Science DMTCS vol. **17:3**, 2016, 263–284

Edge Disjoint Hamilton Cycles in Knödel Graphs

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received 31st July 2014, *revised 3rd Aug. 2015, 24th June 2016*, *accepted 7th July 2016*.

The vertices of the Knödel graph $W_{\Delta,\,n}$ on $n \geq 2$ vertices, n even, and of maximum degree Δ, $1 \leq \Delta \leq \lfloor log_2(n) \rfloor$, are the pairs $(i,\, j)$ with $i = 1,\, 2$ and $0 \leq j \leq \frac{n}{2} - 1$. For $0 \leq j \leq \frac{n}{2} - 1$, there is an edge between vertex $(1,\, j)$ and every vertex $(2,\, j + 2^k - 1\,(mod\ \frac{n}{2}))$, for $k = 0,\, 1,\, 2, \ldots, \Delta - 1$. Existence of a Hamilton cycle decomposition of $W_{k,\,2^k}$, $k \geq 6$ is not yet known, see Discrete Appl. Math. 137 (2004) 173-195. In this paper, it is shown that the k-regular Knödel graph $W_{k,2^k}$, $k \geq 6$ has $\lfloor \frac{k}{2} \rfloor - 1$ edge disjoint Hamilton cycles.

Keywords: Knödel Graphs, Hamilton Cycle Decomposition, Tensor Product, Bieulerian Graph

1 Introduction.

All graphs considered here are simple and finite unless otherwise stated. Let C_k (resp.P_k) denote the cycle (resp. path) on k vertices. For a graph G, if its edge set $E(G)$ can be partitioned into $E_1, E_2, \ldots, E_k$ such that $\langle E_i \rangle \cong H$, for all i, $1 \leq i \leq k$, then we say that H decomposes G. A *k-factor* of G is a k-regular spanning subgraph of it. A *k-factorization* of a graph G is a partition of the edge set of G into $E_1,\, E_2,\, \ldots,\, E_s$ such that $\langle E_i \rangle$, $1 \leq i \leq s$, is a k-factor. We say that a k-regular graph G admits a *Hamilton cycle decomposition*, if the edge set of G can be partitioned into Hamilton cycles or Hamilton cycles together with a 1-factor according as k is even or odd, respectively. If $H_1,\, H_2,\, \ldots,\, H_k$ are edge disjoint subgraphs of G such that $\bigcup_{i=1}^{k} H_i = G$, then we write $G = H_1 \oplus H_2 \oplus \ldots \oplus H_k$. The complete graph on n vertices is denoted by K_n. Let G be a bipartite graph with bipartition (X, Y), where $X = \{x_0, x_1, \ldots, x_{n-1}\}$, $Y = \{y_0, y_1, \ldots, y_{n-1}\}$; the edge $x_i y_{i+\ell}$ is called an edge of jump ℓ from X to Y in G, where addition is taken modulo n; the same edge is called an edge of jump $n - \ell$ from Y to X. If G contains the edges $F_\ell(X, Y) = \{x_i y_{i+\ell} | 0 \leq i \leq n-1,$ where addition in the subscript is taken modulo $n\}$, $0 \leq \ell \leq n-1$, then we say that G has the 1-factor of *jump ℓ from X to Y*. Clearly, if $G = K_{n,n}$, then $E(G) = \bigcup_{i=0}^{n-1} F_i(X, Y)$. Note that $F_i(X, Y) = F_{n-i}(Y, X)$, $0 \leq i \leq n-1$, where we assume $F_n(X, Y) = F_0(X, Y) = F_0(Y,\, X)$.

An *anti-directed path* P is a digraph, whose underlying graph is a path, in which any two consecutive arcs of P are either directed toward or away from the common incident vertex in P. Similarly, we define anti-directed cycles, see Figures 1(a) and 1(b). A digraph $\vec{G} = (V,\, A)$ is denoted by $\vec{G}$. A digraph

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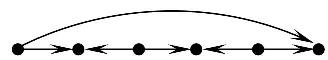
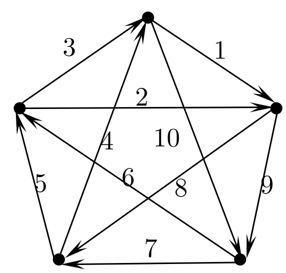
$\vec{G} = (V,\, A)$ is said to be k-*diregular* if $d^+(x) = k = d^-(x)$ for every $x \in V$. If x is the tail, and y is the head of an arc of $\vec{G}$, then it is denoted by $\overrightarrow{xy}$. A digraph $\vec{G}$ is called *aneulerian* if $\vec{G}$ has an Euler tour T such that any two consecutive edges of T are either directed toward or away from the common incident vertex in $\vec{G}$, see Figures 1(c) and 1(d). A directed graph is called *bieulerian* if it is both eulerian (that is, it contains a directed Euler tour) and aneulerian, see Figures 1(c) and 1(d); consequently, a digraph which admits an aneulerian tour cannot be k-diregular for $k \geq 3$.

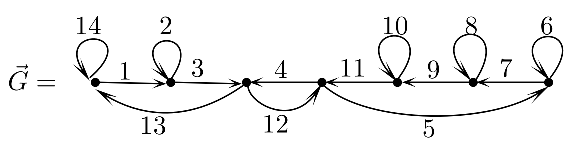
(a). An anti-directed path on 6 vertices



(b). An anti-directed cycle on 6 vertices



(c). An aneulerian tour in $\vec{K}_5$ is described by the arc sequence $1,\, 2,\, 3,\, \ldots,\, 10$, taken in that order.



(d) An aneulerian tour in $\vec{G}$ with directed loops at some of its vertices is given by the arc sequence $1,\, 2,\, \ldots,\, 13,\, 14$, where the directed loops are traversed in the forward or backward direction in order to have an aneulerian tour in $\vec{G}$.

Fig. 1: Examples for anti-directed path, anti-directed cycle and aneulerian tours.

For two graphs G and H their *tensor product*, denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge in $G \times H$ whenever $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. Similarly, for two digraphs $\vec{G}$ and $\vec{H}$ their *tensor product*, denoted by $\vec{G} \times \vec{H}$, has vertex set $V(\vec{G}) \times V(\vec{H})$ in which

$\overrightarrow{(g_1, h_1)(g_2, h_2)}$ is an arc in $\vec{G} \times \vec{H}$ whenever $\overrightarrow{g_1 g_2} \in A(\vec{G})$ and $\overrightarrow{h_1 h_2} \in A(\vec{H})$, where $A(\vec{G})$ denotes the arc set of $\vec{G}$. Note that if xx is a loop at x in G (resp. H), then $x \times V(H)$ (resp. $V(G) \times x$) induces a copy of H (resp. G) in $G \times H$. A *circulant* graph $X = Circ(n;\, L)$ is a graph with vertex set $V(X) = \{u_0,\, u_1,\, \ldots,\, u_{n-1}\}$ and edge set $E(X) = \{u_i u_{i+\ell} \,|\, i \in \mathbb{Z}_n,\, \ell \in L\}$, where $L \subseteq \{1,\, 2,\, \ldots,\, \lfloor \frac{n}{2} \rfloor\}$. The elements of L are called *distances* of the circulant graph and L is called the *set of distances.* A *circulant* digraph $\vec{X} = \overrightarrow{Circ}(n;\, L)$ is a digraph with vertex set $V(\vec{X}) = \{u_0,\, u_1,\, \ldots,\, u_{n-1}\}$ and arc set $A(\vec{X}) = \{\overrightarrow{u_i u_{i+\ell}} \,|\, i \in \mathbb{Z}_n,\, \ell \in L\}$, where $L \subseteq \{1,\, 2,\, \ldots,\, n-1\}$. The elements of L are called *distances* of the circulant digraph. The underlying graph of a digraph $\vec{D}$ is the undirected graph obtained from $\vec{D}$ by simply deleting the orientations of the arcs of $\vec{D}$. For graph theoretical terms not defined here, see [2,6].

Knödel graph was originally introduced in [22]. The family of Knödel graphs has been formally defined by Fraigniaud and Peters [7]. Knödel graph $W_{\Delta,n}$ is a regular graph of even order n and degree Δ, $1 \le \Delta \le \lfloor log_2(n) \rfloor$. Knödel graphs are used as competitors for hypercubes in the domains of broadcasting and gossiping. The gossiping problem, as described by Knödel in [22] is the following: "Given n persons, each having an information, want to distribute their information among them in binary calls, each call taking a constant time, how long must it take before each knows all the information among them?" Broadcasting is a similar problem where only one person (the originator) has all the information that needs to be distributed to a group of people in binary calls. Consequently, they deal with problems in dissemination of information in interconnection networks. Every interconnection network can be represented by means of a graph. If this graph has n vertices, the minimum time required for broadcasting is $\lceil \log_2 n \rceil$. Such graphs are known as minimal broadcasting graphs. For more details on minimal broadcasting and gossiping graphs, see [11,19,20]. There are several papers dealing with Knödel graphs, especially because some subfamilies of Knödel graphs have good properties in terms of broadcasting and gossiping and Fault-Tolerance, see [1,10,14,18,21,23]. In particular, for $n = 2^k$, the Knödel graph $W_{k,\, 2^k}$, of order n and degree k, turns out to be a minimum broadcast graph. It is known that, the diameter of $W_{k,2^k}$ is $\lceil \frac{k+2}{2} \rceil$, see [9]. It is known that $W_{k,\, 2^k}$ is vertex-transitive but not edge-transitive, see [8]. Recently, it has been proved, see [4], that the automorphism group of $W_{k,2^k}$ is the dihedral group $D_{2^{k-1}}$ and in the same paper a short proof is given for the diameter of the Knödel graph $W_{k,2^k}$. For other properties of the Knödel graphs and modified Knödel graphs, see [5,12,13,15–17].

It is known that $W_{k-1,2^k-2}$ is Hamilton cycle decomposable and $W_{k,\, 2^k}$ is bipancyclic, that is, every cycle of length ℓ, $4 \le \ell \le 2^k$ exists, see [8]. A detailed account of various properties the Knödel graphs and comparison with other interconnection networks has been given in the survey [8].

It is not yet known whether the Knödel graphs $W_{k,\, 2^k}$, $k \ge 6$ admit a Hamilton cycle decomposition or not [8]. In this paper, we prove that the k-regular Knödel graph $W_{k,\, 2^k}$, $k \ge 6$ has $\lfloor \frac{k}{2} \rfloor - 1$ edge disjoint Hamilton cycles, that is, the edge set of $W_{k,\, 2^k}$, $k \ge 6$ can be partitioned into $\lfloor \frac{k}{2} \rfloor - 1$ Hamilton cycles together with a 2-factor, if k is even and $\lfloor \frac{k}{2} \rfloor - 1$ Hamilton cycles together with a 3-factor, if k is odd.

2 Definitions and Preliminaries.

Definition 2.1 [8] *The Knödel graph on $n \ge 2$ vertices, n even, and maximum degree Δ, $1 \le \Delta \le \lfloor log_2(n) \rfloor$, is denoted by $W_{\Delta,\, n}$. The vertices of $W_{\Delta,\, n}$ are the pairs $(i,\, j)$ with $i = 1,\, 2$ and $0 \le j \le \frac{n}{2} - 1$. For every $0 \le j \le \frac{n}{2} - 1$, there is an edge between vertex $(1,\, j)$ and every vertex $(2,\, j + 2^k - 1\, (mod\ \frac{n}{2}))$, for $k = 0, 1, 2, \ldots, \Delta - 1$. It is a bipartite graph containing the jump 1-factors*

$2^k - 1,\ k = 0,\ 1,\ 2, \ldots, \Delta - 1$.

For each k, $0 \le k \le \Delta - 1$, the edges $(1,\ j)(2,\ j + 2^k - 1\ (mod\ \frac{n}{2}))$, $0 \le j \le \frac{n}{2} - 1$, induce a 1-factor of jump $2^k - 1$ in $W_{\Delta,\, n}$.

The union of the 1-factors of jumps 0 and 1 from X to Y of $W_{\Delta,\, n}$ is a Hamilton cycle, where we assume that $X = \{(1,\ j) \,|\, 0 \le j \le \frac{n}{2} - 1\}$ and $Y = \{(2,\ j) \,|\, 0 \le j \le \frac{n}{2} - 1\}$ are the bipartition of $W_{\Delta,\, n}$. It is known that the graphs $W_{k,\, 2^k}$, $k \le 5$, are Hamilton cycle decomposable, see [8]. Based on this observation the following problem is raised in [8].

Problem 2.1 [8] *Is $W_{k,\, 2^k}$ Hamilton cycle decomposable for any $k \ge 2$?*

In this paper, we prove that the k-regular Knödel graph $W_{k,\, 2^k}$, $k \ge 6$ has $\lfloor \frac{k}{2} \rfloor - 1$ edge disjoint Hamilton cycles.

Definition 2.2 (Bipartite incident graph [8]). *Let $\vec{G} = (V,\ A)$ be a digraph of order n, with $V = \{0,\ 1,\ \ldots,\ n-1\}$. The bipartite incident graph of $\vec{G}$ is the (undirected) bipartite graph $H = (V_1,\ V_2,\ E)$ of order $2n$, where $V_i = \{0_i,\ 1_i,\ \ldots,\ (n-1)_i\}$, $i = 1,\ 2$, and that for any arc $\overrightarrow{ij}$ of $\vec{G}$, there corresponds an edge $i_1 j_2 \in E(H)$ with $i_1 \in V_1$ and $j_2 \in V_2$ and for each $i \in V$, there is an edge $i_1 i_2 \in E(H)$, where $i_1 \in V_1$ and $i_2 \in V_2$ in H.*

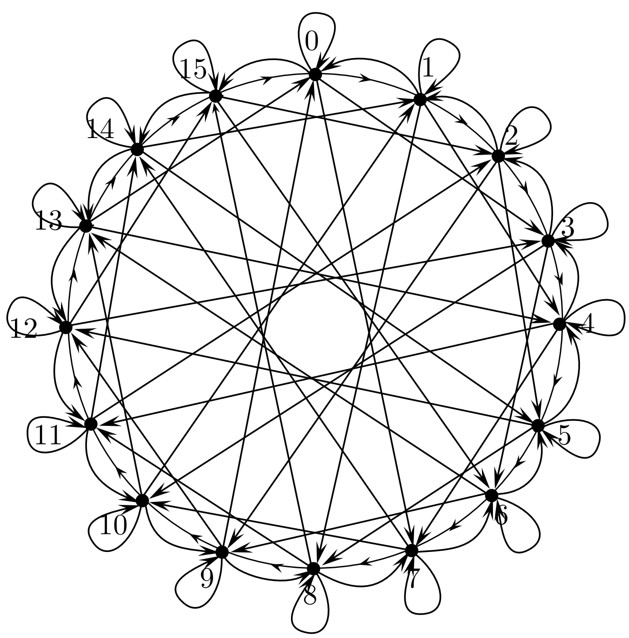


Fig. 2: The Circulant digraph $\vec{G}'_5 = \overrightarrow{\text{Circ}}(16,\ \{0,\ 1,\ 3,\ 7,\ 15\})$.

From the Definition 2.2, it is easy to see that the Knödel graph $W_{\Delta,\,n}$ is the bipartite incident graph of the circulant digraph $\vec{G}$ with vertex set $\{0,\,1,\,2,\,\ldots,\,\frac{n}{2}-1\}$ and arc set consists of the arcs of distance ℓ, $\ell \in L = \{2^k-1,\, 0 \le k \le \Delta-1\}$. In otherwords, $W_{\Delta,\,n}$ is the underlying graph of $\vec{G} \times \vec{K}_2$, where $\vec{K}_2$ denotes an arc.

Let $\vec{G}_k$ denote the circulant digraph $\overrightarrow{Circ}(2^{k-1};\,L)$, where $L = \{2^j-1 \,|\, 1 \le j \le k-1\}$ and let $\vec{G}'_k$ denote the graph obtained from $\vec{G}_k$ by attaching a directed loop at each of its vertices, that is, $\vec{G}'_k = \overrightarrow{Circ}(2^{k-1};\, L \cup \{0\})$, see Figure 2. Clearly, $W_{k,\,2^k}$ is isomorphic to the underlying graph of $\vec{G}'_k \times \vec{K}_2$.

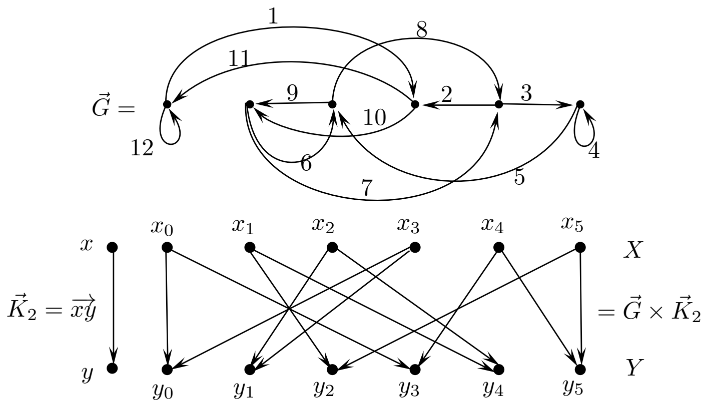
Definition 2.3 [3] *Let $\vec{G}$ be a 2-diregular digraph which does not have a pair of arcs having the same tail and head, but may have directed loops. If $\vec{G}$ admits an uniformly odd aneulerian tour $(UOAT)$, that is, an aneulerian tour in which no proper closed subtrail is of even length, then $\vec{G}$ is called an $UOAT$ digraph (in fact, one can check that if a 2-diregular digraph admits an aneulerian tour T, then each proper closed subtrail of T is of odd length).*

The proof of the following theorem is similar to the proof of Theorem 2.1 of [3].

Theorem 2.1 *Let $\vec{G}$ be a 2-diregular digraph which does not have a pair of arcs having the same tail and head, but may have directed loops. Then the underlying graph of the digraph $\vec{G} \times \vec{K}_2$ is a Hamilton cycle if and only if $\vec{G}$ is an $UOAT$ digraph.*

Proof: Let $\vec{G} = (V,\,A)$ with $V(\vec{G}) = \{v_0,\,v_1,\,v_2,\,\ldots,\,v_n\}$. Let $\vec{K}_2 = \overrightarrow{xy}$. To each arc $\overrightarrow{v_iv_j}$, possibly with $i=j$, in $\vec{G}$ there corresponds an unique arc $\overrightarrow{(v_i,\,x)(v_j,\,y)}$ in $\vec{G} \times \vec{K}_2$ and similarly corresponding to each arc $\overrightarrow{(v_r,\,x)(v_s,\,y)}$, possibly with $r=s$, in $\vec{G} \times \vec{K}_2$ there corresponds an arc $\overrightarrow{v_rv_s}$, may be a directed loop, in $\vec{G}$. Clearly, $\vec{H} = \vec{G} \times \vec{K}_2$ is bipartite digraph with bipartition $(X,\,Y)$, where $X = V(\vec{G}) \times x = \{(v_i,\,x) \,|\, v_i \in V(\vec{G})\}$ and $Y = V(\vec{G}) \times y = \{(v_i,\,y) \,|\, v_i \in V(\vec{G})\}$. In $\vec{H}$ all the vertices in X have only outdegree, namely 2, and all the vertices in Y have only indegree 2. Hence the underlying graph of $\vec{G} \times \vec{K}_2$ is a 2-regular graph. First assume that the underlying graph of $\vec{G} \times \vec{K}_2$ is a spanning cycle C. As each vertex in X has only outdegree and each vertex in Y has only indegree, while we trace along C the corresponding arcs in $\vec{G}$ trace an anti-directed tour T. We claim that T is an aneulerian tour. Observe that if $(v_i,\,x)(v_i,\,y)$ is an edge of C, then its corresponding arc in T is a directed loop at v_i, see Figure 3. We claim that no proper closed subtrail of T is of even length. Let v_k be the origin of a proper closed subtrail T_1 of T. Without loss of generality assume that C visits $(v_k,\,x)$ prior to $(v_k,\,y)$. Hence T_1 arises out of the $(v_k,\,x)$-$(v_k,\,y)$ section of C and this section is of odd length as $\vec{G} \times \vec{K}_2$ is a bipartite digraph. Hence T_1 is of odd length. Hence T is an aneulerian tour. Thus $\vec{G}$ is an $UOAT$ digraph.

Conversely, assume that $\vec{G}$ is an $UOAT$ digraph. Let T be an aneulerian tour of $\vec{G}$ with the origin v_0. Now we describe a Hamilton cycle in the underlying graph of $\vec{G} \times \vec{K}_2$ from the vertex $(v_0,\,x)$ as follows: If $\overrightarrow{v_0v_1}$ is the first arc of T, then take the edge $(v_0,\,x)(v_1,\,y)$ in the underlying graph of $\vec{G} \times \vec{K}_2$ for the Hamilton cycle. The second arc of T would be $\overrightarrow{v_2v_1}$, then take the edge $(v_1,\,y)(v_2,\,x)$ in the underlying graph of $\vec{G} \times \vec{K}_2$. As we move along T, the corresponding edges of the underlying graph of $\vec{G} \times \vec{K}_2$ induce a Hamilton cycle in the underlying graph of $\vec{G} \times \vec{K}_2$; for otherwise, the corresponding arcs of an nonspanning even cycle in the underlying graph of $\vec{G} \times \vec{K}_2$ would yield a proper closed subtrail of even length in T, a contradiction. Hence as we trace along T, the corresponding edges of the underlying graph of $\vec{G} \times \vec{K}_2$ induce a Hamilton cycle.



(i) A Directed Euler tour in $\vec{G}$ is given by the arc sequence $1,\ 10,\ 7,\ 3,\ 4,\ 5,\ 9,\ 6,\ 8,\ 2,\ 11,\ 12$.
(ii) An aneulerian tour in $\vec{G}$ is given by the arc sequence $1,\ 2,\ 3,\ \ldots,\ 11,\ 12$, where the loops are traversed in the forward or backward direction according as our requirement for the existence of an aneulerian tour. Clearly this aneulerian tour is an $UOAT$ of $\vec{G}$.

Fig. 3: $\vec{G} \times \vec{K}_2$, a 2-diregular bieulerian digraph.

This completes the proof of the theorem. □

Theorem 2.2 *Let $\vec{G}$ be a 2-diregular digraph which does not have a pair of arcs having the same tail and head, but may have directed loops. Then the underlying graph of $\vec{G} \times \vec{K}_2$ is a Hamilton cycle if and only if $\vec{G}$ is bieulerian.*

Proof: First assume that the underlying graph $\vec{G} \times \vec{K}_2$ is a Hamilton cycle C. Then by Theorem 2.1 $\vec{G}$ has an $UOAT$. As $\vec{G}$ is diregular, $\vec{G}$ is eulerian. Thus $\vec{G}$ is bieulerian.

Conversely, assume that $\vec{G}$ is bieulerian. Let T be an aneulerian tour of $\vec{G}$. If we proceed as in the proof of Theorem 2.1, when we move along the aneulerian tour T, the corresponding edges in the underlying graph of $\vec{G} \times \vec{K}_2$ induce a Hamilton cycle.

This completes the proof of the theorem. □

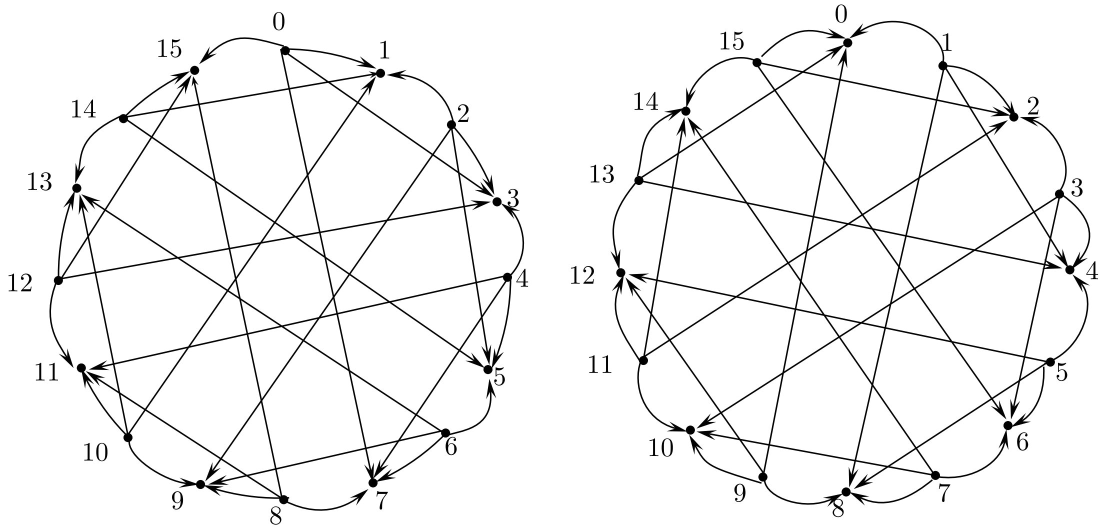
3 Edge Disjoint Hamilton Cycles of $W_{k,\,2^k}$.

In this section we prove that $W_{k,\,2^k}$ contains $\lfloor \frac{k}{2} \rfloor - 1$ edge disjoint Hamilton cycles.

The following Theorem 3.1 is proved in [8].

Theorem 3.1 [8] *For any even m and $1 \le \Delta \le \lfloor log_2(m) \rfloor$, it is possible to construct $W_{\Delta+1,\,2m}$ by taking two disjoint copies of $W_{\Delta,\,m}$ and linking the vertices of the copies by a certain perfect matching.* □

Next we decompose the digraph $\vec{G}_k = \overrightarrow{Circ}(2^{k-1},\, L)$, where $L = \{2^j - 1 \,|\, 1 \le j \le k-1\}$, into two digraphs in the following way to proceed further. Let $V(\vec{G}_k) = \{0,\, 1,\, 2,\, \ldots,\, 2^{k-1} - 1\}$. First we decompose $\vec{G}_k$ into two digraphs $\vec{G}_{k,\,e}$ and $\vec{G}_{k,\,o}$, where $\vec{G}_{k,\,e}$ (resp. $\vec{G}_{k,\,o}$) is the spanning subdigraph



(a). $\overrightarrow{G}_{5,e}$ (b). $\overrightarrow{G}_{5,o}$

Fig. 4: Graphs $\vec{G}_{5,\,e}$ and $\vec{G}_{5,\,o}$.

of $\vec{G}_k$ in which the indegree of each of the even (resp. odd) vertices, namely, $0,\,2,\,\ldots,\,2^{k-1}-2$ (resp. $1,\,3,\,\ldots,\,2^{k-1}-1$), is zero. That is $\vec{G}_{k,\,e}$ (resp. $\vec{G}_{k,\,o}$) is the spanning subdigraph of $\vec{G}_k$ obtained by deleting the set of arcs having their tails at the odd (resp. even) vertices of $\vec{G}_k$. Each arc of $\vec{G}_{k,\,e}$ is directed from an even vertex to an odd vertex and each arc of $\vec{G}_{k,\,o}$ is directed from an odd vertex to an even vertex (see Figure 4 for $k = 5$).

Lemma 3.1 *The underlying graphs of the digraphs $\vec{G}_{k,\,e}$ and $\vec{G}_{k,\,o}$ are isomorphic to $W_{k-1,\,2^{k-1}}$.*

Proof: $\vec{G}_{k,\,e}$ can be viewed as a bipartite graph by placing the even vertices, in the increasing order, in the partite set X and the odd vertices, in the increasing order, in the other partite set Y. Then the arcs of distance $2^\ell - 1$, $1 \le \ell \le k-1$, in $\vec{G}_{k,\,e}$, considering it as a subdigraph of $\vec{G}_k$, become the arcs of jump $2^{\ell-1}-1$ from X to Y in $\vec{G}_{k,\,e}$ in the bipartite structure of it and thus the underlying graph of $\vec{G}_{k,\,e}$ is isomorphic to $W_{k-1,\,2^{k-1}}$, see Figure 5(a), when $k = 5$. To prove the underlying graph of $\vec{G}_{k,\,o}$ is isomorphic to $W_{k-1,\,2^{k-1}}$, we establish an isomorphism between $\vec{G}_{k,\,e}$ and $\vec{G}_{k,\,o}$ by mapping the vertex i of $\vec{G}_{k,\,e}$ to $i+1$, where the addition is taken modulo 2^{k-1}, of $\vec{G}_{k,\,o}$. □

Recall that $\vec{G}'_k$ is the digraph obtained from $\vec{G}_k$ by adding a loop at each of its vertices and, $W_{k,\,2^k}$ is isomorphic to the underlying graph of $\vec{G}'_k \times \vec{K}_2$.

First we explain the proof technique of Theorem 3.2 and the lemmas used in its proof. By Theorem 2.2, finding $\lfloor \frac{k}{2} \rfloor - 1$ Hamilton cycles of Knödel graph $W_{k,\,2^k}$ is equivalent to finding $\lfloor \frac{k}{2} \rfloor - 1$ arc-disjoint spanning 2-diregular subdigraphs of $\vec{G}'_k$ so that each of the subdigraphs is bieulerian. Hence we shall

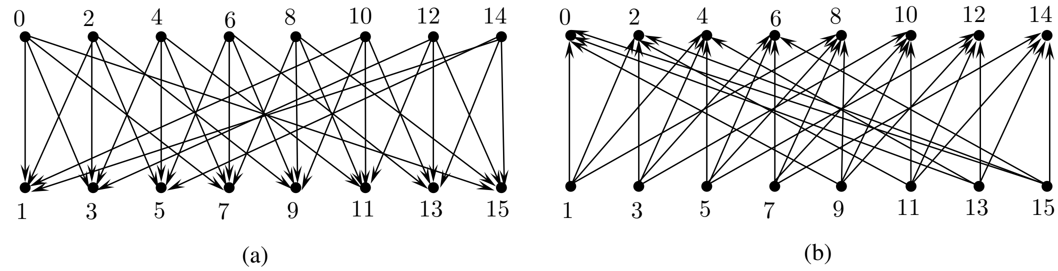


Fig. 5: Redrawings of $\overrightarrow{G}_{5,\,e}$ and $\overrightarrow{G}_{5,\,o}$, in Figure 4, in the "natural" bipartite structure. For example the edges of distance 7 in $\vec{G}_{5,\,e}$ are edges of jump 3 from X to Y, where $X = \{0,\,2,\,4,\,\ldots,\,14\}$ and $Y = \{1,\,3,\,5,\,\ldots,\,15\}$, in the bipartite structure. Actually, an edge of distance $2^{\ell} - 1$ in $\vec{G}_{k,\,e}$ becomes an edge of jump $2^{\ell-1} - 1$, from X to Y, in the bipartite structure. As the graphs of Figures 4(a) and 4(b) are isomorphic, the graphs in Figures 5(a) and 5(b) are isomorphic (the isomorphism is obtained by mapping the vertex i in Figure 5(a) to $i + 1\,(mod\ 2^{k-1})$ of Figure 5(b)).

obtain $\lfloor \frac{k}{2} \rfloor - 1$ arc-disjoint spanning 2-diregular subdigraphs of $\vec{G}'_k$ such that each of which is bieulerian, if k is even, and with a directed 1-factor (that is, $d^+(v) = d^-(v) = 1$ for each v) if k is odd. To establish this, in Lemma 3.2, first we shall obtain $\lfloor \frac{k}{2} \rfloor - 1$ subdigraphs $\vec{H}_{i,\,e}$, $1 \le i \le \lfloor \frac{k}{2} \rfloor - 1$, and another $\lfloor \frac{k}{2} \rfloor - 1$, subdigraphs $\vec{H}_{i,\,o}$, $1 \le i \le \lfloor \frac{k}{2} \rfloor - 1$ out of $\vec{G}_{k,\,e}$ and $\vec{G}_{k,\,o}$, respectively. Obtain the digraph $\vec{H}_i = \vec{H}_{i,\,e} \cup \vec{H}_{i,\,o}$, $1 \le i \le \lfloor \frac{k}{2} \rfloor - 1$. We shall prove that $\vec{H}_i$, $1 \le i \le \lfloor \frac{k}{2} \rfloor - 1$, together with some "appropriate" directed loops at some of its vertices is a spanning 2-diregular bieulerian subdigraph of $\vec{G}'_k$, see Lemma 3.3. Using this, in Theorem 3.2, we prove the existence of $\lfloor \frac{k}{2} \rfloor - 1$ edge disjoint Hamilton cycles in $W_{k,\,2^k}$.

We shall prove what we have told above in the rest of the paper.

Remark 3.1 *Let G be a bipartite graph with bipartition $(X,\,Y)$ and $|\,X\,| = |\,Y\,| = n$. Let $F_i(X,\,Y)$ and $F_j(X,\,Y)$, $i \neq j$, be two 1-factors of jump i and j, respectively, from X to Y in G. Let $H = F_i(X,\,Y) \cup F_j(X,\,Y)$. Note that $F_j(X,\,Y) = F_{n-j}(Y,\,X)$. If $\gcd(n,\,i + n - j\,(mod\ n)) = k$, then H is the union of k disjoint cycles of same length. We shall use this fact in the proof of the next lemma; in particular, if $gcd(n,\,i + n - j\,(mod\ n)) = 1$, then $F_i(X,\,Y) \cup F_j(X,\,Y)$ is a Hamilton cycle of G.*

Lemma 3.2 *There exist $\lfloor \frac{k}{2} \rfloor - 1$ arc-disjoint spanning subdigraphs $\vec{H}_{j,\,e}$, $1 \le j \le \lfloor \frac{k}{2} \rfloor - 1$, of $\vec{G}_{k,\,e}$, where each $\vec{H}_{j,\,e}$ is the union of 2^{j-1} disjoint anti-directed paths whose origins and termini be denoted by $A_j\ (\subset V(\vec{G}_k))$, and, also there exists $\lfloor \frac{k}{2} \rfloor - 1$ arc-disjoint spanning subdigraphs $\vec{H}_{j,\,o}$, $1 \le j \le \lfloor \frac{k}{2} \rfloor - 1$, of $\vec{G}_{k,\,o}$, where each $\vec{H}_{j,\,o}$ is the union of 2^{j-1} disjoint anti-directed paths whose origins and termini are also A_j; that is, the set of origins and termini of the anti-directed paths of $\vec{H}_{j,\,e}$ is the same as the set of origins and termini of the anti-directed paths of $\vec{H}_{j,\,o}$; moreover $A_i \cap A_j = \phi$ for $i \neq j$.*

Proof: $\vec{G}_k$ is the circulant digraph with distance set $\{2^j - 1\,|\,1 \le j \le k - 1\}$. Let $\vec{G}_{k,\,e}$ and $\vec{G}_{k,\,o}$ be the spanning subdigraphs of $\vec{G}_k$ defined as above.

Partition the distance set $\{2^j - 1\,|\,1 \le j \le k - 2\}$ of $\vec{G}_k$ into $\lfloor \frac{k}{2} \rfloor - 1$ 2-subsets $\{a_j = 2^j - 1,\, b_j = 2^{k-1-j} - 1\}$, $1 \le j \le \lfloor \frac{k}{2} \rfloor - 1$, if k is even and, with one more singleton subset having only one distance, namely $2^{\frac{k-1}{2}} - 1$, if k is odd. For each of these pairs of distances, we shall obtain a pair of spanning subdigraphs $\vec{H}_{j,\,e}$ and $\vec{H}_{j,\,o}$ of $\vec{G}_{k,\,e}$ and $\vec{G}_{k,\,o}$, respectively, such that each of the digraphs $\vec{H}_{j,\,e}$ and $\vec{H}_{j,\,o}$ is the union of 2^{j-1} vertex disjoint anti-directed paths.

Let $\vec{G}^{j}_{k,\,e}$, $1 \le j \le \lfloor \frac{k}{2} \rfloor - 1$, be the spanning subdigraph of $\vec{G}_{k,\,e}$, with arc set $A(\vec{G}_{k,\,e}) \cap A(\overrightarrow{Circ}(2^{k-1}; \{2^j - 1,\, 2^{k-1-j} - 1\}))$. The underlying graph of $\vec{G}^{j}_{k,\,e}$, is a bipartite graph, by Lemma 3.1, with 2^{k-2} vertices on each of its partite sets and its edge set consists of the edges of jumps $2^{j-1} - 1$ and $2^{k-j-2} - 1$, from X to Y, where $X = \{0, 2, \ldots, 2^{k-1} - 2\}$ and $Y = \{1, 3, \ldots, 2^{k-1} - 1\}$ (recall that an arc of distance $2^{\ell} - 1$ in $\vec{G}^{j}_{k,\,e}$ will become an arc of jump $2^{\ell-1} - 1$ in the bipartite structure of $\vec{G}^{j}_{k,\,e}$). Clearly, $\vec{G}^{j}_{k,\,e}$ consists of 2^{j-1} anti-directed cycles as $\gcd(2^{k-2},\, 2^{j-1} - 1 + 2^{k-2} - 2^{k-j-2} + 1\,(mod\ 2^{k-2})) = 2^{j-1}$, see Remark 3.1.

We have $a_j = 2^j - 1$ and $b_j = 2^{k-1-j} - 1$; let $x_j = 2(j-1)$. First we exhibit 2^{j-1} disjoint anti-directed cycles $\vec{C}^{i}_{j,\,e}$, $1 \le i \le 2^{j-1}$, in $\vec{G}^{j}_{k,\,e}$, to obtain the required number of anti-directed paths for $\vec{H}_{j,\,e}$.

In the following list of 2^{j-1} anti-directed cycles, the anti-directed cycle $\vec{C}^{i}_{j,\,e}$, $2 \le i \le 2^{j-1}$ is obtained from $\vec{C}^{i-1}_{j,\,e}$ by subtracting $2b_j$ from each of its corresponding vertices and $\vec{C}^{1}_{j,\,e}$ is explicitly described; in the vertices of the following anti-directed cycles, the addition is taken modulo 2^{k-1} and $x_j = 2(j-1)$. We list 2^{j-1} anti-directed cycles and then we prove that they are mutually disjoint.

$$\begin{aligned}
\vec{C}^{1}_{j,\,e} = {} & x_j\,(x_j + a_j)\,(x_j + a_j - b_j)\,(x_j + 2a_j - b_j)\,(x_j + 2a_j - 2b_j)\,\ldots\,(x_j + 2b_j - a_j)\,(x_j + b_j - a_j)\,(x_j + b_j)\,x_j \\
\vec{C}^{2}_{j,\,e} = {} & (x_j - 2b_j)\,(x_j + a_j - 2b_j)\,(x_j + a_j - 3b_j)\,(x_j + 2a_j - 3b_j)\,(x_j + 2a_j - 4b_j)\,\ldots,\,(x_j - a_j) \\
& (x_j - b_j - a_j)\,(x_j - b_j)\,(x_j - 2b_j) \\
\vec{C}^{3}_{j,\,e} = {} & (x_j - 4b_j)\,(x_j + a_j - 4b_j)\,(x_j + a_j - 5b_j)\,(x_j + 2a_j - 5b_j)\,(x_j + 2a_j - 6b_j)\,\ldots\,(x_j - 2b_j - a_j) \\
& (x_j - 3b_j - a_j)\,(x_j - 3b_j)\,(x_j - 4b_j) \\
\vdots \qquad & \qquad \vdots \\
\vec{C}^{i}_{j,\,e} = {} & (x_j - (i-1)2b_j)\,(x_j + a_j - (i-1)2b_j)\,(x_j + a_j - b_j - (i-1)2b_j)\,(x_j + 2a_j - b_j - (i-1)2b_j) \\
& (x_j + 2a_j - 2b_j - (i-1)2b_j)\,\ldots\,(x_j + 2b_j - a_j - (i-1)2b_j)\,(x_j + b_j - a_j - (i-1)2b_j)\,(x_j + b_j - (i-1)2b_j) \\
& (x_j - (i-1)2b_j) \\
\vdots \qquad & \qquad \vdots \\
\vec{C}^{2^{j-1}}_{j,\,e} = {} & (x_j - (2^j - 2)b_j)\,(x_j + a_j - (2^j - 2)b_j)\,(x_j + a_j - (2^j - 1)b_j)\,(x_j + 2a_j - (2^j - 1)b_j) \\
& (x_j + 2a_j - 2^j b_j)\,\ldots\,(x_j - a_j - (2^j - 4)b_j)\,(x_j - a_j - (2^j - 3)b_j) \\
& (x_j - (2^j - 3)b_j)\,(x_j - (2^j - 2)b_j)
\end{aligned}$$

In $\vec{G}^{j}_{k,\,e}$, the vertices $1, 3, 5, \ldots, 2^{k-1} - 1$ are called the odd vertices and $0, 2, 4, \ldots, 2^{k-1} - 2$ are called the even vertices. In $\vec{C}^{1}_{j,\,e}$, the odd vertices and even vertices alternate and an odd vertex is obtained by adding a_j to its preceding (even) vertex, along the anti-directed cycle, and an even vertex is obtained from its preceding (odd) vertex by adding $(-b_j)$ to it. We obtain the required 2^{j-1} anti-directed paths, denoted by $\vec{H}^{i}_{j,\,e}$, $1 \le i \le 2^{j-1}$, from $\vec{C}^{i}_{j,\,e}$, $1 \le i \le 2^{j-1}$, by deleting its last arc, namely, $\overrightarrow{(x_j - (i-1)2b_j)(x_j - (i-1)2b_j + b_j)}$. The origins of the 2^{j-1} anti-directed paths $\vec{H}^{i}_{j,\,e}$, $1 \le i \le 2^{j-1}$, are the origins of $\vec{C}^{i}_{j,\,e}$, namely, $x_j,\, x_j - 2b_j,\, x_j - 4b_j,\, \ldots,\, x_j - (2^j - 2)b_j$. The union of these 2^{j-1} anti-directed paths $\vec{H}^i_{j,\,e}$, $1 \le i \le 2^{j-1}$, is denoted by $\vec{H}_{j,\,e}$, that is, $\vec{H}_{j,\,e} = \bigcup_{i=1}^{2^{j-1}} \vec{H}^i_{j,\,e}$.

Next we prove that the vertices of these anti-directed paths $\vec{H}^i_{j,\,e}$, $1 \le i \le 2^{j-1}$, form a partition of the vertex set of $\vec{G}^j_{k,\,e}$ (and hence that of $\vec{G}_{k,\,e}$).

Claim 1. The origins of the anti-directed paths $\vec{H}^2_{j,\,e}, \vec{H}^3_{j,\,e}, \ldots, \vec{H}^{2^{j-1}}_{j,\,e}$, namely, $(x_j - 2b_j), (x_j - 4b_j), (x_j - 6b_j), \ldots, (x_j - (2^j - 2)b_j)$ are all distinct from the even vertices of $\vec{H}^1_{j,\,e}$.

If $x_j - 2\ell b_j$, the origin of $\vec{H}^{\ell+1}_{j,\,e}$, is in $\vec{H}^1_{j,\,e}$, then, as $x_j - 2\ell b_j$ is even, it should be of the form $x_j + ra_j - rb_j$, for some $r \ne 0$ and $r \le 2^{k-j-1} - 1$; $r \le 2^{k-j-1} - 1$ follows from the fact that each anti-directed cycle is of length 2^{k-j} and among the vertices of the anti-directed cycle, only half of them can be even vertices and further the origin is an even vertex. Hence $x_j - 2\ell b_j \equiv x_j + ra_j - rb_j \ (mod\ 2^{k-1})$, that is,

$$\begin{aligned} rb_j - ra_j - 2\ell b_j &\equiv 0 \,(mod\ 2^{k-1}) \\ r(b_j - a_j) - 2\ell b_j &\equiv 0 \,(mod\ 2^{k-1}),\ \ a\ contradiction, \end{aligned}$$

for, if the congruence holds then, as $2^j \mid b_j - a_j$ and $2^j \mid 2^{k-1}$, we must have $2^j \mid 2\ell b_j$, but this is not the case as b_j is odd and $\ell \le 2^{j-1} - 1$. This implies that the origins of the anti-directed paths $\vec{H}^2_{j,\,e}, \vec{H}^3_{j,\,e}, \ldots, \vec{H}^{2^{j-1}}_{j,\,e}$ are all distinct from the even vertices of $\vec{H}^1_{j,\,e}$.

This completes the proof of Claim 1.

Claim 2. $\vec{H}^{\ell+1}_{j,\,e}$ and $\vec{H}^{\ell'+1}_{j,\,e}$, $\ell \ne \ell'$, are all disjoint.

It is enough to show that the set of even vertices of $\vec{H}^{\ell+1}_{j,\,e}$ and the even vertices of $\vec{H}^{\ell'+1}_{j,\,e}$ are disjoint. Assume that u is an even vertex common to both $\vec{H}^{\ell+1}_{j,\,e}$ and $\vec{H}^{\ell'+1}_{j,\,e}$ for some $\ell' \ne \ell$. Without loss of generality assume that $\ell' < \ell$. As u is an even vertex in $\vec{H}^{\ell+1}_{j,\,e}$, $u = x_j - 2\ell b_j + r(a_j - b_j)$ for some $r \le 2^{k-j-1} - 1$. As u is also in $\vec{H}^{\ell'+1}_{j,\,e}$, the vertex $x_j - 2\ell b_j$ must be in $\vec{H}^{\ell'+1}_{j,\,e}$, since the difference between two even vertices is a multiple of $a_j - b_j$; since $x_j - 2\ell' b_j$ is the origin of $\vec{H}^{\ell'+1}_{j,\,e}$, both $x_j - 2\ell' b_j$ and $x_j - 2\ell b_j$ are in $\vec{H}^{\ell'+1}_{j,\,e}$ and they cannot be equal as $\ell \ne \ell'$. If we add $2\ell' b_j$ to each vertex of $\vec{H}^{\ell'+1}_{j,\,e}$, then the vertex $x_j - 2(\ell - \ell')b_j$ is in $\vec{H}^1_{j,\,e}$, a contradiction to Claim 1 as $x_j - 2(\ell - \ell')b_j$ is the origin of the anti-directed path $\vec{H}^{\ell-\ell'+1}_{j,\,e}$.

This completes the proof of Claim 2.

Clearly, the arcs of $G^j_{k,\,e}$ which are not on the anti-directed paths $\vec{H}^i_{j,\,e}$, $1 \le i \le 2^{j-1}$, are the 2^{j-1} arcs $\overrightarrow{x_j(x_j + b_j)}$, $\overrightarrow{(x_j - 2b_j)(x_j - b_j)}$, $\overrightarrow{(x_j - 4b_j)(x_j - 3b_j)}, \ldots, \overrightarrow{(x_j - (2^j - 2)b_j)(x_j - (2^j - 3)b_j)}$, namely, the last arcs of the anti-directed cycles $\vec{C}^i_{j,\,e}$, $1 \le i \le 2^{j-1}$.

As $\vec{G}_{k,\,e} \cong \vec{G}_{k,\,o}$, it can be verified that $\vec{G}^j_{k,\,e} \cong \vec{G}^j_{k,\,o}$. As $\vec{G}^j_{k,\,e}$ has 2^{j-1} disjoint anti-directed cycles, $\vec{G}^j_{k,\,o}$ also has 2^{j-1} disjoint anti-directed cycles. Similar to $\vec{C}^i_{j,\,e}$, $1 \le i \le 2^{j-1}$, of $\vec{G}^j_{k,\,e}$, we obtain 2^{j-1} disjoint anti-directed cycles $\vec{C}^i_{j,\,o}$, $1 \le i \le 2^{j-1}$, in $\vec{G}^j_{k,\,o}$.

Addition is taken modulo 2^{k-1} in the vertices of the following anti-directed cycles and in the following cycles also $x_j = 2(j-1)$, $a_j = 2^j - 1$ and $b_j = 2^{k-j-1} - 1$.

$$
\begin{aligned}
\vec{C}^{\,1}_{j,\,o} = \;& x_j\,(x_j - a_j)\,(x_j - a_j + b_j)\,(x_j - 2a_j + b_j)\,(x_j - 2a_j + 2b_j)\,\ldots\,(x_j - 2b_j + a_j)\,(x_j - b_j + a_j)\\
& (x_j - b_j)\,x_j\\
\vec{C}^{\,2}_{j,\,o} = \;& (x_j - 2b_j)\,(x_j - a_j - 2b_j)\,(x_j - a_j - b_j)\,(x_j - 2a_j - b_j)\,(x_j - 2a_j)\,\ldots\,(x_j - 4b_j + a_j)\\
& (x_j - 3b_j + a_j)\,(x_j - 3b_j)\,(x_j - 2b_j)\\
\vec{C}^{\,3}_{j,\,o} = \;& (x_j - 4b_j)\,(x_j - a_j - 4b_j)\,(x_j - a_j - 3b_j)\,(x_j - 2a_j - 3b_j)\,(x_j - 2a_j - 2b_j)\,\ldots\,(x_j - 6b_j + a_j)\\
& (x_j - 5b_j + a_j)\,(x_j - 5b_j)\,(x_j - 4b_j)\\
& \vdots\\
\vec{C}^{\,i}_{j,\,o} = \;& (x_j - (i-1)2b_j)\,(x_j - a_j - (i-1)2b_j)\,(x_j - a_j + b_j - (i-1)2b_j)\,(x_j - 2a_j + b_j - (i-1)2b_j)\\
& (x_j - 2a_j + 2b_j - (i-1)2b_j)\,\ldots\,(x_j - 2b_j + a_j - (i-1)2b_j)\,(x_j - b_j + a_j - (i-1)2b_j)\\
& (x_j - b_j - (i-1)2b_j)\,(x_j - (i-1)2b_j)\\
& \vdots\\
\vec{C}^{\,2^{j-1}-1}_{j,\,o} = \;& (x_j - (2^j-4)b_j)\,(x_j - a_j - (2^j-4)b_j)\,(x_j - a_j - (2^j-5)b_j)\,(x_j - 2a_j - (2^j-5)b_j)\\
& (x_j - 2a_j - (2^j-6)b_j)\,\ldots,\,(x_j + a_j - (2^j-2)b_j)\,(x_j + a_j - (2^j-3)b_j)\,(x_j - (2^j-3)b_j)\\
& (x_j - (2^j-4)b_j)\\
\vec{C}^{\,2^{j-1}}_{j,\,o} = \;& (x_j - (2^j-2)b_j)\,(x_j - a_j - (2^j-2)b_j)\,(x_j - a_j - (2^j-3)b_j)\,(x_j - 2a_j - (2^j-3)b_j)\\
& (x_j - 2a_j - (2^j-4)b_j)\,\ldots\,(x_j + a_j - 2^jb_j)\,(x_j + a_j - (2^j-1)b_j)\,(x_j - (2^j-1)b_j)\\
& (x_j - (2^j-2)b_j)
\end{aligned} \tag{1}
$$

The anti-directed cycle $\vec{C}^{i}_{j,\,o}$, $2 \le i \le 2^{j-1}$, is obtained from $\vec{C}^{\,i-1}_{j,\,o}$ by subtracting $2b_j$ from each of its vertices corresponding vertices and $\vec{C}^{1}_{j,\,o}$ is explicitly described.

We obtain 2^{j-1} anti-directed paths, denoted by $\vec{H}^{i}_{j,\,o}$, from $\vec{C}^{i}_{j,\,o}$, $1 \le i \le 2^{j-1}$ by deleting its last arc, namely $\overrightarrow{(x_j - (i-1)2b_j - b_j)(x_j - (i-1)2b_j)}$. We assume that the origin of $\vec{H}^{i}_{j,\,o}$, $1 \le i \le 2^{j-1}$, is the origin of $\vec{C}^{i}_{j,\,o}$; hence the origins are $x_j,\ x_j - 2b_j,\ x_j - 4b_j,\ \ldots,\ x_j - (2^j - 2)b_j$.

Clearly the origin of the path $\vec{H}^{i}_{j,\,e}$ is same as the origin of $\vec{H}^{i}_{j,\,o}$, $1 \le i \le 2^{j-1}$. Similarly the terminus of $\vec{H}^{i}_{j,\,o}$ is the terminus of $\vec{H}^{i+1}_{j,\,e}$ for all i except for $i = 2^{j-1}$. But we want to reconstruct $\vec{H}^{2^{j-1}}_{j,\,o}$, suitably, as an anti-directed path such that the reconstructed path and $\vec{H}^{1}_{j,\,e}$ have the same terminus.

Let S_j and T_j denote the sets of origins and termini of the anti-directed paths $\vec{H}^{i}_{j,\,e}$, $1 \le i \le 2^{j-1}$, respectively.

For the reconstruction of $\vec{H}^{2^{j-1}}_{j,\,o}$, we need to prove that the vertices $x_j + a_j + b_j,\ x_j + b_j,\ x_j + 2b_j$ and $x_j + a_j + b_j + 1$ are on $\vec{C}^{2^{j-1}}_{j,\,o}$ and the three vertices $x_j + a_j + b_j,\ x_j + b_j,\ x_j + 2b_j$ are consecutive vertices along $\vec{C}^{2^{j-1}}_{j,\,o}$. In $\vec{C}^{2^{j-1}}_{j,\,o}$, the vertex $(x_j - a_j - (2^j - 2)b_j)$ is the immediate next vertex of the origin $(x_j - (2^j - 2)b_j)$. Further, $(x_j - a_j - (2^j - 2)b_j) \equiv x_j + 2b_j + 1\,(mod\ 2^{k-1})$, since $a_j = 2^j - 1$, $b_j = 2^{k-j-1} - 1$,

$$\begin{aligned} -a_j - (2^j - 2)b_j &= -(2^j - 1) - (2^j - 2)(2^{k-1-j} - 1) \\ &= 1 - 2^j.2^{k-1-j} + 2.2^{k-1-j} - 2 \\ &= -2^{k-1} + 2.2^{k-1-j} - 1 \\ &\equiv 2.(2^{k-1-j} - 1) + 2 - 1 \,(mod\ 2^{k-1}) \\ &\equiv 2b_j + 1 \,(mod\ 2^{k-1}) \end{aligned}$$

Consequently, $x_j - a_j - (2^j - 2)b_j \equiv x_j + 2b_j + 1 \,(mod\ 2^{k-1})$. Often we recall this congruence $x_j + 2b_j + 1 \equiv x_j - a_j - (2^j - 2)b_j \,(mod\ 2^{k-1})$ in the future. In many places we write $x_j + 2b_j + 1$ instead of $x_j - a_j - (2^j - 2)b_j$.

It is enough to show that $x_j + b_j$ is in $\vec{C}^{2^{j-1}}_{j,\,o}$, since by adding b_j (resp. a_j) to an odd vertex we get its succeeding (resp. preceding) vertex, namely $x_j + 2b_j$ (resp. $x_j + a_j + b_j$), in the anti-directed cycle. Now we prove that $x_j + b_j$ is in $\vec{C}^{2^{j-1}}_{j,\,o}$. Clearly, $b_j - a_j \equiv 0 \,(mod\ 2^j)$ as $j \leq \lfloor \frac{k}{2} \rfloor - 1$; let $b_j - a_j = m2^j$, where m is odd.

$$\begin{aligned} Now\ a_j b_j + a_j + b_j + 1 &= (a_j + 1)(b_j + 1) \\ &= (2^j)(2^{k-j-1}) \\ &= 2^{k-1} \equiv 0 \,(mod\ 2^{k-1}) \end{aligned}$$

Hence

$$\begin{aligned} b_j + 2^j &= b_j + a_j + 1 \\ &\equiv -a_j b_j (mod\ 2^{k-1}) \end{aligned} \tag{2}$$

As x_j is even, $x_j + b_j$ and the last vertex $x_j - (2^j - 1)b_j = x_j - a_j b_j$ of the anti-directed cycle $\vec{C}^{2^{j-1}}_{j,\,o}$ are odd. Hence the vertex $x_j + b_j$ is in $\vec{C}^{2^{j-1}}_{j,\,o}$ if and only if $x_j + b_j + \ell(b_j - a_j) \equiv x_j - a_j b_j \,(mod\ 2^{k-1})$, for some $\ell < 2^{k-j-1}$, since the difference between two odd vertices along $\vec{C}^{2^{j-1}}_{j,\,o}$ is a multiple of $b_j - a_j$ and there can be only 2^{k-j-1} odd vertices in an anti-directed cycles.

$$\begin{aligned} & x_j + b_j\ is\ in\ \vec{C}^{2^{j-1}}_{j,\,o} \\ &\iff x_j + b_j + \ell(b_j - a_j) \equiv x_j - a_j b_j \,(mod\ 2^{k-1}) \\ &\iff b_j + \ell(b_j - a_j) \equiv -a_j b_j \,(mod\ 2^{k-1}) \\ &\iff b_j + \ell m 2^j \equiv -a_j b_j \,(mod\ 2^{k-1}),\ \text{since } b_j - a_j = m2^j. \end{aligned}$$

$$\begin{aligned} \iff b_j + \ell m 2^j &\equiv b_j + 2^j \,(mod\ 2^{k-1}),\ \text{by (2)} \\ \iff \ell m 2^j &\equiv 2^j \,(mod\ 2^{k-1}) \\ \iff \ell m &\equiv 1 \,(mod\ 2^{k-j-1}) \end{aligned} \tag{3}$$

As $b_j - a_j = m2^j$ and $b_j - a_j$ are known, m is known. Thus (3) gives that $x_j + b_j$ is in $\vec{C}^{2^{j-1}}_{j,\,o}$ if and only if the congruence in (3) has a solution. As m is odd and $\gcd(m,\, 2^{k-j-1}) = 1$, the congruence in (3) has a unique solution. Thus $x_j + b_j$ must be in $\vec{C}^{2^{j-1}}_{j,\,o}$.

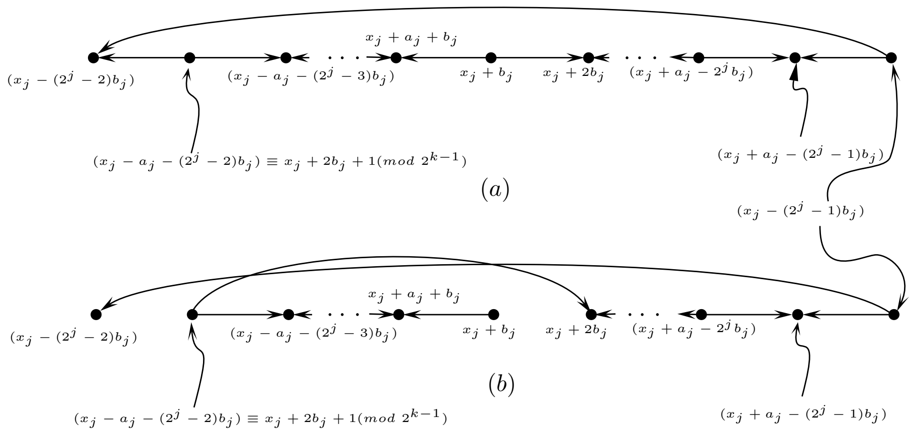


Fig. 6: Figure 6(a) shows the cycle $\vec{C}^{2^{j-1}}_{j,\,o}$ as in (4) above and Figure 6(b) shows the reconstructed anti-directed path $\vec{H}^{2^{j-1}}_{j,\,o}$ with origin $x_j - (2^j - 2)b_j$ and terminus $x_j + b_j$.

Clearly, the vertex next to $x_j + b_j$ along $\vec{C}^{2^{j-1}}_{j,\,o}$ is $x_j + 2b_j$; therefore, $\vec{C}^{2^{j-1}}_{j,\,o}$ must be of the form

$$\begin{aligned}\vec{C}^{2^{j-1}}_{j,\,o} = \quad & (x_j - (2^j - 2)b_j)\,(x_j - a_j - (2^j - 2)b_j)\,(x_j - a_j - (2^j - 3)b_j)\,(x_j - 2a_j - (2^j - 3)b_j)\\ & (x_j - 2a_j - (2^j - 4)b_j)\ \ldots,\ (x_j + a_j + b_j)\,(x_j + b_j)\,(x_j + 2b_j)\,(x_j + 2b_j - a_j)\ \ldots\ (x_j + a_j - 2^j b_j)\\ & (x_j + a_j - (2^j - 1)b_j)\,(x_j - (2^j - 1)b_j)\,(x_j - (2^j - 2)b_j)\end{aligned} \tag{4}$$

Hence the reconstructed last anti-directed path $\vec{H}^{2^{j-1}}_{j,\,o}$ is obtained from (4) by deleting the first arc $\overrightarrow{(x_j - a_j - (2^j - 2)b_j)(x_j - (2^j - 2)b_j)}$ and the arc $\overrightarrow{(x_j + b_j)(x_j + 2b_j)}$ and, adding the arc $\overrightarrow{(x_j - a_j - (2^j - 2)b_j)(x_j + 2b_j)}$; recall that $(x_j - a_j - (2^j - 2)b_j) \equiv (x_j + 2b_j + 1)\,(mod\ 2^{k-1})$; see Figure 6. In fact, the arc $\overrightarrow{(x_j + 2b_j + 1)(x_j + 2b_j)}$ is never used in the previous anti-directed paths $\vec{H}^{i}_{j,\,o}$, $1 \le i \le 2^{j-1} - 1$, as it is an arc of distance $2^{k-1} - 1$ in $\vec{G}_k$, which is neither in $\vec{G}^{j}_{k,\,e}$ nor in $\vec{G}^{j}_{k,\,o}$, $1 \le j \le \lfloor \frac{k}{2} \rfloor - 1$, see Figure 6, that is,

$$\begin{aligned}\vec{H}^{2^{j-1}}_{j,\,o} = \quad & (x_j - (2^j - 2)b_j)\,(x_j - (2^j - 1)b_j)\,(x_j + a_j - (2^j - 1)b_j)\,(x_j + a_j - 2^j b_j)\ \ldots\\ & (x_j + 2b_j)\,(x_j - a_j - (2^j - 2)b_j)\,(x_j - a_j - (2^j - 3)b_j)\,(x_j - 2a_j - (2^j - 3)b_j)\\ & (x_j - 2a_j - (2^j - 4)b_j)\ \ldots,\ (x_j + b_j)\end{aligned} \tag{5}$$

In (5) note that the vertex $(x_j - a_j - (2^j - 2)b_j) \equiv (x_j + 2b_j + 1)\,(mod\ 2^{k-1})$. For all our future reference $\vec{H}^{2^{j-1}}_{j,\,o}$ will refer to the reconstructed anti-directed path in (5).

This path contains the arc $\overrightarrow{(x_j + 2b_j + 1)(x_j + 2b_j)}$ which is an arc of distance $2^{k-1} - 1$ of $\vec{G}_k$. Consequently, the arcs of $\vec{G}^{j}_{k,\,o}$ which are not on the anti-directed paths $\vec{H}^{i}_{j,\,o}$, $1 \le i \le 2^{j-1}$ are the $2^{j-1} + 1$ arcs

$\overrightarrow{(x_j-b_j)x_j}$, $\overrightarrow{(x_j-3b_j)(x_j-2b_j)}$, $\overrightarrow{(x_j-5b_j)(x_j-4b_j)}, \ldots, \overrightarrow{(x_j-(2^j-3)b_j)(x_j-(2^j-4)b_j)}$, $\overrightarrow{(x_j+b_j)(x_j+2b_j)}$, $\overrightarrow{(x_j-a_j-(2^j-2)b_j)(x_j-(2^j-2)b_j)}$, that is, these are the last arcs of the $2^{j-1}-1$ anti-directed cycles $\vec{C}^{\,i}_{j,\,o}$, $1 \le i \le 2^{j-1}-1$, and the two arcs deleted from $\vec{C}^{\,2^{j-1}}_{j,\,o}$ to obtain the reconstructed $\vec{H}^{2^{j-1}}_{j,\,o}$.

Let the union of these 2^{j-1} anti-directed paths $\vec{H}^i_{j,\,o}$, $1 \le i \le 2^{j-1}$, be denoted by $\vec{H}_{j,\,o}$, that is, $\vec{H}_{j,\,o} = \bigcup_{i=1}^{2^{j-1}} \vec{H}^i_{j,\,o}$.

Thus for each pair of distances $(a_j = 2^j-1,\, b_j = 2^{k-j-1}-1)$ of $\vec{G}_k$ and $x_j = 2(j-1)$, we have the pair of digraphs $\vec{H}_{j,\,e}$ and $\vec{H}_{j,\,o}$, $1 \le j \le \lfloor \frac{k}{2} \rfloor - 1$.

Recall that S_j and T_j denote the origins and termini of the anti-directed paths $\vec{H}^i_{j,\,e}$, $1 \le i \le 2^{j-1}$, respectively, and also they are the origins and termini of the anti-directed paths of $\vec{H}^i_{j,\,o}$. Let $A_j = S_j \cup T_j$ and set $t = \lfloor \frac{k}{2} \rfloor - 1$.

Next we prove that A_j, $1 \le j \le \lfloor \frac{k}{2} \rfloor - 1$ are mutually disjoint. We know that the origin and terminus of $\vec{H}^i_{j,\,e}$, $1 \le i \le 2^{j-1}$, are $x_j-(i-1)2b_j$ and $x_j-(i-1)2b_j+b_j\,(=x_j-(2i-3)b_j)$, respectively, and they are independent of a_j.

By our choice $x_j = 2(j-1)$, $x_{j-1} = 2(j-1-1) = 2(j-1)-2 = x_j-2$; thus, recursively we can write x_ℓ in terms of x_j for all $\ell < j$. Further, as $a_j = 2^j-1$, $a_{j-1} = 2^{j-1}-1 = \frac{2^j-2}{2} = \frac{a_j-1}{2}$ and $b_j = 2^{k-j-1}-1$ implies $b_{j-1} = 2^{k-j}-1 = 2.2^{k-j-1}-1 = 2(2^{k-j-1}-1)+1 = 2b_j+1$. Thus, recursively we can write both a_ℓ and b_ℓ in terms of a_j and b_j, respectively, for all $\ell < j$. Hence all x_j's b_j's and a_j's can be written in terms of x_t, b_t and a_t, respectively, where $t = \lfloor \frac{k}{2} \rfloor - 1$. If $j = t-r$ for some $r > 0$, then $x_j = x_{t-r} = x_{t-r+1}-2$. Again $x_{t-r+1} = x_{t-r+2}-2$, thus $x_j = x_{t-r} = x_{t-r+1}-2 = x_{t-r+2}-4$. Proceeding like this, we get $x_j = x_{t-r} = x_t - 2r$. Similarly for $j = t-r$, recursively applying the above relation, we get $b_j = b_{t-r} = 2b_{t-r+1}+1 = 2(2b_{t-r+2}+1)+1 = 4b_{t-r+2}+3 = 4(2b_{t-r+3}+1)+3 = 8b_{t-r+3}+7$, *etc.* Proceeding like this, we get $b_j = b_{t-r} = 2^r b_t + 2^r - 1$. Also for $j = t-r$, recursively applying the above relation, we get $a_j = a_{t-r} = \frac{a_{t-r+1}-1}{2} = \frac{\frac{a_{t-r+2}-1}{2}-1}{2} = \frac{a_{t-r+2}-3}{4} = \frac{\frac{a_{t-r+3}-1}{2}-3}{4} = \frac{a_{t-r+3}-7}{8} = etc.$ Proceeding like this, we get $a_j = a_{t-r} = \frac{a_t-2^r+1}{2^r}$.

For our convenience, we shall denote the origin and terminus of an anti-directed path $\vec{H}^\ell_{j,\,e}$ by the ordered pair $(r,\, s)$ where r is the origin and s is the terminus of $\vec{H}^\ell_{j,\,e}$, respectively. For $t = \lfloor \frac{k}{2} \rfloor - 1$, by the above notation, the ordered pairs $(x_t,\, x_t+b_t)$, $(x_t-2b_t,\, x_t-b_t)$, $(x_t-4b_t,\, x_t-3b_t)$, $(x_t-6b_t,\, x_t-5b_t), \ldots (x_t-(2^t-4)b_t,\, x_t-(2^t-5)b_t)$, $(x_t-(2^t-2)b_t,\, x_t-(2^t-3)b_t)$ denote the origins and termini of the anti-directed paths $\vec{H}^i_{t,\,e}$, $1 \le i \le 2^{t-1}$, in $\vec{H}_{t,\,e}$. Similarly, $(x_{t-1},\, x_{t-1}+b_{t-1})$, $(x_{t-1}-2b_{t-1},\, x_{t-1}-b_{t-1})$, $(x_{t-1}-4b_{t-1},\, x_{t-1}-3b_{t-1})$, $(x_{t-1}-6b_{t-1},\, x_{t-1}-5b_{t-1}), \ldots (x_{t-1}-(2^{t-1}-4)b_{t-1},\, x_{t-1}-(2^{t-1}-5)b_{t-1})$, $(x_{t-1}-(2^{t-1}-2)b_{t-1},\, x_{t-1}-(2^{t-1}-3)b_{t-1})$ denote the origins and termini of the anti-directed paths $\vec{H}^i_{t-1,\,e}$, $1 \le i \le 2^{t-2}$, in $\vec{H}_{t-1,\,e}$. But we have seen above that $b_{t-1} = 2b_t+1$ and $x_{t-1} = x_t-2$. If we write x_{t-1} and b_{t-1} in terms of x_t and b_t to the origins and termini of the anti-directed paths of $\vec{H}_{t-1,\,e}$, the origins and termini of the anti-directed paths in $\vec{H}_{t-1,\,e}$ are $(x_t-2,\, x_t+2b_t-1)$, $(x_t-4b_t-4,\, x_t-2b_t-3)$, $(x_t-8b_t-6,\, x_t-6b_t-5)$, $(x_t-12b_t-8,\, x_t-10b_t-7), \ldots, (x_t-(2^t-8)b_t-2^{t-1}+2,\, x_t-(2^t-10)b_t-2^{t-1}+3)$, $(x_t-(2^t-4)b_t-2^{t-1},\, x_t-(2^t-6)b_t-2^{t-1}+1)$. Proceeding as above, the origin and terminal vertices of each of the anti-direted paths of $\vec{H}_{j,\,e}$ can be given in terms of x_t and b_t.

A_t	A_{t-1}	A_{t-2}	$\dots$	A_2	A_1
$(x_t,\, x_t + b_t)$	$(x_t - 2,\, x_t + 2b_t - 1)$	$(x_t - 4,\, x_t + 4b_t - 1)$	$\cdots$	$(2,\, 2^{k-3} + 1)$	$(0,\, 2^{k-2} - 1)$
$(x_t - 2b_t,\, x_t - b_t)$					
$(x_t - 4b_t,\, x_t - 3b_t)$	$(x_t - 4b_t - 4,\, x_t - 2b_t - 3)$				
$(x_t - 6b_t,\, x_t - 5b_t)$					
$(x_t - 8b_t,\, x_t - 7b_t)$	$(x_t - 8b_t - 6,\, x_t - 6b_t - 5)$	$(x_t - 8b_t - 10,\, x_t - 4b_t - 7)$			
$(x_t - 10b_t,\, x_t - 9b_t)$					
$(x_t - 12b_t,\, x_t - 11b_t)$	$(x_t - 12b_t - 8,\, x_t - 10b_t - 7)$				
$(x_t - 14b_t,\, x_t - 13b_t)$					
$(x_t - 16b_t,\, x_t - 15b_t)$	$(x_t - 16b_t - 10,\, x_t - 14b_t - 9)$	$(x_t - 16b_t - 16,\, x_t - 12b_t - 13)$	$\cdots$		
$(x_t - 18b_t,\, x_t - 17b_t)$					
$(x_t - 20b_t,\, x_t - 19b_t)$	$(x_t - 20b_t - 12,\, x_t - 18b_t - 11)$				
$(x_t - 22b_t,\, x_t - 21b_t)$					
$(x_t - 24b_t,\, x_t - 23b_t)$	$(x_t - 24b_t - 14,\, x_t - 22b_t - 13)$	$(x_t - 24b_t - 22,\, x_t - 20b_t - 19)$			
$(x_t - 26b_t,\, x_t - 25b_t)$					
$(x_t - 28b_t,\, x_t - 27b_t)$	$(x_t - 28b_t - 16,\, x_t - 26b_t - 15)$				
$(x_t - 30b_t,\, x_t - 29b_t)$					
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$(2 - 2(2^{k-3} - 1),\, 2 - (2^{k-3} - 1))$	
$(x_t - (2^t - 8)b_t,\, x_t - (2^t - 9)b_t)$	$(x_t - (2^t - 8)b_t - (2^{t-1} - 2),\, x_t - (2^t - 9)b_t + b_t + 1 - (2^{t-1} - 2))$	$(x_t - (2^t - 8)b_t - (2^{t-1} - 2) - 2^{t-2},\, x_t - (2^t - 9)b_t + b_t + 1 - (2^{t-1} - 2) + 2b_t + 2 - 2^{t-2})$	$\cdots$		
$(x_t - (2^t - 6)b_t,\, x_t - (2^t - 7)b_t)$					
$(x_t - (2^t - 4)b_t,\, x_t - (2^t - 5)b_t)$	$(x_t - (2^t - 4)b_t - 2^{t-1},\, x_t - (2^t - 5)b_t + b_t + 1 - 2^{t-1})$				
$(x_t - (2^t - 2)b_t,\, x_t - (2^t - 3)b_t)$					

Tab. 1: The set of origins and termini of the anti-directed paths in $\bigcup_{j=1}^{\lfloor \frac{k}{2} \rfloor - 1} \vec{H}_{j,\,e}$ is given as ordered pairs, that is, the ith ordered pair in column j is the origin and terminus of $\vec{H}^{i}_{j,\,e}$. Actually, A_j denotes the union of all the vertices in the column j, where we consider each ordered pair as a set with the two vertices when we consider their union.

In general, the origins and termini of anti-directed paths of $\vec{H}_{j,\,e}$ are $(x_j,\,x_j+b_j)$, $(x_j-2b_j,\,x_j-b_j)$, $(x_j-4b_j,\,x_j-3b_j)$, $\ldots$, $(x_j-(i-1)2b_j,\,x_j-(2i-3)b_j)$, $\ldots$, $(x_j-(2^j-2)b_j,\,x_j-(2^j-3)b_j)$. If $j=t-r$, then by the relation obtained above, $x_j=x_t-2r$ and $b_j=2^rb_t+2^r-1$. Thus the origins and termini of anti-directed paths of $\vec{H}_{j,\,e}$ in terms of x_t and b_t are given by the ordered pairs $(x_t-2r,\,x_t-2r+2^rb_t+2^r-1)$, $(x_t-2r-2(2^rb_t+2^r-1),\,x_t-2r-(2^rb_t+2^r-1))$, $(x_t-2r-4(2^rb_t+2^r-1),\,x_t-2r-3(2^rb_t+2^r-1))$, $\ldots$, $(x_t-2r-(i-1)2(2^rb_t+2^r-1),\,x_t-2r-(2i-3)(2^rb_t+2^r-1))$, $\ldots$, $(x_t-2r-(2^j-2)(2^rb_t+2^r-1),\,x_t-2r-(2^j-3)(2^rb_t+2^r-1))$. Hereafter we shall represent the elements of A_j in terms of x_t and b_t.

We list the elements of A_j, $1\le j\le\lfloor\frac{k}{2}\rfloor-1$, in terms of x_t and b_t, in ordered pairs in the Table 1 below with the elements of A_t appearing in the first column, the elements of A_{t-1} in the second column etc., and the last column contains the elements of A_1.

Next we shall prove that $A_i\cap A_j=\phi$, $i\ne j$; the intersection here is the intersection of elements of A_i and A_j (but not as intersection of ordered pairs of A_i and A_j described in the Table 1). For this it is enough to prove that the even vertices in the Table 1 are distinct. For, let $i<j$ with $i=t-s$ and $j=t-r$; hence $r<s$. By the above recursive relations we get $x_j=x_t-2r$, $x_i=x_t-2s$, $b_j=2^rb_t+2^r-1$ and $b_i=2^sb_t+2^s-1$. Suppose the origins in the $(\ell+1)$th pair of A_j in Table 1, that is, $x_j-2\ell b_j$ and the $(\ell'+1)$th pair of A_i, that is, $x_i-2\ell'b_i$ are same then

$$\begin{aligned}
x_j-2\ell b_j &\equiv x_i-2\ell'b_i\,(mod\ 2^{k-1});\ \ this\ \ implies\\
x_t-2r-2\ell(2^rb_t+2^r-1) &\equiv x_t-2s-2\ell'(2^sb_t+2^s-1)\,(mod\ 2^{k-1})\\
that\ is,\ 2r+2^{r+1}\ell b_t+2^{r+1}\ell-2\ell &\equiv 2s+2^{s+1}\ell'b_t+2^{s+1}\ell'-2\ell'\,(mod\ 2^{k-1})\\
that\ is,\ 2(r-s)+2^{r+1}\ell(b_t+1)-2\ell &\equiv 2^{s+1}\ell'(b_t+1)-2\ell'\,(mod\ 2^{k-1})\\
that\ is,\ 2(r-s)+2(\ell'-\ell) &\equiv 2^{s+1}\ell'2^{k-t-1}-2^{r+1}\ell 2^{k-t-1}\,(mod\ 2^{k-1})\\
&\equiv A2^{r+k-t}\,(mod\ 2^{k-1})\text{ for some constant } A\text{, that is,}\\
A2^{r+k-t}-2(r-s+\ell'-\ell) &\equiv 0\,(mod\ 2^{k-1})
\end{aligned}$$

However, we shall show that this congruence does not hold. We assume that the congruence holds and we obtain a contradicition. Observe that as $0\le r<s\le\lfloor\frac{k}{2}\rfloor-1$ and $t=\lfloor\frac{k}{2}\rfloor-1$, we have $r+k-t\le\lfloor\frac{k}{2}\rfloor-2+k-\lfloor\frac{k}{2}\rfloor+1=k-1$.

First we suppose that A is odd. Since A is odd, as 2^{r+k-t} divides both $A2^{r+k-t}$ and 2^{k-1}, then it must divide $2[r-s+\ell'-\ell]$, that is, $2[r-s+\ell'-\ell]\equiv 0\,(mod\ 2^{r+k-t})$, which is a contradiction, as $0\le r<s\le\lfloor\frac{k}{2}\rfloor-1$, $0\le\ell\le 2^{t-r-1}-1$ and $0\le\ell'\le 2^{t-s-1}-1$.

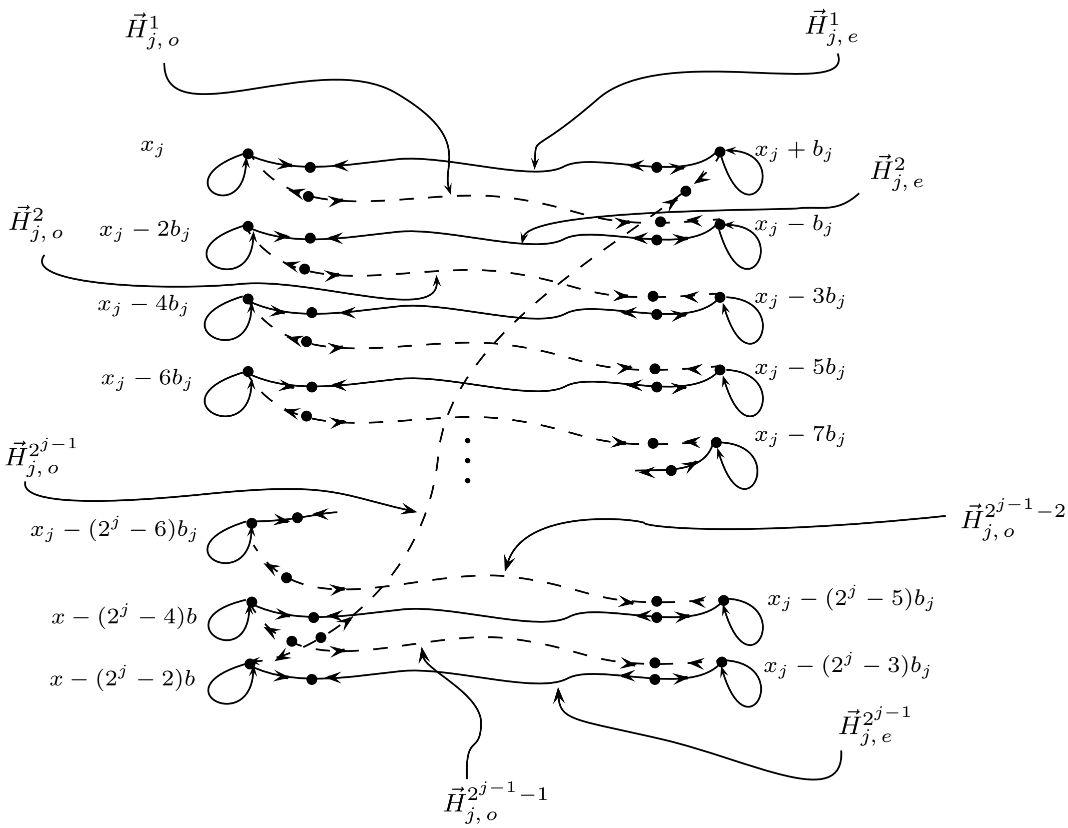
Next we suppose that A is even. Let d be the maximum integer such that $2^d\,|\,A2^{r+k-t}$. Now we consider the case $d\le k-1$; then the proof follows similar to the case when A is odd by replacing $r+k-t$ by d. Next we assume that $d>k-1$. Since $d>k-1$, $2^{k-1}\,|\,2^d$. Then 2^{k-1} divides $2[r-s+\ell'-\ell]$, that is, $2[r-s+\ell'-\ell]\equiv 0\,(mod\ 2^{k-1})$, which is a contradiction, as $0\le r<s\le\lfloor\frac{k}{2}\rfloor-1$, $0\le\ell\le 2^{t-r-1}-1$ and $0\le\ell'\le 2^{t-s-1}-1$.

Hence irrespective of the parity of A, the congruence does not hold. Thus the origins, that is, the even vertices of the ordered pairs listed in the columns A_i and A_j, $i\ne j$, of Table 1, are distinct and hence the odd vertices of the ordered pairs listed in the columns A_i and A_j, $i\ne j$, of Table 1 are also distinct.
This completes the proof of the lemma. □

Lemma 3.3 *There exist* $\lfloor \frac{k}{2} \rfloor - 1$ *arc disjoint* 2*-diregular bieulerian digraphs* $\vec{H}_j$, $1 \leq j \leq \lfloor \frac{k}{2} \rfloor - 1$, *each of which is a spanning subdigraph of* $\vec{G}'_k$.

Proof: By Lemma 3.2, there exist $\lfloor \frac{k}{2} \rfloor - 1$ arc disjoint spanning subdigraphs $\vec{H}_{j,\,e}$ and $\vec{H}_{j,\,o}$, $1 \leq j \leq \lfloor \frac{k}{2} \rfloor - 1$, of $\vec{G}_{k,\,e}$ and $\vec{G}_{k,\,o}$, respectively. Clearly, each of the digraphs $\vec{H}_{j,\,e}$ and $\vec{H}_{j,\,o}$ consists of union of 2^{j-1} anti-directed paths with their origins and termini at $A_j = S_j \cup T_j$.

Next we construct $\lfloor \frac{k}{2} \rfloor - 1$ spanning 2-diregular bieulerian digraphs using $\vec{H}_{j,\,e}$ and $\vec{H}_{j,\,o}$, $1 \leq j \leq \lfloor \frac{k}{2} \rfloor - 1$. For a fixed j, we construct a spanning 2-diregular bieulerian digraph from $\vec{H}_{j,\,e}$ and $\vec{H}_{j,\,o}$ by



The anti-directed paths $\vec{H}^i_{j,\,e}$, $1 \leq i \leq 2^{j-1}$, of $\vec{H}_{j,\,e}$ are shown in solid lines and the anti-directed paths $\vec{H}^i_{j,\,o}$, $1 \leq i \leq 2^{j-1}$, of $\vec{H}_{j,\,o}$ are shown in broken lines. For clarity of the figure, $\vec{H}^i_{j,\,o}$ is shown as vertex disjoint from $\vec{H}^i_{j,\,e}$, but it is not the case as both $\vec{H}_{j,\,e}$ and $\vec{H}_{j,\,o}$ are spanning subdigraphs of $\vec{G}_k$.

Fig. 7: 2-diregular spanning bieulerian digraph $\vec{H}_j$ constructed from $\vec{H}_{j,\,e}$ and $\vec{H}_{j,\,o}$ and loops added at each of the vertices of A_j.

concatenating the anti-directed paths of them appropriately and adding some appropriate loops to it. That is, the spanning bieulerian digraph is $H^1_{j,e} \cup H^{2^{j-1}}_{j,o} \cup H^{2^{j-1}}_{j,e} \cup H^{2^{j-1}-1}_{j,o} \cup H^{2^{j-1}-1}_{j,e} \cup H^{2^{j-1}-2}_{j,o} \cup H^{2^{j-1}-2}_{j,e} \cup H^{2^{j-1}-3}_{j,o} \cup H^{2^{j-1}-3}_{j,e} \cup \ldots \cup H^2_{j,o} \cup H^2_{j,e} \cup H^1_{j,o} \cup \{$directed loops at each of the vertices of $A_j\}$, see Figure 7. We denote the graph $\vec{H}_{j,e} \cup \vec{H}_{j,o}$ together with directed loops added at the vertices of A_j by $\vec{H}_j$, see Figure 7. Clearly, $\vec{H}_j$ is eulerian and, $\vec{H}_j$ is aneulerian follows by moving along the anti-directed paths of $\vec{H}_{j,e}$ and $\vec{H}_{j,o}$, alternately, and the directed loops at the vertices of A_j; while obtaining an aneulerian tour, as and when a directed loop of A_j is encountered when we visit through $\vec{H}_{j,e}$ or $\vec{H}_{j,o}$, the directed loop is visited in the clockwise or anticlockwise direction according to the requirement for the existence of an aneulerian tour.

This completes the proof of the lemma. □

Next we prove that $W_{k,\,2^k}$, $k \geq 6$, has $\lfloor \frac{k}{2} \rfloor - 1$ edge disjoint Hamilton cycles.

Theorem 3.2 *$W_{k,\,2^k}$ contains $\lfloor \frac{k}{2} \rfloor - 1$ edge disjoint Hamilton cycles.*

Proof: By Lemma 3.3, $\vec{G}'_k$ has $\lfloor \frac{k}{2} \rfloor - 1$ arc disjoint 2-diregular spanning bieulerian subdigraphs $\vec{H}_j$, $1 \leq j \leq \lfloor \frac{k}{2} \rfloor - 1$. Let $\vec{H} = \bigcup_{j=1}^{\lfloor \frac{k}{2} \rfloor - 1} \vec{H}_j$. Now $\vec{H} \times \vec{K}_2 = \bigoplus_{j=1}^{\lfloor \frac{k}{2} \rfloor - 1} (\vec{H}_j \times \vec{K}_2)$. But the underlying graph of $\vec{H}_j \times \vec{K}_2$, $1 \leq j \leq \lfloor \frac{k}{2} \rfloor - 1$, is a Hamilton cycle, by Theorem 2.2. Thus the underlying graph of $\vec{H} \times \vec{K}_2$ is Hamilton cycle decomposable. As $\vec{H}$ is a spanning subdigraph of $\vec{G}'_k$ and the underlying graph of $\vec{G}'_k \times \vec{K}_2$ is isomorphic to $W_{k,\,2^k}$, $W_{k,\,2^k}$ contains $\lfloor \frac{k}{2} \rfloor - 1$ edge disjoint Hamilton cycles. □

To prove the existence of a Hamilton cycle decomposition of $W_{k,\,2^k}$, it is enough to prove that $\vec{G}''_k = \vec{G}'_k - A(\vec{H})$, when k is even or, $\vec{G}''_k = \vec{G}'_k - \{A(\vec{H}) \cup \vec{F}\}$, where $\vec{F}$ stands for the set of arcs of distance $\frac{k+1}{2}$ of $\vec{G}'_k$, when k is odd, is a bieulerian digraph. If $\vec{G}''_k$ is bieulerian, then by Theorem 2.2, the underlying graph of $\vec{G}''_k \times \vec{K}_2$ is Hamiltonian. The following remark explains the difficulty in proving that $\vec{G}''_k$ is bieulerian.

Remark 3.2 *In fact, we know a method by which we can prove that $\vec{G}''_k$ is bieulerian. But proving $\vec{G}''_k$ is bieulerian is complicated and too long and hence we have omitted the proof. A sketch of the proof of $\vec{G}''_k$ is bieulerian using a "reduction technique"is described in the Appendix I of the Ph.D. thesis of the second author, see [24]. Also, using the technique described therein, we have proved that G''_8 is bieulerian, see Appendix II of [24]. This proves that $W_{8,2^8}$ is Hamilton cycle decomposable. Using the method described in the Appendix I mentioned above, and Lemma 3.3, of this paper, here we obtain three arc disjoint spanning bieulerian subdigraphs of $\vec{G}'_6$, see Figures 8, 9 and 10. Thus $W_{6,\,2^6} = W_{6,\,64}$ is Hamilton cycle decomposable, by Theorem 2.2. We have not given the arc disjoint bieulerian subdigraphs of G'_k for some smaller values of $k \geq 7$ as they have too many vertices and edges to draw.* □

We present three edge disjoint Hamilton cycles of $W_{6,2^6}$, where we assume that (X, Y) is the bipartition with $X = \{(1, j) \,|\, 0 \leq j \leq 31\}$ and $Y = \{(2, j) \,|\, 0 \leq j \leq 31\}$. The three Hamilton cycles H_1, H_2 and H_3 of $W_{6,64}$ given below are obtained using the arc disjoint bieulerian subdigraphs (shown in Figures 8 to 10) of G'_6.

$$\begin{aligned} H_1 = {} & (1,0)(2,7)(1,4)(2,11)(1,8)(2,15)(1,12)(2,19)(1,16)(2,23)(1,20)(2,20)(1,13)(2,16) \\ & (1,17)(2,24)(1,21)(2,28)(1,25)(2,0)(1,29)(2,4)(1,1)(2,8)(1,5)(2,12)(1,9)(2,9)(1,6) \\ & (2,13)(1,10)(2,17)(1,14)(2,21)(1,18)(2,25)(1,22)(2,29)(1,26)(2,1)(1,30)(2,5)(1,2) \\ & (2,2)(1,31)(2,6)(1,3)(2,10)(1,7)(2,14)(1,11)(2,18)(1,15)(2,22)(1,19)(2,26)(1,23) \\ & (2,30)(1,27)(2,27)(1,24)(2,31)(1,28)(2,3)(1,0). \end{aligned}$$

$$\begin{aligned} H_2 = {} & (1,0)(2,1)(1,18)(2,19)(1,4)(2,5)(1,22)(2,23)(1,8)(2,9)(1,26)(2,27)(1,12)(2,13) \\ & (1,30)(2,31)(1,16)(2,17)(1,2)(2,3)(1,20)(2,21)(1,6)(2,7)(1,24)(2,25)(1,10)(2,11) \\ & (1,28)(2,29)(1,14)(2,15)(1,15)(2,16)(1,1)(2,2)(1,19)(2,20)(1,5)(2,6)(1,23)(2,24) \\ & (1,9)(2,10)(1,27)(2,28)(1,13)(2,14)(1,31)(2,30)(1,29)(2,12)(1,11)(2,26)(1,25) \\ & (2,8)(1,7)(2,22)(1,21)(2,4)(1,3)(2,18)(1,17)(2,0)(1,0). \end{aligned}$$

$$\begin{aligned} H_3 = {} & (1,0)(2,31)(1,31)(2,0)(1,1)(2,1)(1,2)(2,9)(1,10)(2,10)(1,11)(2,11)(1,12)(2,12)(1,13) \\ & (2,13)(1,14)(2,14)(1,15)(2,30)(1,30)(2,29)(1,29)(2,28)(1,28)(2,27)(1,20)(2,19)(1,19) \\ & (2,18)(1,18)(2,17)(1,17)(2,20)(1,21)(2,21)(1,22)(2,22)(1,23)(2,23)(1,24)(2,24)(1,25) \\ & (2,25)(1,26)(2,26)(1,27)(2,2)(1,3)(2,3)(1,4)(2,4)(1,5)(2,5)(1,6)(2,6)(1,7)(2,7)(1,8) \\ & (2,8)(1,9)(2,16)(1,16)(2,15)(1,0). \end{aligned}$$

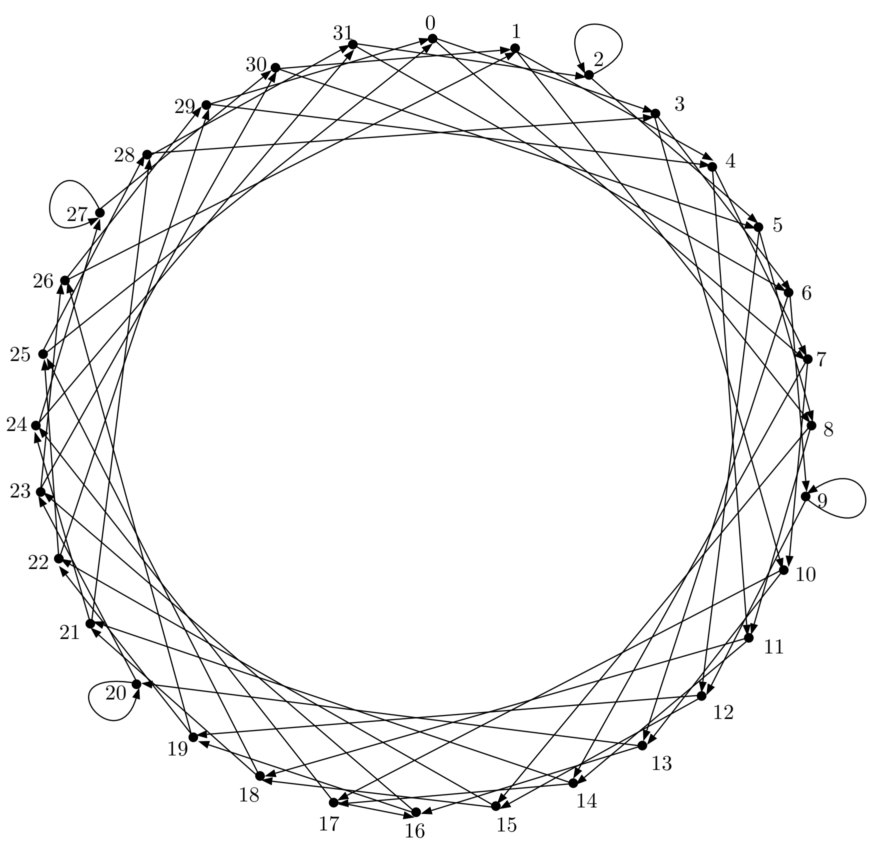


Fig. 8: A spanning 2-diregular bieulerian subdigraph $\vec{H}_2$ of $\vec{G}'_6$.

Acknowledgements

The second author would like to thank the Department of Science and Technology, Government of India, New Delhi and Annamalai University for awarding Junior Research Fellowship under the PURSE Programme. The second author also thank the Management and Principal, SSN College of Engineering, Chennai.

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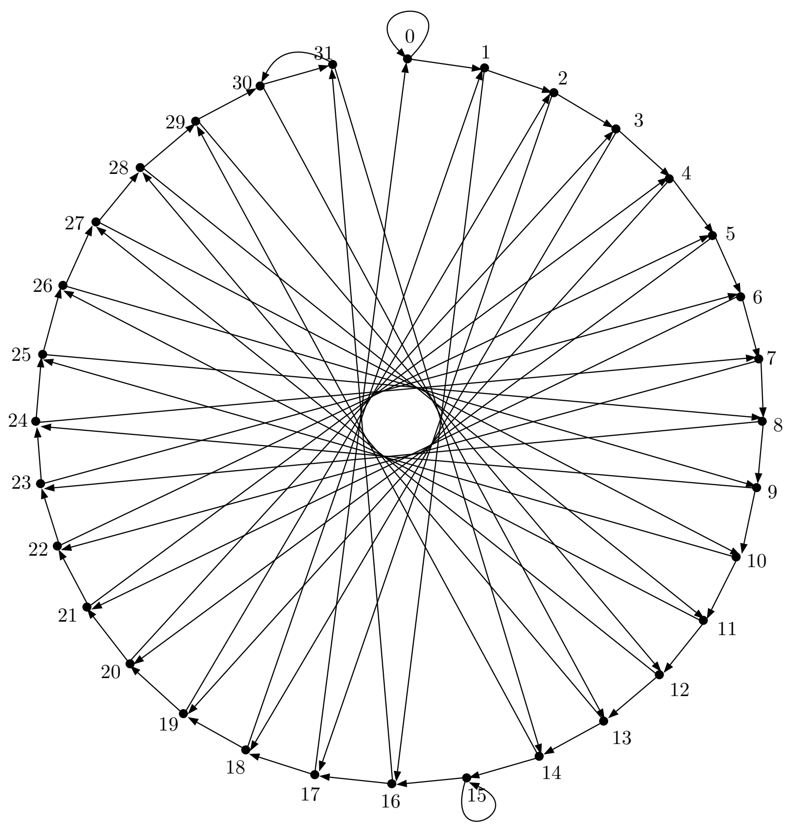

Fig. 9: A spanning 2-diregular bieulerian subdigraph $\vec{H}_1$ of $\vec{G}'_6$.

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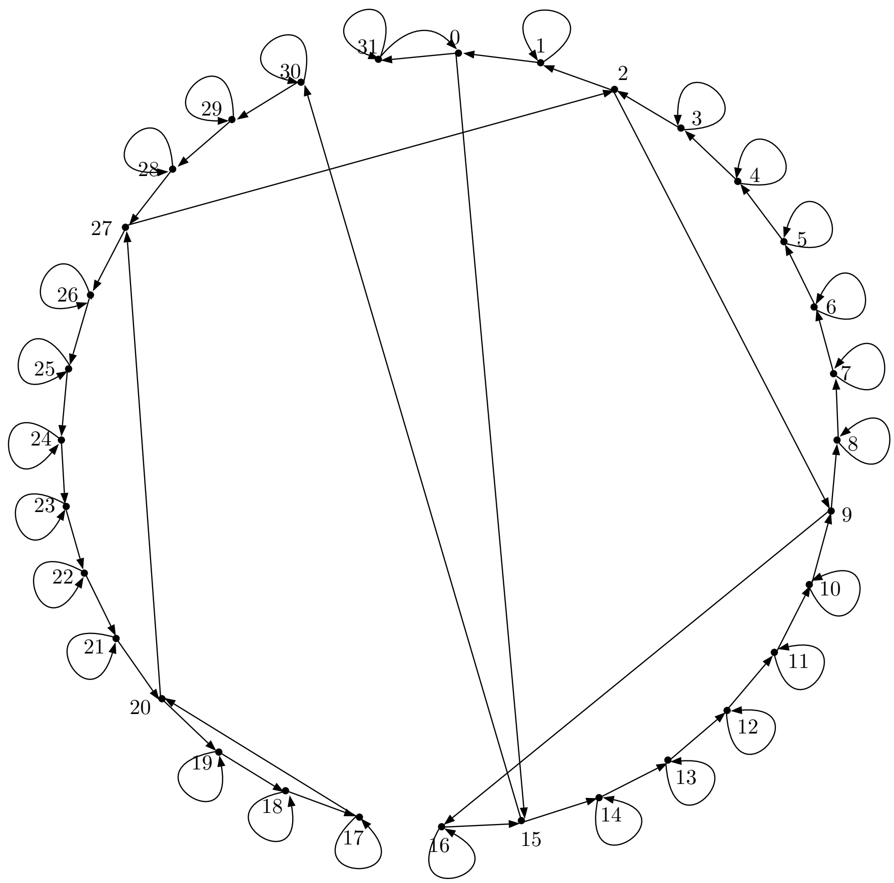

Fig. 10: A spanning 2-diregular bieulerian subdigraph $\vec{H}_0$ of $\vec{G}'_6$.

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