Discrete Mathematics and Theoretical Computer Science DMTCS vol. **17:1**, 2015, 307–308

A Note on a Recent Attempt to Improve the Pin-Frankl Bound *

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received 3rd Dec. 2014, *revised 2nd Apr. 2015, 8th Apr. 2015*, *accepted 8th Apr. 2015*.

We provide a counterexample to a lemma used in a recent tentative improvement of the Pin-Frankl bound for synchronizing automata. This example naturally leads us to formulate an open question, whose answer could fix the line of the proof, and improve the bound.

Keywords: Automata, Synchronization, Černý's conjecture.

This short note studies a problem related with synchronizing automata and Černý's conjecture, formulated in [2]. A good survey on the topic is given in [10]. See [1], [4], [5] for recent work on the subject.

A (deterministic, finite state, complete) *automaton* (DFA) is a triplet (Q, Σ, δ) with Q the set of *states*, Σ the alphabet of *letters* and δ the transition function $\delta : Q \times \Sigma \to Q$ defining the effect of the letters on the states. For $q_i, q_j \in Q$ and $l \in \Sigma$, we write $q_i l = q_j$ if $\delta(q_i, l) = q_j$. We call a *word* w of *length* m a sequence of m letters $l_1 \ldots l_m$, $l_i \in \Sigma, 1 \leq i \leq m$. We write Σ^m the set of words of length m. For $q_i, q_j \in Q$ and $w = l_1 \ldots l_m \in \Sigma^m$, we write $q_i w = q_j$ if $\delta(\ldots\delta(\delta(q_i, l_1), l_2)\ldots, l_m) = q_j$. For an automaton with n states and a word w, we note $Qw = \{q_j \mid q_i w = q_j,\ 1 \leq i \leq n\}$ the set of states that are in the image of w. We can represent an automaton as a directed graph. Each state is represented as a vertex, and the effect of each letter on each state is represented as a directed edge. We call a DFA *strongly connected* if its graph representation is a strongly connected graph.

A word w is called a *synchronizing word* if, for any states $q_i, q_j \in Q$, $q_i w = q_j w$. A DFA is called a *synchronizing automaton* if it has a synchronizing word.

Černý's conjecture [2] states that any *synchronizing automaton* with n states has a *synchronizing word* of length at most $(n-1)^2$.

So far the best proven bound is $(n^3 - n)/6$, obtained more than 30 years ago in [3] and [7], and rediscovered independently in [6]. Recently, a tentative improvement to $n(7n^2 + 6n - 16)/48$ has been proposed in [8]. However, as mentioned later by the author on ArXiv [9], there is a flaw in the proof. Nevertheless, since the publication of [8], many new papers are citing this result, and no publication clearly

*This work was also supported by the communauté francaise de Belgique - Actions de Recherche Concertées and by the Belgian Program on Interuniversity Attraction Poles initiated by the Belgian Federal Science Policy Office. Emails: francois.gonze@uclouvain.be, raphael.jungers@uclouvain.be, trakht@macs.biu.ac.il

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confirms that the proof is not valid. In this note, we make this point clear by providing a counterexample to Lemma 3 in [8]. The lemma is the following:

Lemma 1 (Lemma 3 in [8]) *Let Q be the set of states of a synchronizing strongly connected n-state DFA. Then for any state q there exists a word w of length not greater than n such that $q \notin Qw$. For any $k < n$ there are at least k states $q_1, ..., q_k$ and words $w_1, ..., w_k$ of length not greater than k such that $q_i \notin Qw_i$, $1 \leq i \leq k$.*

We refute the lemma by exhibiting an automaton such that, for one state q_0, there is no word w of length smaller than or equal to the number of states with the property that $q_0 \notin Qw$.

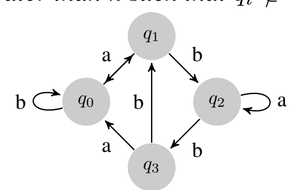

Fig. 1: A counterexample to Lemma 1

Counterexample The automaton represented in Fig. 1 is a synchronizing automaton, as the word $abbababba$ is a synchronizing word. However, the shortest word w such that $q_0 \notin Qw$ is $w = abbaba$. Since the automaton has only 4 states and w is 6 letters long, this contradicts Lemma 1.

Lemma 1 was a key step in the improvement on the maximal length of a shortest synchronizing word. We observe that a weaker version of Lemma 1 could still improve the Pin-Frankl bound. In fact, any value proportional to the number of states of the automaton would lead to an improvement of the bound. This motivates us to raise the following open question.

Open question *Let Q be the set of states of a synchronizing strongly connected $n-$state DFA.*

Is there a constant c such that, for any state $q \in Q$, there exists a word w of length not greater than cn such that $q \notin Qw$?

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