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# *The distribution of ascents of size $d$ or more in compositions*

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A composition of a positive integer $n$ is a finite sequence of positive integers $a_1,\, a_2,\, \ldots,\, a_k$ such that $a_1 + a_2 + \cdots + a_k = n$. Let $d$ be a fixed nonnegative integer. We say that we have an ascent of size $d$ or more if $a_{i+1} \geq a_i + d$. We determine the mean, variance and limiting distribution of the number of ascents of size $d$ or more in the set of compositions of $n$. We also study the average size of the greatest ascent over all compositions of $n$.

**Keywords:** compositions, distributions, generating functions, ascents

## 1 Introduction

A composition of a positive integer $n$ is a finite sequence of positive integers $a_1,\, a_1,\, \ldots,\, a_k$ such that $a_1 + a_2 + \cdots + a_k = n$. It is well known that there are $2^{n-1}$ compositions of $n$. The compositions (denoted by $a_1 a_2 a_3 \ldots a_k$) for $n = 1, 2, \ldots, 5$ are:

| $n$ | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | | | | | | | |
| 2 | 11 | 2 | | | | | | | | | | | | | | |
| 3 | 111 | 12 | 21 | 3 | | | | | | | | | | | | |
| 4 | 1111 | 13 | 31 | 22 | 112 | 121 | 211 | 4 | | | | | | | | |
| 5 | 11111 | 14 | 41 | 23 | 32 | 113 | 131 | 311 | 122 | 212 | 221 | 1112 | 1121 | 1211 | 2111 | 5 |

Let $d \geq 0$ be a fixed integer. We say, that we have an ascent of size $d$ or more, whenever $a_{i+1} \geq a_i + d$. For example, there are 3 ascents of size 2 or more that occur in the compositions of 5: $\underline{14}$, $1\underline{13}$, and $\underline{13}1$ .

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In Sections 2, 3 and 4, respectively, we determine the mean, variance and asymptotic distribution of the number of ascents of size $d$ or more in compositions of $n$. Ordinary ascents (the cases $d = 0$ and $d = 1$) have previously been studied by Carlitz (1) and more recently by Chinn, Heubach and Grimaldi in (2). Finally in Section 5 we investigate the maximum value of $d$ for which compositions of $n$ can expect to have an ascent of size $d$. That is, we find the average value of the *largest ascent* that occurs in the compositions of $n$. For example, the compositions of 4 in the table have maximum ascents of sizes $0, 2, 0, 0, 1, 1, 0, 0$, respectively, giving an average largest ascent size per composition of $1/2$ when $n = 4$.
We note that the asymptotic expression for the maximum ascent size in Theorem 4 involves a fluctuating function of $n$ of mean zero. A similar phenomenon has been observed when studying averages of certain other statistics for compositions, such as the largest part size (11) or the number of distinct part sizes (8), (7).

## 2 The average number of ascents of size $d$ or more in compositions

For fixed $d \geq 0$ we wish to find the average number of ascents of size $d$ or more per composition of $n$.
We use the "adding-the-slice" technique which was originally used by Flajolet and Prodinger in (4) and more recently, for example, by Knopfmacher and Prodinger in (9).
Let $j$ be the value of the last component of the composition with $k$ parts, i.e. $a_k = j$. We proceed from a composition with $k$ parts to a composition with $k + 1$ parts. We denote by $f_k(z, u, v)$ the generating function where $z$ marks the size $n$, $u$ the value of $j$ and $v$ the number of ascents of size $d$ or more in compositions with $k$ parts.
In moving from a composition with $k$ parts to a composition with $k + 1$ parts, where $a_k = j$, we have an ascent whenever the new last integer has any value from $j + d$ onwards. This gives the following rule for adding a new part or "slice" to the end of the composition:

$$\begin{aligned} u^j \longrightarrow & zu + (zu)^2 + (zu)^3 + \cdots + (zu)^{j+d-1} + v\left\{(zu)^{j+d} + (zu)^{j+d+1} + \cdots\right\} \\ &= zu\frac{1-(zu)^{j+d-1}}{1-zu} + v(zu)^{j+d}\frac{1}{1-zu}. \end{aligned}$$

This implies that

$$\begin{aligned} f_{k+1}(z,u,v) &= \frac{zu}{1-zu}f_k(z,1,v) - \frac{(zu)^d}{1-zu}f_k(z,zu,v) + \frac{v(zu)^d}{1-zu}f_k(z,zu,v) \\ &= \frac{zu}{1-zu}f_k(z,1,v) - \frac{(1-v)(zu)^d}{1-zu}f_k(z,zu,v). \end{aligned} \tag{2.1}$$

Now define $F(z,u,v) := \sum_{k\geq 1} f_k(z,u,v)$ . Then summing (2.1) over $k \geq 1$ gives

$$F(z,u,v) - f_1(z,u,v) = \frac{zu}{1-zu}F(z,1,v) - \frac{(1-v)(zu)^d}{1-zu}F(z,zu,v)\,,$$

so that

$$F(z,u,v) = \frac{zu}{1-zu}F(z,1,v) + \frac{zu}{1-zu} - \frac{(1-v)(zu)^d}{1-zu}F(z,zu,v)\,,$$

where we have used

$$f_1(z,u,v) = zu + (zu)^2 + (zu)^3 + \cdots = \frac{zu}{1-zu}\,.$$

At this stage we iterate the recursion for $F(z,u,v)$.

$$\begin{aligned}
F(z,u,v) &= \frac{zu}{1-zu}F(z,1,v) + \frac{zu}{1-zu} - \frac{(1-v)(zu)^d}{1-zu} \times \\
&\qquad \times \left\{ \frac{z^2u}{1-z^2u}F(z,1,v) + \frac{z^2u}{1-z^2u} - \frac{(1-v)(z^2u)^d}{1-z^2u}F(z,z^2u,v) \right\} \\
&= \left[ \frac{zu}{1-zu} - \frac{(1-v)z^2u(zu)^d}{(1-zu)(1-z^2u)} \right][F(z,1,v)+1] + \frac{(1-v)^2(zu)^d(z^2u)^d}{(1-zu)(1-z^2u)} \times \\
&\qquad \times \left\{ \frac{z^3u}{1-z^3u}F(z,1,v) + \frac{z^3u}{1-z^3u} - \frac{(1-v)(z^3u)^d}{1-z^3u}F(z,z^3u,v) \right\} \\
&= \left[ \frac{zu}{1-zu} - \frac{(1-v)z^2u(zu)^d}{(1-zu)(1-z^2u)} + \frac{(1-v)^2\, z^3u(zu)^d(z^2u)^d}{(1-zu)(1-z^2u)(1-z^3u)} \right][F(z,1,v)+1] \\
&\qquad - \frac{(1-v)^3(zu)^d(z^2u)^d(z^3u)^d}{(1-zu)(1-z^2u)(1-z^3u)}F(z,z^3u,v).
\end{aligned}$$

We keep iterating, noting that $F(z,z^m u,v) \to 0$ as $m \to \infty$ for $|z| < \frac{1}{2}$ and $u$, $v$ in a suitable small neighbourhood of 1, and put $u = 1$ to obtain

$$F(z,1,v) = \sum_{i\ge1} \frac{(-1)^{i-1}z^i(1-v)^{i-1}z^{d\binom{i}{2}}}{(1-z)(1-z^2)\cdots(1-z^i)}[F(z,1,v)+1]\,. \tag{2.2}$$

By adding the term 1 for the empty composition we obtain the bivariate generating function for compositions according to the number of ascents of size $d$ or more as

$$F(z,v) := 1 + F(z,1,v) = \frac{1}{1-\tau(z,v)}\,, \tag{2.3}$$

where

$$\tau(z,v) := \sum_{i\ge1} \frac{(-1)^{i-1}z^i(1-v)^{i-1}z^{d\binom{i}{2}}}{(1-z)(1-z^2)\cdots(1-z^i)}\,. \tag{2.4}$$

The expected value of the number of ascents of size $d$ or more is $\frac{[z^n]\frac{\partial F(z,v)}{\partial v}\big|_{v=1}}{2^{n-1}}$. For this we shall need

$$\tau(z,1) = \sum_{i\ge1} \frac{(-1)^{i-1}z^i(1-v)^{i-1}z^{d\binom{i}{2}}}{(1-z)(1-z^2)\cdots(1-z^i)}\Bigg|_{v=1} = \frac{z}{1-z}, \tag{2.5}$$

and

$$\frac{\partial\tau(z,v)}{\partial v}\Big|_{v=1} = \sum_{i\ge2} \frac{(-1)^i z^i(i-1)(1-v)^{i-2}z^{d\binom{i}{2}}}{(1-z)(1-z^2)\cdots(1-z^i)}\Bigg|_{v=1} = \frac{z^{2+d}}{(1-z)(1-z^2)}\,. \tag{2.6}$$

In particular, the generating function for all compositions is $F(z,1) = \frac{1}{1-\tau(z,1)} = \frac{1-z}{1-2z}$.
Now

$$\left.\frac{\partial F(z,v)}{\partial v}\right|_{v=1} = \left.\frac{\frac{\partial \tau(z,v)}{\partial v}}{(1-\tau(z,v))^2}\right|_{v=1} = \frac{z^{2+d}}{(1+z)(1-2z)^2}$$
$$= z^d\left[\frac{1}{9(1+z)} + \frac{1}{6(1-2z)^2} - \frac{5}{18(1-2z)}\right].$$

So that

$$[z^n]\left.\frac{\partial F(z,v)}{\partial v}\right|_{v=1} = \frac{(-1)^{n-d}}{9} + \frac{(n-d+1)2^{n-d}}{6} - \frac{5\,2^{n-d}}{18}$$
$$= \frac{(-1)^{n-d}}{9} + \frac{(3n-3d-2)2^{n-d}}{18}.$$

After dividing by $2^{n-1}$, the total number of compositions of $n$, we have

**Theorem 1** *The expected number of ascents of size $d$ or more in the compositions of $n$ is*

$$\mathbb{E}(n) := \frac{2^{-d}}{9}(3n-3d-2) + \frac{2}{9}\frac{(-1)^{n-d}}{2^n}\,, \textit{ for } n \geq d.$$

*Hence for fixed $d$, as $n \to \infty$,*

$$\mathbb{E}(n) = \frac{2^{-d}}{3}\,n + O(1)\,.$$

Previously Chinn, Heubach and Grimaldi found the number of ascents for $d = 0$ and $d = 1$ in (2). The case $d = 0$ corresponds to the number of rises plus the number of levels, whereas $d = 1$ corresponds to the number of rises.

# 3 Variance of the number of ascents of size $d$ or more in compositions

To find the variance we first need to compute $\left.\frac{\partial^2 F(z,v)}{\partial v^2}\right|_{v=1}$. In addition to formulas (2.3) to (2.6) from Section 2 we require

$$\left.\frac{\partial^2 \tau(z,v)}{\partial v^2}\right|_{v=1} = \sum_{i\geq 3}\left.\frac{(-1)^{i-1}z^i\,(i-1)(i-2)(1-v)^{i-3}z^{d\binom{i}{2}}}{(1-z)(1-z^2)\cdots(1-z^i)}\right|_{v=1} = \frac{2z^{3(1+d)}}{(1-z)(1-z^2)(1-z^3)}\,.$$

Then

$$\left.\frac{\partial^2 F(z,v)}{\partial v^2}\right|_{v=1} = \left.\frac{(1-\tau)\frac{\partial^2\tau(z,v)}{\partial v^2} + 2\tau'^{\,2}}{(1-\tau)^3}\right|_{v=1}.$$

Computing the $n$th coefficient of the second derivative amounts to expanding and combining binomial series.
Finally, after adding the expectation and subtracting the square of the expectation we find

**Theorem 2** *The variance of the expected number of ascents of size $d$ or more in the compositions of $n$ is*

$$\begin{aligned}\mathbb{V}(n) &:= 2^{-d}\left\{\frac{-2}{9}+\frac{n}{3}-\frac{d}{3}\right\}+2^{-2d}\left\{\frac{80}{81}+\frac{10d}{9}+\frac{d^2}{3}-\frac{13n}{27}-\frac{2nd}{9}\right\}\\ &+2^{-3d}\left\{-\frac{352}{441}+\frac{8n}{21}-\frac{8d}{7}\right\}+2^{-n}\left\{(-1)^n\left(\frac{5}{27}+\frac{2n}{27}-\frac{4d}{27}\right)-\frac{1}{3}+2\,\alpha(n)\right\}\\ &+2^{-n-d}(-1)^{n-d}\left\{\frac{4d}{27}-\frac{4n}{27}+\frac{8}{81}\right\}-2^{-2n}\frac{4}{81},\end{aligned}$$

*for $n \geq 3d$, where*

$$\alpha(n)=\begin{cases}\frac{26}{147} & \text{if } n=3m,\\ \frac{-4}{147} & \text{if } n=3m-2,\\ \frac{-22}{147} & \text{if } n=3m-1, \quad \text{for } m\in\mathbb{N}.\end{cases}$$

*For fixed $d$ we have*

$$\mathbb{V}(n)\sim n\left\{\frac{2^{-d}}{3}-2^{-2d}\left(\frac{13}{27}+\frac{2d}{9}\right)+\frac{2^{-3d}\,8}{21}\right\} \text{ as } n\to\infty\,.$$

# 4 Limiting distribution

We are interested in finding the limiting distribution of our random variable. We make use of Theorem IX.9 from Flajolet and Sedgewick (5). A short version is as follows:

*Let $F(z,u)$ be a bivariate function that is bivariate analytic at $(z,u)=(0,0)$ and has nonnegative coefficients there. Assume that $F(z,1)$ is meromorphic in $z\leq r$ with only a simple pole at $z=\rho$ for some positive $\rho<r$. Then, under further conditions stated in (5), the random variable with probability generating function*

$$p_n(u)=\frac{[z^n]F(z,u)}{[z^n]F(z,1)}$$

*converges in distribution to a Gaussian variable with a speed of convergence that is $O(n^{-1/2})$.*
Let us introduce the notation

$$c_{i,j}:=\frac{\partial^{i+j}}{\partial z^i\partial u^j}\,C(z,u)\bigg|_{(\rho,1)}. \tag{4.1}$$

From Theorem IX.9 we need to show that

$$c_{0,1}c_{1,0}\neq 0. \tag{4.2}$$

In addition, we must show that

$$\rho c_{1,0}^2c_{0,2}-\rho c_{1,0}c_{1,1}c_{0,1}+\rho c_{2,0}c_{0,1}^2+c_{0,1}^2c_{1,0}+c_{0,1}c_{1,0}^2\rho\neq 0. \tag{4.3}$$

For our specific problem

$$F(z,v)=\frac{1}{1-\tau(z,v)}\equiv\frac{B(z,v)}{C(z,v)},$$

so that

$$C(z,v) = 1 - \sum_{i\geq 1} \frac{(-1)^{i-1} z^i (1-v)^{i-1} z^{d\binom{i}{2}}}{(1-z)(1-z^2)\cdots(1-z^i)}.$$

We have $\rho(1) = \rho = \frac{1}{2}$ and using (4.1),

$$c_{0,1} = -\frac{2^{1-d}}{3}, \qquad c_{1,0} = -4, \qquad c_{1,1} = -\frac{3d+11}{9} 2^{2-d}, \qquad c_{0,2} = -\frac{2^{4-3d}}{21}, \qquad c_{2,0} = -16.$$

We are now in a position to check the conditions listed in the theorem.
Equation (4.2) is satisfied since

$$c_{0,1} c_{1,0} = 8 \frac{2^{-d}}{3} \neq 0.$$

Equation (4.3) is equivalent to

$$2^{4-3d} \left( \frac{2^3}{21} + 2^d \frac{3d+11}{27} - \frac{2^{1+d}}{9} - \frac{2^{2+2d}}{9} \right) \neq 0$$

for non-negative integer values of $d$. Thus we deduce

**Theorem 3** *The distribution of the number of ascents of size* $d$ *or more in compositions of* $n$ *converges to a Gaussian distribution with a speed of convergence of* $O(n^{-1/2})$ *with the mean* $\mu_n$ *and the variance* $\sigma_n^2$ *are as given in Theorems 1 and 2.*

**Remark** In Flajolet and Sedgewick (5) it is also shown that under the conditions of Theorem IX.9, the mean $\mu_n$ and variance $\sigma_n^2$ are of the form

$$\mu_n = \mathfrak{m}\left(\frac{\rho(1)}{\rho(u)}\right) n + O(1), \qquad \sigma_n^2 = \mathfrak{v}\left(\frac{\rho(1)}{\rho(u)}\right) n + O(1),$$

where

$$\mathfrak{m}(f) = \frac{f'(1)}{f(1)} \quad \text{and} \quad \mathfrak{v}(f) = \frac{f''(1)}{f(1)} + \frac{f'(1)}{f(1)} - \left(\frac{f'(1)}{f(1)}\right)^2.$$

These asymptotic expressions are easily checked to be in agreement with the exact results for the mean and variance found previously in Theorems 1 and 2.

# 5 Size of the maximum ascent

Given a composition $a_1\, a_2 \ldots a_k$ of $n$ we shall study the size of the maximum ascent, that is, the parameter $X$ where

$$X := \max\{a_{i+1} - a_i \mid 1 \leq i < k \text{ and } a_{i+1} \geq a_i\}.$$

We assign a value of 0 if no pair of consecutive integers satisfies this condition. The mean value of $X$ is given by the expression

$$\sum_{d=0}^{n} \mathbb{P}(X > d) = \sum_{d=0}^{n} (1 - \mathbb{P}(X \leq d)).$$

Therefore for each fixed $d$ we need to compute the probability that a composition of $n$ has maximum ascent $X \le d$.
We already know the generating function for compositions with no ascents of size $d$ or more, which we will denote by $F_d(z)$. For this we use (2.3) with $d+1$ instead of $d$ and $v=0$, giving

$$F_{d+1}(z) := \frac{1}{1-\tau(z,0)} = \frac{1}{1-\sum_{i\ge 1}\frac{(-1)^{i-1}z^{i+(d+1)\binom{i}{2}}}{(1-z)(1-z^2)\cdots(1-z^i)}} .$$

We want the generating function of compositions with $X \le d$. This is equivalent to the generating function of the compositions with no ascent of size $d+1$ or more, which is $F_{d+1}(z)$. We now need to study the dominant poles, $\rho_d$, of $F_{d+1}(z)$, that is, the dominant zeros of

$$1-\sum_{i\ge 1}\frac{(-1)^{i-1}z^{i+(d+1)\binom{i}{2}}}{(1-z)(1-z^2)\cdots(1-z^i)} = 0 . \tag{5.1}$$

We follow the approach used by Gourdon and Prodinger in (6), which is also analogous to the one found in (10). Now for $|z| \le \frac{3}{5}$, say, using the first two terms of the series above, the root $\rho_d$ above can be approximated by the smallest positive root of

$$1-\frac{z}{1-z}+\frac{z^{d+3}}{(1-z)(1-z^2)}+O(z^{3d}) = 0 ,$$

since the omitted terms in (5.1) are $O\left(z^{3d}\right)$. This error will be majorized by subsequent $O$ terms below. That is we want the root of

$$1-2z+\frac{z^{d+3}}{1-z^2}+O(z^{3d}) = 0 .$$

The bootstrapping method gives a suitable approximation to $\rho_d$.
Let $\rho_d := \frac{1}{2}+\varepsilon_d$, then

$$1-2\left(\frac{1}{2}+\varepsilon_d\right)+\frac{4}{3}\,2^{-d-3} = O\left(\left(\frac{3}{5}\right)^{3d}\right)+O\left(\frac{d}{2^{2d}}\right) ,$$

from which we find

$$\varepsilon_d = \frac{2^{-d}}{12}+O\left(\frac{d}{2^{2d}}\right) \text{ as } d\to\infty.$$

As $F_{d+1}(z)$ has a simple pole at $\rho_d$, and by means of Rouche's Theorem, see IX.6.2 in (5), by comparing $|\frac{1}{F_{d+1}(z)}|$ with $|1-\frac{z}{1-z}|$ on the circle $|z| = \frac{3}{4}$, we see that $\frac{1}{F_{d+1}(z)}$ has no other zeros in $|z| \le 3/4$.

It follows that

$$[z^n]F_{d+1}(z) = [z^n]\frac{A_d}{1-z/\rho_d}+O\left(\left(\frac{4}{3}\right)^n\right) \text{ with } A_d = \frac{1}{\rho_d\left.\frac{d\tau(z,0)}{dz}\right|_{z=\rho_d}} = \frac{2+O(2^{-d})}{\left.\frac{d\tau(z,0)}{dz}\right|_{z=\rho}} ,$$

where from (2.4) as $d \to \infty$

$$\frac{d\tau(z,0)}{dz}\Big|_{z=\rho} = \frac{\partial}{\partial z}\sum_{i\geq 1}\frac{(-1)^{i-1}z^{i+(d+1)\binom{i}{2}}}{(z;z)_i}\Bigg|_{z=\rho} = \left(\frac{z}{(1-z)^2}+O(z^d)\right)\Big|_{z=\rho} = 4+O(2^{-d})\,.$$

Therefore as $d \to \infty$, $A_d = \frac{1}{2}+O(2^{-d})$ . Let us now restrict our attention to those $d$ for which $n^{-3} \leq 2^{-d} \leq \frac{\log n}{n}$. The probability that $X \leq d$ is then approximated as $n \to \infty$ by

$$\begin{aligned} A_d\,\rho_d^{-n} = A_d\left(\frac{1}{2}+\frac{2^{-d}}{12}+O\left(\frac{d}{2^{2d}}\right)\right)^{-n} &= 2^{n-1}\exp\left(-\frac{2^{-d}\,n}{6}\right)\left(1+O\left(2^{-d}\right)+O\left(n\,d\,2^{-2d}\right)\right) \\ &= 2^{n-1}\exp\left(-\frac{2^{-d}\,n}{6}\right)\left(1+O\left(\frac{\log^3 n}{n}\right)\right)\,. \end{aligned}$$

So after dividing by $2^{n-1}$ we have for $n \to \infty$ and $n^{-3} \leq 2^{-d} \leq \frac{\log n}{n}$,

$$\mathbb{P}(X \leq d) = \exp\left(-\frac{n}{2^d\,6}\right)\left(1+O\left(\frac{\log^3 n}{n}\right)\right)\,. \tag{5.2}$$

Turning now to smaller values of $d \geq 1$, that is, $d$ such that $2^{-d} > \frac{\log n}{n}$, a similar computation shows that (5.2) remains valid in this range, although now the probabilities $\mathbb{P}(X \leq d)$ are small, since for such $d$, $\exp\left(-\frac{n}{2^d\,6}\right) = O(\frac{1}{n})$ as $n \to \infty$. Finally we must consider larger values of $d \leq n$ that is, $d$ for which $n^{-3} > 2^{-d}$, or equivalently, $d \geq 3\log_2 n$. In this range we find that

$$\mathbb{P}(X \leq d) = 2^{n-1}\exp\left(-\frac{2^{-d}\,n}{6}\right)\left(1+O\Big(\frac{1}{n^2}\Big)\right)\,. \tag{5.3}$$

In view of the $O$ estimates in (5.2) and (5.3) we may deduce that the mean value of $X$ satisfies

$$\mathbb{E}_{max}(n) := \sum_{d=0}^{n}(1-\mathbb{P}(X\leq d)) = \left(\sum_{d=0}^{n}\left(1-\exp\left(-\frac{n}{2^d\,6}\right)\right)\right)\left(1+O\left(\frac{\log^4 n}{n}\right)\right) \quad \text{as } n\to\infty.$$

We now use the Mellin transform, (see (3)), to estimate the function

$$f(t) = \sum_{d\geq 0}\left(1-\exp\left(-\frac{t}{2^d\,6}\right)\right) \text{ as } t\to\infty.$$

The Mellin transform of $f(t)$ is

$$f^*(s) = -\sum_{d\geq 0}\left(2^d\,6\right)^s\,\Gamma(s) = -\frac{\Gamma(s)\,6^s}{1-2^s} \quad \text{for } -1<\Re s<0\,.$$

Next we apply the Mellin inversion formula to recover $f(t)$:

$$f(t) = -\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\Gamma(s)\frac{6^s}{1-2^s}\,t^{-s}\,ds\,.$$

We then shift the line of integration to the right and collect the (negative) residues at $s = \frac{2k\pi i}{L}$ where $k \in \mathbb{Z}$ and $L = \log 2$.

$$f(t) = \sum_{k\geq 0} \operatorname{Res} \Gamma(s) \frac{6^s}{1-2^s}\, t^{-s} \bigg|_{s=\frac{2k\pi i}{L}} + \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s) \frac{6^s}{1-2^s}\, t^{-s}\, ds\,.$$

To evaluate the residue at the double pole $s = 0$, we need the expansion of the terms in the integrand to two terms as $s \to 0$, (here $\gamma$ denotes Euler's constant),

$$\Gamma(s) \sim \frac{1}{s} - \gamma\,,$$
$$6^s \sim 1 + s\log 6\,,$$
$$t^{-s} \sim 1 - s\log t\,,$$
$$\frac{1}{1-2^s} \sim \frac{1}{2} - \frac{1}{sL}\,.$$

Hence the (negative) residue at $s = 0$ is

$$\begin{aligned}&[s^{-1}]\left\{\left(\frac{1}{2} - \frac{1}{sL}\right)(1 + s\log 6)\left(\frac{1}{s} - \gamma\right)(1 - s\log t)\right\}\\ &= \log_2 t - \frac{1}{2} - \log_2 3 + \frac{\gamma}{L}\,.\end{aligned}$$

For the residue at $s = \frac{2k\pi i}{L}$ for $k \neq 0$, let $\varepsilon = s - \frac{2k\pi i}{L}$ we have

$$[\varepsilon^{-1}]\Gamma\left(\frac{2k\pi i}{L}\right)\left(\frac{t}{6}\right)^{-\frac{2k\pi i}{L}} \frac{-1}{\varepsilon L} = -\frac{1}{L}\,\Gamma\left(\frac{2k\pi i}{L}\right)\left(\frac{t}{6}\right)^{-\frac{2k\pi i}{L}}\,.$$

The remainder integral $\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}$ is of smaller order, so we have found

**Theorem 4** *The mean value of the size of the greatest ascent in the compositions of $n$ satisfies as $n \to \infty$,*

$$\mathbb{E}_{max}(n) \sim \log_2 n - \frac{1}{2} - \log_2 3 + \frac{\gamma}{L} - \delta\left(\log_2 \frac{n}{6}\right)$$

*where $\delta(x)$ is a continuous periodic function of mean zero, period one and small amplitude with Fourier series*

$$\delta(x) = \frac{1}{L}\sum_{k\neq 0} \Gamma\left(-\frac{2k\pi i}{L}\right) e^{2k\pi i x}\,.$$

Computations show that $\delta(x) < 1.7 \times 10^{-6}$, as a result of the fast decrease of the gamma function with imaginary argument.

The diagram below shows the plot of $\delta(\log_2 \frac{n}{6})$ for $1 \leq n \leq 100$.

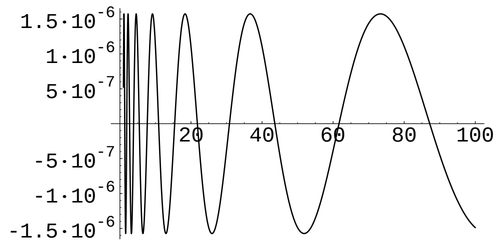


**Fig. 1:** $\delta(\log_2 \frac{n}{6})$ for $1 \leq n \leq 100$

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