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# *Noncommutative Symmetric Functions Associated With a Code, Lazard Elimination, and Witt Vectors*

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The construction of the universal ring of Witt vectors is related to Lazard's factorizations of free monoids by means of a noncommutative analogue. This is done by associating to a code a specialization of noncommutative symmetric functions.

**Keywords:** Witt vectors, codes, symmetric functions, factorizations of free monoids

## 1 Introduction

Among Ernst Witt's many contributions to mathematics, one finds two apparently unrelated ideas, both published in 1937 in two consecutive issues of Crelle's journal. The first one, the ring of Witt vectors [19] (a generalisation of $p$-adic numbers), solves a problem in commutative algebra, whilst the second one, the introduction of the free Lie algebra [20], definitely pertains to noncommutative mathematics.

The aim of this note is to point out a close connection between both constructions, through the notion of noncommutative symmetric functions. What comes out is that the natural noncommutative analogues of the symmetric functions classically associated to the construction of universal Witt vectors can be related to another classical topic in the combinatorics of free Lie algebras: Lazard's factorizations of free monoids. This relation manifests itselfs when the elementary noncommutative symmetric functions are specialized by means of a code.

Our notations will be those of [11] and [7]. If $S$ is a set of words, we denote by $\underline{S} = \sum_{w\in S} w$ its characteristic series.

## 2 Noncommutative Witt symmetric functions

Let us first recall the construction of the universal ring $W(R)$ of Witt vectors associated to a commutative ring $R$. This is not Witt's original construction, but a simpler one that he found much later, and that he communicated to S. Lang, who published it as a series of exercises in his famous textbook of Algebra [9].

The ring $W(R)$ can be characterized by the following properties [4] (see also [8, 5]):

(W1) As a set, $W(R) = \{\mathbf{a} = (a_n)_{n\geq 1}\, ,\ a_n \in R\}$, and for any ring homomorphism $f:\ R \to R'$, the map $W(f):\ \mathbf{a} \mapsto (f(a_n))_{n\geq 1}$ is a ring homomorphism.

(W2) The maps $w_n:\ \mathbf{a} \mapsto \sum_{d|n} d a_d^{n/d}$ are ring homomorphisms $W(R) \to R$.

Operations on Witt vectors are better understood in terms of symmetric functions. Let $X = \{x_n\, ,\ n \geq 1\}$ be an infinite set of commuting indeterminates (called here an *alphabet*), and following [14], define symmetric functions $q_n(X)$ by

$$\prod_{n\geq 1} \frac{1}{1 - t^n q_n(X)} = \sigma_t(X) := \sum_{n\geq 0} t^n h_n(X) \tag{1}$$

where the complete homogeneous functions $h_n(X)$ are defined by (*cf.* [11])

$$\sigma_t(X) = \prod_{n\geq 1} (1 - t x_n)^{-1}.$$

The $q_n$'s are connected to the power-sums

$$p_n(X) = \sum_i x_i^n,$$

by

$$p_n = \sum_{d|n} d q_d^{n/d}, \tag{2}$$

and condition (W2) can be regarded as expressing the familiar properties of power sums

$$p_n(X + Y) = p_n(X) + p_n(Y),$$

$$p_n(XY) = p_n(X) p_n(Y),$$

where we use the $\lambda$-ring notation (an alphabet is identified with the formal sum of its elements). The transformation (1) is also used in [2, 12].

Reutenauer [14] studied the symmetric functions $q_n(X)$ and made the conjecture that for $n \geq 2$, $(-q_n)$ is Schur positive. This conjecture was proved in [17], in a generalised noncommutative form. Denoting as in [7] the noncommutative complete homogeneous symmetric functions of an alphabet $A$ by $S_n(A)$, one introduces noncommutative Witt symmetric functions $Q_n(A)$ by the identity

$$\overrightarrow{\prod_{n\geq 1}} \frac{1}{1 - t^n Q_n(A)} = \sigma_t(A) := \sum_{n\geq 0} t^n S_n(A). \tag{3}$$

Then, it is proved in [17] that for $n \geq 2$, $(-Q_n)$ is a positive, multiplicity free, sum of noncommutative ribbon Schur functions $R_I$.

## 3 Noncommutative symmetric functions associated to a code

Let $A$ be an alphabet and $C \subset A^+$ a *code*, *i.e.* a minimal generating set of a free submonoid $C^*$ in $A^*$. Suppose we have a decomposition of $C$ as a disjoint union

$$C = \coprod_{n \geq 1} C_n, \tag{4}$$

of possibly empty subsets $C_n$. In what follows, we shall in general assume that $C_n = C \cap A^n$, but this restriction in not necessary. In any case, we will consider that the elements of $C_n$ have degree $n$.

We will denote by $l(w)$ the length of a word $w$. Recall that, as a graded algebra, $\mathbf{Sym} = \mathbb{Q}\langle \Lambda_1, \Lambda_2, \dots \rangle$ where $\Lambda_n$ has degree $n$. We can define a specialization $\mathbf{Sym}[C]$ of the algebra of noncommutative symmetric functions by setting

$$\Lambda_n[C] = (-1)^{n-1} \underline{C}_n \,. \tag{5}$$

With this choice of signs, the complete symmetric functions are then given by

$$S_n[C] = \sum_{w \in (C^*)_n} w \,, \tag{6}$$

the sum of all elements of degree $n$ in $C^*$. We call *$C$-Witt symmetric function* the value $Q_i[C]$ of $Q_i$ under this specialization. Let us give the first $Q_i[C]$ for three examples

**Example 1** The Fibonacci prefix code $C = \{b, ab\}$.

$$\begin{aligned}
Q_1[C] &= b \\
Q_2[C] &= ab \\
Q_3[C] &= ab^2 \\
Q_4[C] &= ab^3 \\
Q_5[C] &= ab^2ab + ab^4 \\
Q_6[C] &= ab^3ab + ab^5 \\
Q_7[C] &= ab^6 + ab^2(ab)^2 + ab^4ab + ab^3ab^2
\end{aligned}$$

**Example 2** The infinite prefix code $C = ba^*$.

$$\begin{aligned}
Q_1[C] &= b \\
Q_2[C] &= ba \\
Q_3[C] &= ba^2 + bab \\
Q_4[C] &= bab^2 + ba^3 + ba^2b \\
Q_5[C] &= bab^2a + bab^3 + ba^2b^2 + ba^2b^2 + ba^4 + ba^2ba \\
Q_6[C] &= ba^4b + ba^3b^2 + ba^2b^2a + bab^3a + bab^4 + ba^3ba + ba^5 + ba^2b^3
\end{aligned}$$

**Example 3** The Dyck code $D$ (for the Dyck language $\underline{D^*} = 1 + a\underline{D^*}b\underline{D^*}$.)

$$\begin{array}{l} Q_1[D] = Q_3[D] = Q_5[D] = Q_7[D] = 0 \\ Q_2[D] = ab \\ Q_4[D] = aabb \\ Q_6[D] = aaabbb + aababb + aabbab \\ Q_8[D] = aaaabbbb + aaababbb + aabaabbb + aaabbbab + aaabbabb \\ +aabbabab + aabababb + aababbab \end{array}$$

On these examples, we remark that each $Q_i$ is multiplicity free and is the characteristic series of a code. In the following section, we prove that it is always the case and give a characterization of these codes in terms of Lazard's factorizations.

# 4 Witt symmetric functions and factorization

## 4.1 Lazard elimination process

We recall that a factorization of a monoid $\mathbb{M}$ is an ordered family of monoids $\mathbb{F} = (\mathbb{M}_i)_{i\in I}$ such that each element $m \in \mathbb{M}$ admits an unique decomposition

$$m = m_{i_1} \cdots m_{i_k} \tag{7}$$

where $i_1 > \cdots > i_k$ and $m_{i_1} \in \mathbb{M}_{i_1}, \dots, m_{i_k} \in \mathbb{M}_{i_k}$. In the case of free monoids $\mathbb{M} = A^*$, this property can be stated in terms of generating series

$$\sum_{w\in A^*} w = \overleftarrow{\prod_i} \sum_{w\in\mathbb{M}_i} w \tag{8}$$

A submonoid $\mathbb{M}' \subset \mathbb{M}$ can be characterized by its generating set $\mathbb{M}' \setminus \mathbb{M}'^2$. In the sequel, a factorization will be denoted by the sequence of the generating sets of its components, and for a factorization $\mathbb{F} = (C_i)_{i\in I}$ of $A^*$ into free submonoids, (8) reads

$$\frac{1}{1-\underline{A}} = \overleftarrow{\prod_i} \frac{1}{1-\underline{C_i}}. \tag{9}$$

Factorisations of free monoids have been extensively studied and the reader can refer to [3, 18] for a survey. A simple but relevant example is a *Lazard bisection*: considering a subalphabet $B \subset A$, one has

$$A^* = B^*((A \setminus B)B^*)^*. \tag{10}$$

The pair $(B, (A \setminus B)B^*)$ is a factorization of $A^*$ (*cf.*[18, 3], and [15] for applications to free Lie algebras). Now, from (10) we can obtain a trisection (*i.e.* a factorization in three submonoids) by iterating the process on the left or the right factor, and so on. Factorisations which can be generated applying only Lazard bisections on the right factors will be called *Lazard right compositions*. This is a simple particular case of a *locally finite right regular factorization* [18]. Let $A$ be an alphabet (possibly infinite) and $\omega : A \to \mathbb{N} \setminus 0$ a weight function. This function can be extended uniquely as a morphism $\omega : A^* \to (\mathbb{N}, +)$. We can associate a factorization $\mathbb{F}(A,\omega)$ to this weight as follows. Let $(Z_i)_{i\geq 1}$ and $(C_i)_{i\geq 1}$ be the sequences of codes defined by the recurrence relations

1. $Z_1 = A$,
2. for each integer $i > 0, C_i = Z_i \cap \{w \in A^* | \omega(w) = i\}$,
3. for each integer $i > 0, Z_{i+1} = (Z_i \setminus C_i)C_i^*$.

The sequence $\mathbb{F}(A,\omega) = (C_i)_{i\in I}$ obtained by omitting the empty elements in $(C_i)_{i\geq 1}$ is a Lazard right composition. Since each code $C \in A^*$ admits the length of words as weight function, we define the *right length factorization* $\mathbb{F}(C^*)$ of $C^*$ by

$$\mathbb{F}(C^*) = \mathbb{F}(C, l). \tag{11}$$

We can remark that a code is homogeneous if and only if $\mathbb{F}(C^*) = (C)$.

## 4.2 Computation of the $C$-Witt symmetric functions

The equality between formal series (3) can be rewritten as

$$\frac{1}{1-\underline{C}} = \frac{1}{1-Q_1}\frac{1}{1-Q_2}\frac{1}{1-Q_3}\cdots, \tag{12}$$

and in this section, we prove that this factorization of series is the weight right factorization of $C^*$.

In a more general setting, T. Scharf and one of the authors [17], gave a recursive algorithm for computing the $Q_n's$. Recall that the algebra **Sym** of noncommutative symmetric functions is the free associative algebra $\mathbb{Q}\langle S_1, S_2, \cdots\rangle$ generated by an infinite sequence of non commuting variables $S_n$, graded by the weight function $\omega(S_n) = n$. If $I = (i_1, \cdots, i_n)$, one defines

$$\tilde{S}^I = (-1)^n S_{i_1}\cdots S_{i_n}. \tag{13}$$

The term $Q_i$ can be computed following the rules given in [17]:

1. $F_1 = -\sum_i \tilde{S}_i$ ,
2. $F_{n+1} = F_n + Q_n(1 - F_n)$,
3. $Q_n$ is the term of weight $n$ in $F_n$ multiplied by $-1$.

Setting $Z_n = 1 - (1 - F_n)^{-1}$, we obtain

$$F_{n+1} = 1 - (1 - Z_{n+1})^{-1}, \tag{14}$$

and

$$\begin{aligned} F_n + Q_n(1-F_n) &= 1 - (1-Z_n)^{-1} + Q_n(1-Z_n)^{-1} \\ &= 1 - (1-Q_n)(1-Z_n)^{-1}. \end{aligned} \tag{15}$$

This implies

$$\begin{array}{rcl} Z_{n+1} & = & 1-(1-Z_n)(1-Q_n)^{-1} \\ & = & (Z_n-Q_n)(1-Q_n)^{-1}. \end{array} \tag{16}$$

Following [17], each $Q_i$ is multiplicity free on the $\tilde{S}_I$.

Let $Z_i[C]$ and $F_i[C]$ be respectively the values of $Z_i$ and $F_i$ under the specialization $S_n = S_n[C]$. The definition of noncommutative complete functions gives

$$\sigma_1[C] = \sum S_n[C] = \frac{1}{1-\underline{C}}. \tag{17}$$

Hence, one has

$$Z_1[C] = 1-(\sigma_1[C])^{-1} = \underline{C}, \tag{18}$$

$$Q_1[C] = S_1[C] = \underline{C_1}, \tag{19}$$

and

$$Z_1[C] - Q_1[C] = \underline{C} - \underline{C_1}. \tag{20}$$

It follows that $Z_1[C]$ and $Q_1[C]$ are the characteristic series of the codes $Z_1 = C$ and $Q_1 = C_1$. By induction, for each $n > 0$ the series $Z_{n+1}[C]$ and $Q_{n+1}[C]$ are the characteristic series of $Z_{n+1} = (Z_n \setminus Q_n)Q_n^*$ and $Q_n^* = Z_n \cap A^{\leq n}$. Hence, the following statement holds.

**Proposition 1** *Let $C$ be a code. Each $C$-Witt symmetric function $Q_i[C]$ is the characteristic series of a code, and the sequence obtained by deleting the empty set from $(Q_1, Q_2, \ldots, Q_n, \ldots)$ is the right length factorization of $C^*$.*

**Example 4** The sequence $(a, Q_1[ba^*], Q_2[ba^*], \ldots)$ is a factorization of $A^*$. The same method is applicable to compute homogeneous factorizations of non-homogeneous alphabets. For example, considering the alphabet $A = \mathbb{N} \setminus 0$ with the weight $\omega = id$, one finds

$$\begin{array}{l} Q_1[A] = 1, \\ Q_2[A] = 2, \\ Q_3[A] = 21+3, \\ Q_4[A] = 211+31+4, \\ Q_5[A] = 2111+212+311+32+41+5, \\ Q_6[A] = 21111+51+2112+6+3111+312+42+411. \end{array}$$

It is easy to see that this can be obtained from Example 2 by the morphism $ba^n \to n+1$.

One has the following decomposition

**Corollary 2** *Let $w \in C^*$, then either $w \in C$ either there exist $w_1, w_2 \in C^*$ such that $w = w_1w_2$, with $\omega(w_2) < \omega(w_1) < \omega(w)$. Furthermore, if $w_1 \notin C$ then $w_1 = w_1'w_1''$ with $w_1', w_1'' \in C^*$ and $\omega(w_1'') \leq \omega(w_2)$.*

This follows from the standard bracketing process of regular factorizations, which is described in [18].

### 4.3 Noncommutative elementary symmetric functions and Lazard elimination

The link between Lazard elimination and noncommutative Witt symmetric functions can be better understood in terms of elementary symmetric functions. The generating function of noncommutative elementary symmetric function is

$$\lambda_t = \sum_{k\geq 0} \Lambda_k t^k. \tag{21}$$

These are related to noncommutative complete functions by

$$\sigma_t = \frac{1}{\lambda_{-t}} = \frac{1}{1 - \Lambda_1 t + \Lambda_2 t^2 - \cdots + (-1)^n \Lambda_n t^n + \cdots}. \tag{22}$$

If we set $\tilde{\Lambda}_n = (-1)^{n+1}\Lambda_n$, the series $\sigma_1$ can be considered as the characteristic series of the free monoid $\mathbf{\Lambda}^* = \{\tilde{\Lambda}_1, \tilde{\Lambda}_2, \ldots, \tilde{\Lambda}_n \ldots\}^*$. We endow this monoid with the weight function defined by $\omega(\tilde{\Lambda}_n) = n$. Then,

**Theorem 3** *One has*

$$\mathbb{F}(\mathbf{\Lambda}, \omega) = (Q_1[\mathbf{\Lambda}], Q_2[\mathbf{\Lambda}], \ldots) = (Q_1, Q_2, \ldots) \tag{23}$$

This provides a simple algorithm for computing the decomposition of the noncommutative symmetric Witt functions on the basis of elementary symmetric functions. Let us gives the computation of the first $Q_i$'s,

$$\begin{aligned}
Q_1 =\ & \Lambda_1,\\
Q_2 =\ & -\Lambda_2,\\
Q_3 =\ & -\Lambda_2\Lambda_1 + \Lambda_3,\\
Q_4 =\ & -\Lambda_2\Lambda_1\Lambda_1 + \Lambda_3\Lambda_1 - \Lambda_4,\\
Q_5 =\ & -\Lambda_2\Lambda_1^3 + \Lambda_2\Lambda_1\Lambda_2 + \Lambda_3\Lambda_1\Lambda_1 - \Lambda_3\Lambda_2 - \Lambda_4\Lambda_1 + \Lambda_5,\\
Q_6 =\ & -\Lambda_2\Lambda_1^4 + \Lambda_5\Lambda_1 + \Lambda_2\Lambda_1\Lambda_1\Lambda_2 - \Lambda_6 + \Lambda_3\Lambda_1^3 - \Lambda_3\Lambda_1\Lambda_2 + \Lambda_4\Lambda_2\\
& -\Lambda_4\Lambda_1\Lambda_1,\\
Q_7 =\ & -\Lambda_3\Lambda_1^2\Lambda_2 + \Lambda_3\Lambda_1^4 - \Lambda_3\Lambda_1\Lambda_2\Lambda_1 + \Lambda_3\Lambda_1\Lambda_3 + \Lambda_3\Lambda_2^2 - \Lambda_4\Lambda_1^3\\
& +\Lambda_4\Lambda_1\Lambda_2 - \Lambda_4\Lambda_3 + \Lambda_4\Lambda_2\Lambda_1 + \Lambda_5\Lambda_1^2 - \Lambda_5\Lambda_2 - \Lambda_6\Lambda_1 + \Lambda_7\\
& -\Lambda_2\Lambda_1^5 + \Lambda_2\Lambda_1^3\Lambda_2 + \Lambda_2\Lambda_1^2\Lambda_2\Lambda_1 - \Lambda_2\Lambda_1^2\Lambda_3 - \Lambda_2\Lambda_1\Lambda_2^2.
\end{aligned}$$

(compare Examples 2 and 4).

The decomposition of the elementary functions on the basis $Q_I = Q_{i_1} \cdots Q_{i_n}$ is obtained by inspection of the series

$$\frac{1}{\sigma_t} = \lambda_{-t} = 1 - \sum_n \tilde{\Lambda}_n t^n = \overleftarrow{\prod_i}(1 - Q_i t^i). \tag{24}$$

One finds

$$\tilde{\Lambda}_n = \sum_k (-1)^k \sum_{\substack{i_1 > \cdots > i_k \\ i_1 + \cdots + i_k = n}} Q_{i_1} \cdots Q_{i_k}. \tag{25}$$

# 5 Examples involving Gaudier's $*$-multinomials

## 5.1 Gaudier's $*$ multinomials

Classical (commutative) Witt vectors give rise to examples involving interesting integer sequences. Let us recall for example the construction of $*$-multinomial coefficients and $*$-factorials given by Gaudier in [6]. If $R$ is a $\mathbb{Q}$-algebra, there is a commutative diagram

$$\begin{array}{ccc} W(R) & \xrightarrow{e} & \Lambda(R) = 1 + tR[[t]] \\ {\scriptstyle w}\Big\downarrow & & \Big\downarrow{\scriptstyle \partial} \\ R^{\mathbb{N}^*} & \xrightarrow{\iota} & R[[t]] \end{array}$$

where

$$\partial = \frac{d}{dt}\ln,$$

$$l(c_1, \cdots, c_n, \cdots) = \sum_{n\geq 1} c_n t^{n-1},$$

$$w = (w_1, \cdots, w_n, \cdots),$$

$$e(a_1, \dots, a_n, \dots) = \overrightarrow{\prod} \frac{1}{1 - a_n t^n}.$$

These maps are all isomorphisms. Let $i_1, i_2, \dots, i_k$ be $k$ positive integers and $n = i_1 + \cdots + i_k$. In [6] Gaudier has defined the $*$-multinomial coefficient $*\binom{n}{i_1,\dots,i_k} \in W(\mathbb{Q})$ as the Witt vector such that $w_p = \binom{np}{pi_1,\dots,pi_k}$. In the same paper, he has defined the $*$-factorial $*n!/n!$ by $w_p(*n!/n!) = \frac{1}{n!}\frac{(pn)!}{p!^n}$. In particular, he has computed $e(*2!/2!) = e(\frac{1}{2} * \binom{2}{1})$ in closed form and related it to the Catalan numbers

$$e(*2!/2!) = \left(\frac{1+\sqrt{1-4t}}{2}\right)^{-1}. \qquad (26)$$

This raised the question whether it was possible to find similar expressions for other $*$-factorials and $*$-multinomials, and to give combinatorial interpretations.

## 5.2 Non-commutative analogues of $\frac{1}{n} * \binom{n}{1}$

Consider the Dyck code of example 3. The free monoid $D^*$ is a submonoid of $\{aa, ab, bb\}^*$, graded by $\rho(aa) = \rho(bb) = \rho(ab) = 1$. Under the $D$-specialization,

$$\Lambda_n = (-1)^{n+1} \sum_{\substack{w\in D \\ \rho(w)=n}} w. \qquad (27)$$

Recall that the noncommutative power sums of the second kind $\Phi_n$ are defined by

$$\sum_{n\geq 1} \Phi_n \frac{t^n}{n} = \log \sigma_t\,. \qquad (28)$$

If one interprets Dyck words as binary trees, the $\Phi_n$ can be regarded as sums over forests

$$\Phi_n = n \sum_k \frac{1}{k} \sum_{\substack{w_1,\dots,w_k \in D \\ \rho(w_1)+\cdots+\rho(w_k)=n}} w_1 \cdots w_k. \tag{29}$$

**Example 5** The first values of $\Phi_n$ are
$\Phi_1 = ab$,
$\Phi_2 = 2aabb + ab.ab$,
$\Phi_3 = 3aaabbb + 3aababb + \frac{3}{2}aabb.ab + \frac{3}{2}ab.aabb + ab.ab.ab$,

$$\begin{aligned}\Phi_4 \;=\;& 4aaaabbbb + 4aaabbabb + 4aabaabbb + 4aabababb + 4aaababbb \\ &+2aaabbb.ab + 2aababb.ab + 2ab.aaabbb + 2ab.aababb + \tfrac{4}{3}aabb.ab.ab \\ &+\tfrac{4}{3}ab.aabb.ab + \tfrac{4}{3}ab.ab.aabb + ab.ab.ab.ab.\end{aligned}$$

See Example 3 for the first values of $Q_n$.

Setting $\pi(w) = 1$ for all words $w$, one has

$$\pi(\Phi_n) = w_n(\frac{1}{2!} * 2!) = w_n(\frac{1}{2} * \binom{2}{1}), \tag{30}$$

and one recovers (26).

More generally, for $*\frac{1}{n}\binom{n}{1}$, an easy application of the Lagrange inversion formula gives that

$$e\left(\frac{1}{n} * \binom{n}{1}\right) = \sum_{k\geq 0} \frac{\binom{nk}{k}}{(n-1)k+1} t^k, \tag{31}$$

is also the generating series of $n$-ary trees. Consider the free monoid $F_n = T_n^*$ over the alphabet $T_n = \{a_{\mathbf{t}}\}$ whose letters are labelled by $n$-ary trees and weighted by $\rho(a_{\mathbf{t}}) = E(\mathbf{t})/n$, where $E(\mathbf{t})$ is the number of edges of $\mathbf{t}$. Instead of specializing $\Lambda_k$ as in the case $n = 2$, we set $S_k = \sum_{\rho(\mathbf{t})=k} a_{\mathbf{t}}$. Then, we obtain

$$\Phi_k = k \sum_p \frac{(-1)^{p+1}}{p} \sum_{\substack{\mathbf{t}_1,\dots,\mathbf{t}_p \\ \rho(\mathbf{t}_1)+\cdots+\rho(\mathbf{t}_p)=k}} a_{\mathbf{t}_1} \cdots a_{\mathbf{t}_p}. \tag{32}$$

Again, applying the morphism $\pi(w) = 1$, we obtain

$$\pi(\Phi_k) = w_k(\frac{1}{n} * \binom{n}{1}), \tag{33}$$

that is, we recover (31). Remark that this implies the identity

$$\sum_p \frac{(-1)^{p+1}}{p} \sum_{i_1+\cdots+i_p=k} \prod_m \frac{\binom{ni_m}{i_m}}{(n-1)i_m+1} = \frac{1}{nk}\binom{nk}{k}. \tag{34}$$

### 5.3 Combinatorial interpretations of some $*$-binomial coefficients

Remark that, as in section 5.2, the specialization $S_k = \sum_{\rho(\mathbf{t})=k+1} a_{\mathbf{t}}$ gives a non-commutative analogue of

$$e(*\binom{n}{1})(z) = \sum_{k=1}^{\infty} \frac{\binom{nk}{k}}{(n-1)k+1} t^{k-1} = \frac{e(\frac{1}{n} * \binom{n}{1}) - 1}{z}. \tag{35}$$

This last equality can be generalized as follows.

**Proposition 5.1** *Let $p \geq 1$ and $\omega = e^{2i\pi/p}$.*

$$\begin{aligned} e(*\binom{np}{p}) &= \frac{-1}{z} \prod_{k=0}^{p-1} \left(1 - e(\frac{1}{n} * \binom{n}{1}\right)(\omega^k z^{\frac{1}{p}})) & (36) \\ &= \prod_{k=0}^{p-1} e\left(*\binom{n}{1}\right)(\omega^k z^{\frac{1}{p}}). & (37) \end{aligned}$$

**Proof:** We first prove (37), by computing

$$\begin{aligned} \ln\left(\prod_{k=0}^{p-1} e\left(*\binom{n}{1}\right)(\omega^k z^{\frac{1}{p}})\right) &= \\ \sum_{k=0}^{p-1} \ln\left(e\left(*\binom{n}{1}\right)(\omega^k z^{\frac{1}{p}})\right) &= \\ \sum_{k=0}^{p-1}\sum_{j\geq 1} \frac{\binom{jn}{j}}{j} \omega^{jk} z^{\frac{j}{p}} &= \sum_{k\geq 1} \frac{\binom{knp}{kp}}{k} z^k \\ &= \ln(e\left(*\binom{np}{p}\right). \end{aligned}$$

Next (36) follows from the equality

$$e\left(*\binom{np}{p}\right) = \prod_{k=0}^{p-1} \frac{e\left(\frac{1}{n} * \binom{n}{1}\right)(\omega^k z^{\frac{1}{p}}) - 1}{\omega^k z^{\frac{1}{p}}}.$$

$\square$

From [13] (Theorem 1, p.7), the series $e(*\binom{np}{p})$ is the generating series of the number $q_m$ of lattice paths from $(0,0)$ to $((n-1)pm, pm)$ that never go above the path $(\uparrow^{(n-1)p}\rightarrow^p)^m$ (lattice paths are represented by words over the alphabet $F = \{\rightarrow, \uparrow\}$, where $\rightarrow$ means the elementary horizontal path $(1,0)$ and $\uparrow$ the elementary vertical path $(0,1)$).

**Example 6** The series

$$\begin{aligned} e(*\binom{4}{2})(z) &= 1 + 6z + 53z^2 + 554z^3 + 6362z^4 + \cdots \\ &= e(*\binom{2}{1})(z^{\frac{1}{2}}) e(*\binom{2}{1})(-z^{\frac{1}{2}}) \end{aligned},$$

is also the generating series of ordered trees on $2n$ nodes with every subtree at the root having an even number of edges(see Sloane [16] `ID number:A066357`). See also [1] for another enumeration.

$$\begin{array}{rcl} e(*\binom{6}{2})(z) & = & 1+15z+360z^2+10463z^3+\cdots \\ & = & e(*\binom{3}{1})(z^{\frac{1}{2}})e(*\binom{3}{1})(-z^{\frac{1}{2}}), \\ e(*\binom{6}{3})(z) & = & 1+20z+662z^2+26780z^3+\cdots \\ & = & e(*\binom{2}{1})(z^{\frac{1}{3}})e(*\binom{2}{1})(\exp\{\frac{2i\pi}{3}\}z^{\frac{1}{3}})\times \\ & & \times e(*\binom{2}{1})(\exp\{\frac{4i\pi}{3}\}z^{\frac{1}{3}}), \\ e(*\binom{12}{3})(z) = & = & 1+220z+91498z^2+47961320z^3+\cdots \\ & = & e(*\binom{4}{1})(z^{\frac{1}{3}})e(*\binom{4}{1})(\exp\{\frac{2i\pi}{3}\}z^{\frac{1}{3}})\times \\ & & \times e(*\binom{4}{1})(\exp\{\frac{4i\pi}{3}\}z^{\frac{1}{3}}). \end{array}$$

One can construct noncommutative analogues of the $*\binom{np}{p}$ by specializing $S_m$ to the sum of the words coding the lattice paths from $(0,0)$ to $((n-1)mp, mp)$ that never go above the path $(\uparrow^{(n-1)p}\rightarrow^{p})^m$.

**Example 7** Let us consider the non-commutative analogue of $*\binom{2}{1}$. Under this specialization, the first values of the $S_n$ are

$S_1 = \uparrow\rightarrow + \rightarrow\uparrow$,

$S_2 = \rightarrow^2\uparrow^2 + (\rightarrow\uparrow)^2 + \rightarrow\uparrow^2\rightarrow + \uparrow\rightarrow^2\uparrow + (\uparrow\rightarrow)^2$,

$$\begin{array}{rcl} S_3 & = & \rightarrow^3\uparrow^3 + \rightarrow^2\uparrow\rightarrow\uparrow^2 + \rightarrow^2\uparrow^2\rightarrow\uparrow + \rightarrow^2\uparrow^3\rightarrow \\ & & + \rightarrow\uparrow\rightarrow^2\uparrow^2 + (\rightarrow\uparrow)^3 + (\rightarrow\uparrow)^2\uparrow\rightarrow + \rightarrow\uparrow^2\rightarrow^2\uparrow \\ & & + \rightarrow\uparrow^2\rightarrow\uparrow\rightarrow + \uparrow\rightarrow^3\uparrow^2 + \uparrow\rightarrow^2\uparrow\rightarrow\uparrow + \uparrow\rightarrow^2\uparrow^2\rightarrow + \uparrow\rightarrow\uparrow\rightarrow^2\uparrow + (\uparrow\rightarrow)^3, \end{array}$$

and the first values of the $\Lambda_n$ are

$\Lambda_1 = \uparrow\rightarrow + \rightarrow\uparrow$,

$\Lambda_2 = -\rightarrow^2\uparrow^2$,

$\Lambda_3 = \rightarrow^2\uparrow\rightarrow\uparrow^2 + \rightarrow^3\uparrow^3$ .

More generally, from

$$\Lambda_m = (-1)^{m+1}\sum_{k=1}^{m}(-1)^k \sum_{i_1+\cdots+i_k=m} S_{i_1}\cdots S_{i_k},$$

one obtains

$$\Lambda_m = (-1)^{m+1}\sum_w w, \tag{38}$$

where the sum is over all lattice paths from $(0,0)$ to $((n-1)mp, mp)$ that never go above $(\uparrow^{(n-1)p}\rightarrow^{p})^m$ and which avoid the points $((n-1)kp, kp)$ for $0<k<m$. Hence,

$$\Phi_m = m\sum_w \frac{w}{c(w)+1}, \tag{39}$$

where the sum is over the lattice paths from $(0,0)$ to $((n-1)mp, mp)$ below $(\uparrow^{(n-1)p}\rightarrow^{p})^m$, and $c(w)$ is the number of points $((n-1)kp, kp)$ belonging to the path $w$, with $0<k<m$.

**Example 8** Let us consider the noncommutative analogue of $*\binom{4}{2}$. The first values of the $\Lambda_n$ are

$$\Lambda_1 = \to^2\uparrow^2 + (\to\uparrow)^2 + \to\uparrow^2\to + \uparrow\to^2\uparrow + \uparrow\to\uparrow\to + \uparrow^2\to^2,$$

$$\begin{aligned}\Lambda_2 \quad = \quad & -(\to^4\uparrow^4 + \to^3\uparrow\to\uparrow^3 + \to^3\uparrow^2\to\uparrow^2 + \to^3\uparrow^3\to\uparrow + \to^3\uparrow^4\to \\ & + \to^2\uparrow\to^2\uparrow^3 + \to^2\uparrow\,(\to\uparrow)^2\uparrow + \to^2\uparrow\to\uparrow^2\to\uparrow + \to^2\uparrow\to\uparrow^3\to \\ & + \to\uparrow\to^3\uparrow^3 + \to\uparrow\to^2\uparrow\to\uparrow^2 + \to\uparrow\to\uparrow^2\to\uparrow + \to\uparrow\to\uparrow^3\to \\ & + \uparrow\to^4\uparrow^3 + \uparrow\to^3\uparrow\to\uparrow^2 + \uparrow\to^2\uparrow^2\to\uparrow + \uparrow\to^2\uparrow^3\to).\end{aligned}$$

A natural question is whether it is possible to find similar interpretations for other $*$-binomials.

### 5.4 Shuffle analogues of $*$-multinomials in non commutative Witt vectors

We shall now construct another noncommutative analogue of $*$-multinomials. Let $A$ be an alphabet and $w_1, \cdots, w_k$ be $k$ words of respective lengths $m_1, \ldots, m_k$. Let $w = w_1 \cdot w_2 \cdots w_k$ and $m = m_1 + \cdots + m_k$. The shuffle $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, w_k$ contains $\binom{m}{m_1,\ldots,m_k}$ terms, and we can denote it by $\binom{w}{w_1,\ldots,w_k}$ to emphasize this point. Then, we introduce the noncommutative Witt vector analogue

$$*\binom{w}{w_1,\ldots,w_k} = (Q_1, Q_2, \cdots, Q_n, \cdots), \tag{40}$$

given by the sequence of Witt symmetric functions under the specialization

$$\Phi_n = w_1^n \sqcup\!\sqcup\, w_2^n \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, w_k^n. \tag{41}$$

**Example 9** If $w_i = a^{p_i}$, one has

$$\begin{aligned} *\binom{w}{w_1,\ldots,w_k} &= (w_1(*\binom{p_1+\cdots+p_k}{p_1,\ldots,p_k}a^{p_1+\cdots+p_k}, \\ & \cdots, w_n(*\binom{(p_1+\cdots+p_k)}{p_1,\ldots,p_k}a^{n(p_1+\cdots+p_k)}, \cdots). \end{aligned} \tag{42}$$

Sending $a$ to 1, one recovers $*$-multinomials.

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