Enumeration of words by the sum of differences between adjacent letters

Toufik Mansour

Department of Mathematics, University of Haifa, 31905 Haifa, Israel. Center for Combinatorics, LPMC, Nankai University, 300071 Tianjin, P.R. China.

received August 15, 2007, *revised January 9, 2009, January 25, 2009, February 23, 2009*, *accepted February 22, 2009*.

We consider the sum u of differences between adjacent letters of a word of n letters, chosen uniformly at random from a given alphabet. This paper obtains the enumerating generating function for the number of such words with respect to the sum u, as well as explicit formulas for the mean and variance of u.

Keywords: k-ary words, statistics, generating functions, kernel method, Chebyshev polynomials

1 Introduction

Given a totally ordered set of k letters $[k] = \{1, 2, \ldots, k\}$, we refer to the elements of $[k]^n$ as k-ary words of length n, or just words. Let σ be any k-ary word of length n, we define $u(\sigma) = u_{n,k}(\sigma)$ to be the sum of the differences between each pair of adjacent letters of σ, that is,

$$u(\sigma) = \sum_{i=1}^{n-1} |\sigma_i - \sigma_{i+1}|,$$

and u is the *total variation* (or just *variation*) of σ. For example, the total variation of the word $\sigma = 1121415221 \in [5]^{10}$ is $u(\sigma) = 0+1+1+3+3+4+3+0+1 = 16$.

Several authors have studied the statistic $u(\sigma)$ in different contexts. For example, Curtis [5] discussed the problem of finding a word $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ over the alphabet $\{a_1, a_2, \ldots, a_k\}$ which maximizes the *risk function* $\sum_{j=1}^{n} |\sigma_i - \sigma_{i-1}|$, where $\sigma_0 = \sigma_n$. Chao and Liang [3] studied the permutations of n distinct letters arranged on a circle or a line and found the arrangements that maximize the risk function. Later, Cohen and Tonkes [4] (see also [5, 9]) analyzed optimal permutations for multisets.

Motivated by the above results, we pose the following problem:

- Given a nonnegative integer m and a multiset of n letters $M = \{a_1, a_2, \ldots, a_n\}$, enumerate all permutations σ of M with $u(\sigma) = m$.

This paper addresses the restricted case of the problem in which $\sigma \in [k]^n$ for a fixed integer $k > 0$. More precisely, let $f_k(n, m)$ be the number of k-ary words σ of length n with $u(\sigma) = m$. Clearly, $u(\sigma) \leq u(1k1k\cdots) = (n-1)(k-1)$ for any $\sigma \in [k]^n$. We denote the correspond generating function by $F_k(n; q)$, that is,

$$F_k(n;q) = \sum_{m=0}^{(n-1)(k-1)} f_k(n,m)q^m = \sum_{\sigma\in[k]^n} q^{u(\sigma)},$$

and define

$$F_k(x,q) = \sum_{n\geq 0} F_k(n;q)x^n = \sum_{n\geq 0} x^n \sum_{\sigma\in[k]^n} q^{u(\sigma)}.$$

In the next section we find two different expressions for the bivariate generating function $F_k(x, q)$. The first expression is obtained by using the scanning element algorithm as described in [6] which it is given by the following result.

Theorem 1.1 *The bivariate generating function $F_k(x, q)$ is given by*

$$F_k(x,q) = 1 + \frac{k(1-q)x}{1-q-x(1+q)} - \frac{2x^2q\frac{1-t^k(q)}{1-t(q)}}{(1-q-x(1+q))\left(1-x\frac{1-qt^k(q)}{1-qt(q)}\right)},$$

where

$$t(q) = \frac{1-x+q^2(1+x)-\sqrt{(1-q)(1+q)(1+q-x(1-q))(1-q-x(1+q))}}{2q}.$$

A crucial device in proving Theorem 1.1 is the idea of a new statistic on the set of k-ary words of length n defined as follows. The *complete total variation* (or just *complete variation*) of $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in [k]^n$ is defined as $v(\sigma) = v_{n,k}(\sigma) = u(\sigma) + \sigma_1$. For example, the complete variation of the word $\sigma = 1121415221 \in [5]^{10}$ is $v(\sigma) = 1 + u(\sigma) = 17$.

The bivariate generating function $F_k(x, q)$ can be expressed in terms of Chebyshev polynomials of the second kind, see Theorem 1.2. *Chebyshev polynomials of the second kind* are defined by $U_r(\cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta}$. For example, $U_0(x) = 1$, $U_1(x) = 2x$, $U_2(x) = 4x^2 - 1$, and in general,

$$U_r(x) = 2xU_{r-1}(x) - U_{r-2}(x), \quad r = 0, \pm 1, \pm 2, \ldots. \tag{1.1}$$

Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, combinatorics, and number theory (see [12]).

Theorem 1.2 *The bivariate generating function $F_k(x, q)$ is given by*

$$F_k(x,q) = 1 + \sum_{i=1}^{k}(x_i - qx_{i-1}),$$

where

$$x_i = \frac{x(1-q)}{q}\sum_{j=i}^{k-1}\frac{U_{i-1}(\xi/2)\sum_{s=0}^{j-1}U_s(\xi/2)}{U_{j-1}(\xi/2)U_j(\xi/2)} + \frac{xU_{i-1}(\xi/2)}{U_{k-1}(\xi/2)}\left(\frac{U_{k-1}(\xi/2)+(1-q)/q\sum_{j=0}^{k-1}U_j(\xi/2)}{U_k(\xi/2)-q(1+x)U_{k-1}(\xi/2)}\right),$$

and $\xi = \frac{1+q^2-x(1-q^2)}{q}$.

Theorem 1.2 with $q = 1$ ($\xi = 2$, $U_j(1) = j+1$ and $U_k(\xi/2) - q(1+x)U_{k-1}(\xi/2) = 1 - kx$) gives that $x_i = \frac{ix}{1-kx}$ and $F_k(x,1) = 1 + \frac{kx}{1-kx} = \frac{1}{1-kx}$, which is the generating function for all k-ary words of length n. Moreover, using [7, Theorem IX.9 (Meromorphic schema)] with $\rho = \frac{1}{k}$ we obtain that the random variable $u_{n,k}$ with probability generating function $\frac{F_k(n;q)}{k^n}$ after standardization, converges in distribution to a Gaussian variable, with a speed of convergence that is $O(1/\sqrt{n})$, and the mean and the standard deviation of $u_{n,k}$ are asymptotically linear in n.

In our next theorem, we determine the mean and the variance of the total variation exactly.

Theorem 1.3 *The mean and the variance of the total variation of a uniformly chosen k-ary word of length n are given by*

$$\frac{(n-1)(k+1)(k-1)}{3k},$$

and

$$\frac{(k-1)(k+1)(k^2(6n-7)+2(3n-1))}{90k^2},$$

respectively.

We present two proofs of the above theorem. The first proof relies on Theorem 1.1, while the second consists of an elegant derivation based solely on counting special types of subsets in $[k]^n$ and the additivity of the expected value operator, see Section 3.

2 Proof of Theorems 1.1 and 1.2

In this section we establish the formula for the generating function $F_k(x,q)$ asserted in Theorems 1.1 and 1.2. To this end, let $F_k(x,q \mid i_1 i_2 \cdots i_s)$ denote the generating function for the number of k-ary words $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ enumerated with respect to the variation statistic $u(\sigma)$ such that $\sigma_1\sigma_2\cdots\sigma_s = i_1 i_2 \cdots i_s$, $s \in [n]$,

$$F_k(x,q \mid i_1 i_2 \cdots i_s) = \sum_{n\geq 0}\sum_{\sigma = i_1 i_2 \cdots i_s \sigma' \in [k]^n} x^n q^{u(\sigma)}.$$

Our first result is a recurrence relation for the functions $F_k(x,q \mid i)$.

Lemma 2.1 *For any $i = 1, 2, \ldots, k$,*

$$F_k(x,q \mid i) = x + x\sum_{j=1}^{k} q^{|i-j|}F_k(x,q \mid j).$$

Proof: From the definitions we can state that $F_k(x,q \mid i) = x + \sum_{j=1}^{k} F_k(x,q \mid ij)$ and $F_k(x,q \mid ij) = q^{|i-j|}F_k(x,q \mid j)$. Thus, $F_k(x,q \mid i) = x + x\sum_{j=1}^{k} q^{|i-j|}F_k(x,q \mid j)$, as claimed. □

Now let us solve the recurrence relation in the statement of Lemma 2.1 either by using kernel method technique (see [2, 6, 8]) or by using tridiagonal matrix algorithm (see [17]).

2.1 Kernel method technique

In order to solve the recurrence relation in the statement of Lemma 2.1, we define

$$F_k(x,q;t) = 1 + \sum_{i=1}^{k} F_k(x,q \mid i)t^{i-1}.$$

Clearly, $F_k(x,q;1) = F_k(x,q)$. So, in order to specify $F_k(x,q)$, we need to study a functional relation for $F_k(x,q;t)$.

Lemma 2.2

$$\begin{aligned} F_k(x,q;t) &= 1 + x\left(\tfrac{1-t^k}{1-t} - \tfrac{1-q^kt^k}{1-qt}\right) \\ &+\tfrac{x}{1-qt}\left(F_k(x,q;t) - q^kt^kF_k(x,q;1/q)\right) + \tfrac{x}{1-t/q}\left(F_k(x,q;q) - F_k(x,q;t)\right). \end{aligned}$$

Proof: Lemma 2.1 gives

$$F_k(x,q \mid i) = x + x\sum_{j=1}^{i} q^{i-j}F_k(x,q \mid j) + x\sum_{j=i+1}^{k} q^{j-i}F_k(x,q \mid j),$$

for all $i = 1, 2, \ldots, k$. Multiplying through by t^{i-1} and summing over i, $i \in [k]$, we obtain

$$\begin{aligned} &\sum_{i=1}^{k} F_k(x,q \mid i)t^{i-1} \\ &\quad = x\sum_{i=1}^{k} t^{i-1} + \sum_{i=1}^{k}\left(t^{i-1}\sum_{j=1}^{k} q^{|j-i|}F_k(x,q \mid j)\right) \\ &\quad = x\tfrac{1-t^k}{1-t} + x\sum_{j=1}^{k} t^{j-1}\tfrac{1-(qt)^{k+1-j}}{1-qt}F_k(x,q \mid j) + x\sum_{j=1}^{k} q^{j-1}\tfrac{1-(t/q)^{j-1}}{1-t/q}F_k(x,q \mid j). \end{aligned}$$

Using $F_k(x,q;t) = 1 + \sum_{j=1}^{k} t^{j-1}F_k(x,q \mid j)$ we get that

$$\begin{aligned} F_k(x,q;t) - 1 = x\tfrac{1-t^k}{1-t} &+ \tfrac{x}{1-qt}\left(F_k(x,q;t) - 1 - q^kt^kF_k(x,q;1/q) + q^kt^k\right) \\ &+ \tfrac{x}{1-t/q}\left(F_k(x,q;q) - F_k(x,q;t)\right), \end{aligned}$$

which completes the proof. □

We note the following useful restatement of Lemma 2.2:

$$\begin{aligned} &\left(1 - \tfrac{x}{1-qt} + \tfrac{x}{1-t/q}\right)F_k(x,q;t) \\ &\qquad = 1 + x\left(\tfrac{1-t^k}{1-t} - \tfrac{1-q^kt^k}{1-qt}\right) - \tfrac{xq^kt^k}{1-qt}F_k(x,q;1/q) + \tfrac{x}{1-t/q}F_k(x,q;q). \end{aligned} \tag{2.1}$$

We need a result relating $F_k(x, q; 1/q)$ and $F_k(x, q; q)$. We claim that

$$F_k(x, q; t) - 1 = t^{k-1}\left(F_k(x, q; 1/t) - 1\right). \tag{2.2}$$

To see this, define the complement of $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ to be $\sigma' = (k+1-\sigma_1)(k+1-\sigma_2)\cdots(k+1-\sigma_n)$. Then it is immediate from the definitions that σ' is a k-ary word of length n with $u(\sigma') = u(\sigma)$ and $\sigma'_1 = k+1-\sigma_1$. Hence,

$$\begin{aligned} &t^{k-1}(F_k(x, q; 1/t) - 1) \\ &\quad = t^{k-1}\sum_{n\ge1}\sum_{i=1}^{k}\sum_{\sigma=i\theta\in[k]^n} x^n(1/t)^{i-1}q^{u(\sigma)} = \sum_{n\ge1}\sum_{i=1}^{k}\sum_{\sigma=i\theta\in[k]^n} x^n t^{k-i}q^{u(\sigma)} \\ &\quad = \sum_{n\ge1}\sum_{i=1}^{k}\sum_{\sigma'=(k+1-i)\theta'\in[k]^n} x^n t^{i-1}q^{u(\sigma')} = F_k(x, q; t) - 1, \end{aligned}$$

as claimed.

Equation (2.2) for $t = q$ gives that $F_k(x, q; 1/q) = 1 + \frac{1}{q^{k-1}}(F_k(x, q; q) - 1)$. Hence, (2.1) reduces to

$$\begin{aligned} &\left(1 - \tfrac{x}{1-qt} + \tfrac{x}{1-t/q}\right) F_k(x, q; t) \\ &\quad = 1 + x\left(\tfrac{1-t^k}{1-v} - \tfrac{1-q^kt^k}{1-qv}\right) - \tfrac{xq^kt^k}{1-qt}\left(1 + \tfrac{1}{q^{k-1}}(F_k(x, q; q) - 1)\right) + \tfrac{x}{1-t/q}F_k(x, q; q), \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\left(1 - \tfrac{x}{1-qt} + \tfrac{x}{1-t/q}\right) F_k(x, q; t) \\ &\quad = 1 + x\left(\tfrac{1-t^k}{1-t} - \tfrac{1-qt^k}{1-qt}\right) - \left(\tfrac{xqt^k}{1-qt} - \tfrac{x}{1-t/q}\right) F_k(x, q; q), \end{aligned} \tag{2.3}$$

This type of functional equation can be solved using the *kernel method* (see [2, 8]). In this case, if we assume that

$$t = t(q) = \frac{1 - x + q^2(1+x) - \sqrt{(1-q)(1+q)(1+q-x(1-q))(1-q-x(1+q))}}{2q},$$

then (2.3) gives

$$F_k(x, q; q) = \frac{1 + x\left(\frac{1-t^k(q)}{1-t(q)} - \frac{1-qt^k(q)}{1-qt(q)}\right)}{\frac{xqt^k(q)}{1-qt(q)} - \frac{x}{1-t(q)/q}} = 1 + \frac{x\frac{1-t^k(q)}{1-t(q)}}{1 - x\frac{1-qt^k(q)}{1-qt(q)}}. \tag{2.4}$$

We can formulate the following theorem. Let $G_k(x, q)$ be the generating function for the number of k-ary words $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ of length n having (complete variation) $v(\sigma) = m$, that is,

$$G_k(x, q) = \sum_{n\ge0} x^n \sum_{\sigma\in[k]^n} q^{v(\sigma)}.$$

Theorem 2.3 *The generating function $G_k(x,q) = 1 - q + qF_k(x,q;q)$ is given by*

$$1 + \frac{qx\frac{1-t^k(q)}{1-t(q)}}{1 - x\frac{1-qt^k(q)}{1-qt(q)}},$$

where

$$t(q) = \frac{1 - x + q^2(1+x) - \sqrt{(1-q)(1+q)(1+q-x(1-q))(1-q-x(1+q))}}{2q}.$$

Proof: Using the definition $v(\sigma) = \sigma_1 + u(\sigma)$, we have that the contribution for nonempty k-ary words in $G_k(x,q)$ is $q(F_k(x,q;q) - 1)$ and the contribution of the empty word in $G_k(x,q)$ is 1. Thus $G_k(x,q) = 1 + q(F_k(x,q;q) - 1)$. The rest follows from (2.4). □

For example, Theorem 2.3 gives

$$\begin{aligned}
G_1(x,q) &= 1 + \tfrac{x}{1-x}q, \\
G_2(x,q) &= \tfrac{1-x+xq^2}{1-x-xq} = 1 + \tfrac{x}{1-x}q + \sum_{j\geq 2} \tfrac{x^{j-1}}{(1-x)^j}q^j, \\
G_3(x,q) &= \tfrac{(1-x)^2+x(1-x)q-x^2q^2+x(1+x)q^3}{(1-x)^2-xq^2(1+x)} \\
&= 1 + \tfrac{x}{1-x}q + \sum_{j\geq 0} \tfrac{x^j(1+x)^{j-1}}{(1-x)^{2j}}q^{2j} + \sum_{j\geq 1} \tfrac{x^j(1+x)^j}{(1-x)^{2j+1}}q^{2j+1}.
\end{aligned}$$

We are now ready to establish Theorem 1.1.

Proof *Proof of Theorem 1.1:* By substituting $t = 1$ in (2.3), we obtain

$$\left(1 - \frac{x(1+q)}{1-q}\right)F_k(x,q;1) = 1 + (k-1)x - \frac{2xq}{1-q}F_k(x,q;q),$$

which implies

$$F_k(x,q;1) = \frac{1-q+(k-1)(1-q)x}{1-q-x(1+q)} - \frac{2x}{1-q-x(1+q)}(G_k(x,q) - 1 + q).$$

Then by applying Theorem 2.3 we obtain the second part of Theorem 1.1. □

For example, Theorem 1.1 gives

$$\begin{aligned}
F_1(x,q) &= \tfrac{1}{1-x}, \\
F_2(x,q) &= \tfrac{1+x-xq}{1-x-xq} = 1 + 2\sum_{j\geq 0} \tfrac{x^{j+1}}{(1-x)^{j+1}}q^j, \\
F_3(x,q) &= \tfrac{(1+2x)(1-x)+4x^2q-x(1+2x)q^2}{(1-x)^2-xq^2(1+x)} \\
&= \tfrac{1+2x}{1-x} + 2(1+2x)\sum_{j\geq 1} \tfrac{x^{j+1}(1+x)^{j-1}}{(1-x)^{2j+1}}q^{2j} + 4\sum_{j\geq 0} \tfrac{x^{j+2}(1+x)^j}{(1-x)^{2j+2}}q^{2j+1}.
\end{aligned}$$

2.2 Tridiagonal matrix algorithm

Let $\mathbf{A}_k = \mathbf{A}_k(x,q) = (\mathbf{A}_k(x,q|i,j))_{1\le i,j\le k}$ be the $k\times k$ matrix

$$\mathbf{A}_k(x,q|i,j) = \begin{cases} 1-x, & i=j \\ -xq^{|i-j|}, & i\neq j. \end{cases}$$

Rewriting the equations in the statement of Lemma 2.1 in matrix form we get that

$$\mathbf{A}_k \begin{pmatrix} F_k(x,q\mid 1) \\ \vdots \\ F_k(x,q\mid k) \end{pmatrix} = x \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

which is equivalent to

$$\mathbf{B}_k \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = x \begin{pmatrix} 1-q \\ \vdots \\ 1-q \\ 1 \end{pmatrix}, \tag{2.5}$$

where $F_k(x,q\mid i) = x_i - qx_{i-1}$ (here we assume that $x_0 = 0$) and $\mathbf{B}_k = (b_{ij})_{1\le i,j\le k}$ is a $k\times k$ tridiagonal matrix

$$b_{ij} = \begin{cases} 0, & |i-j|>1 \\ -q, & |i-j|=1 \\ 1+q^2-x(1-q^2), & i=j<k \\ 1-x, & i=j=k. \end{cases}$$

By using the tridiagonal matrix algorithm on (2.5), as described in [17], we get that

$$x_k = d_k \text{ and } x_i = d_i - c_i x_{i+1}, \text{ for all } i=k-1,k-2,\dots 1, \tag{2.6}$$

where

$$d_i = \begin{cases} \frac{x(1-q)/q}{\xi}, & i=1 \\ \frac{x(1-q)/q+d_{i-1}}{\xi+c_{i-1}}, & 2\le i\le k-1 \\ \frac{x+qd_{k-1}}{1-x+qc_{k-1}}, & i=k, \end{cases} \qquad c_i = \begin{cases} -\frac{1}{\xi}, & i=1 \\ -\frac{1}{\xi+c_{i-1}}, & 2\le i\le k-1, \end{cases} \tag{2.7}$$

and $\xi = \frac{1+q^2-x(1-q^2)}{q}$. Thus, by Equations (1.1) and (2.7) together with induction on i, we obtain that

$$c_i = -\frac{U_{i-1}(\xi/2)}{U_i(\xi/2)}, \quad d_i = \begin{cases} \frac{x(1-q)}{qU_i(\xi/2)} \sum\limits_{j=0}^{i-1} U_j(\xi/2), & i<k \\ \frac{xU_{k-1}(\xi/2)+x(1-q)/q\sum_{j=0}^{k-1}U_j(\xi/2)}{U_k(\xi/2)-q(1+x)U_{k-1}(\xi/2)}, & i=k. \end{cases} \tag{2.8}$$

On the other hand, Equation (2.6) together with induction on $i=k,k-1,\ldots,1$ give that $x_i=\sum_{j=i}^k d_j\prod_{s=i}^{j-1}(-c_s)$. Thus, by using (2.8) we have that

$$x_i=\frac{x(1-q)}{q}\sum_{j=i}^{k-1}\frac{U_{i-1}(\xi/2)\sum_{s=0}^{j-1}U_s(\xi/2)}{U_{j-1}(\xi/2)U_j(\xi/2)} + \frac{xU_{i-1}(\xi/2)}{U_{k-1}(\xi/2)}\left(\frac{U_{k-1}(\xi/2)+(1-q)/q\sum_{j=0}^{k-1}U_j(\xi/2)}{U_k(\xi/2)-q(1+x)U_{k-1}(\xi/2)}\right),$$

for all $i=1,2,\ldots,k$. Hence, by using $F_k(x,q\mid i)=x_i-qx_{i-1}$, $i=1,2,\ldots,k$, together with $F_k(x,q)=1+\sum_{i=1}^k F_k(x,q\mid i)$, we complete the proof of Theorem 1.2.

3 Proof of Theorem 1.3

Let $[k]^n$ be the set of k-ary words of length n, where we assume that all words occur equally likely. Then $u_{n,k}=u_{n,k}(\sigma)$ and $v_{n,k}=v_{n,k}(\sigma)$ where $\sigma\in[k]^n$ are discrete random variables for the variation and the complete variation that take values $0,1,\ldots,(n-1)(k-1)$ and $1,2,\ldots,(n-1)(k-1)+k$, respectively.

3.1 Generating function techniques

As a corollary of Theorem 1.1 we get the following two results.

Theorem 3.1 *The mean of the random variables $u_{n,k}$ and $v_{n,k}$ are given by*

$$\mu_{n,k}=\frac{1}{k^n}\sum_{\sigma\in[k]^n}u_{n,k}(\sigma)=\frac{(n-1)(k+1)(k-1)}{3k},$$

and

$$\nu_{n,k}=\frac{1}{k^n}\sum_{\sigma\in[k]^n}v_{n,k}(\sigma)=\frac{(k+1)((2n+1)k-2(n-1))}{6k}.$$

Proof: After simple algebraic operations, Theorem 2.3 gives

$$\sum_{n\geq0}\nu_{n,k}x^n=\frac{\partial}{\partial q}G_k(x,q)\big|_{q=1}=\frac{\binom{k+1}{2}x-\binom{k+2}{3}x^2}{(1-kx)^2},$$

and Theorem 1.1 gives

$$\sum_{n\geq0}\mu_{n,k}x^n=\frac{\partial}{\partial q}F_k(x,q)\big|_{q=1}=\frac{2\binom{k+1}{3}x^2}{(1-kx)^2},$$

which implies that $\nu_{n,k}=\frac{(k+1)((2n+1)k-2(n-1))}{6k}$ and $\mu_{n,k}=\frac{(n-1)(k+1)(k-1)}{3k}$, as claimed. □

Theorem 3.2 *The variance of the random variables $u_{n,k}$ and $v_{n,k}$ are given by*

$$\frac{(k-1)(k+1)(k^2(6n-7)+2(3n-1))}{90k^2},$$

and

$$\frac{(k-1)(k+1)(12n(k^2+1)+k^2-4)}{180k^2},$$

respectively.

Proof: After simple algebraic operations, Theorem 1.1 gives

$$\sum_{n\geq 0}\mu_{n,k}(\mu_{n,k}-1)x^n = \frac{\partial^2}{\partial q^2}F_k(x,q)\big|_{q=1} = \frac{4x^2\left(\binom{k+1}{4}+\frac{k^2+5k-4}{10}\binom{k+1}{3}x-\frac{k}{3}\binom{k+2}{5}x^2\right)}{(1-kx)^3},$$

which implies that the ratio of the second factorial moment of the random variable $u_{n,k}$ by k^n is given by

$$\mu'_{n,k} = \tfrac{1}{k^n}\left(4k^{n-2}\binom{k+1}{4}\binom{n}{2}+\tfrac{2k^{n-3}(k^2+5k-4)}{5}\binom{k+1}{3}\binom{n-1}{2}-\tfrac{4k^{n-3}}{3}\binom{k+2}{5}\binom{n-2}{2}\right).$$

Using this expression, the variance of the random variable $u_{n,k}$ is given by

$$\mu'_{n,k}+\mu_{n,k}-(\mu_{n,k})^2 = \tfrac{(k-1)(k+1)(k^2(6n-7)+2(3n-1))}{90k^2}.$$

Similarly, Theorem 2.3 gives

$$\begin{aligned}\sum_{n\geq 0}\nu_{n,k}(\nu_{n,k}-1)x^n &= \tfrac{\partial^2}{\partial q^2}G_k(x,q)\big|_{q=1}\\ &= \tfrac{kx(k-1)(k+1)(30-15kx+6(k^2-4)x^2-k(k^2-4)x^3}{90(1-kx)^3},\end{aligned}$$

which implies that the ratio of the second factorial moment of the random variable $v_{n,k}$ by k^n is given by

$$\nu'_{n,k} = \tfrac{(k-1)(k+1)(16k^2(n+3)+10(k^2-1)n^2+26n-12)}{90k^2}.$$

Using this expression, the variance of the random variable $v_{n,k}$ is given by

$$\begin{aligned}&\nu'_{n,k}+\nu_{n,k}-(\nu_{n,k})^2\\ &= \tfrac{(k-1)(k+1)(12n(k^2+1)+k^2-4)}{180k^2},\end{aligned}$$

as required. □

3.2 Additivity of the expected value operator

In this section, we give an alternative proof of Theorems 3.1 and 3.2 (see Theorem 1.3), which consists of an elegant derivation based solely on counting special types of subsets in $[k]^n$ and the additivity of the expected value operator.

In order to do that, let $X_i = X_i(\sigma)$, $i=1,2,\ldots,n-1$ and $\sigma\in[k]^n$, be the discrete random variable for the absolute difference $|\sigma_i-\sigma_{i+1}|$ between values in the i-th and $(i+1)$-st positions, and $X_0 = X_0(\sigma)$ be the discrete random variable for the value of the first letter of $\sigma\in[k]^n$. From the definitions we obtain that

- $P(X_i = 0) = \frac{k^{n-1}}{k^n} = \frac{1}{k}$,
- $P(X_i = m) = \frac{2(k-m)k^{n-2}}{k^n} = \frac{2(k-m)}{k^2}$, for all $m = 1, 2, \ldots, k-1$,
- $P(X_0 = m) = \frac{1}{k}$ for every $m \in [k]$, for all $m = 1, 2, \ldots, k$,

where $P(Y = m)$ denotes the probability that the discrete random variable Y equals m. Then all X_i's, $i = 1, 2, \ldots, n-1$, have the same distribution and $u_{n,k} = \sum_{i=1}^{n-1} X_i$. Thus, the expected value of $u_{n,k}$ is given by $E(u_{n,k}) = \sum_{i=1}^{n-1} E(X_i) = (n-1)E(X_1)$, where

$$E(X_1) = \sum_{m=0}^{k-1} mP(X_1 = m) = \sum_{m=0}^{k-1} \frac{2m(k-m)}{k^2} = \frac{(k+1)(k-1)}{3k}.$$

Therefore,

$$E(u_{n,k}) = \frac{(n-1)(k+1)(k-1)}{3k}. \tag{3.1}$$

The discrete random variable $v_{n,k}$ for the complete total variation is given by $v_{n,k} = u_{n,k} + X_0$ and its expected value equals $E(v_{n,k}) = E(u_{n,k}) + E(X_0)$, where

$$E(X_0) = \sum_{m=1}^{k} mP(X_0 = m) = \sum_{m=1}^{k} \frac{m}{k} = \frac{k+1}{2}.$$

Thus, by (3.1) we have that

$$E(v_{n,k}) = \frac{(n-1)(k+1)(k-1)}{3k} + \frac{k+1}{2}. \tag{3.2}$$

Hence, Equations (3.1) and (3.2) complete the proof of Theorem 3.1.

Now we ready to prove Theorem 3.2. The random variables X_i and X_j are independent if $|i-j| > 1$ (that is, the covariance $Cov(X_i, X_j)$ equals 0 for all $|i-j| > 1$), thus the variance of $u_{n,k}$ is given by

$$\begin{aligned} Var(u_{n,k}) &= \sum_{i=1}^{n-1} Var(X_i) + 2\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} Cov(X_i, X_j) \\ &= (n-1)Var(X_1) + 2(n-2)Cov(X_1, X_2) \end{aligned}$$

with

$$\begin{aligned} Var(X_1) &= E(X_1^2) - E(X_1)^2 \\ &= \sum_{m=0}^{k-1} m^2 P(X_1 = m) - \frac{(k+1)^2(k-1)^2}{9k^2} \\ &= \sum_{m=0}^{k-1} \frac{2m^2(k-m)}{k^2} - \frac{(k+1)^2(k-1)^2}{9k^2} \\ &= \frac{(k+1)(k-1)(k^2+2)}{18k^2}, \end{aligned}$$

and

$$\begin{aligned}
E(X_1X_2) &= 2\sum_{1\le i\le j\le k}\frac{j(i-j)(k-i)}{k^3} + 2\sum_{1\le j\le k}\frac{j^2(k-j)}{k^3} + 4\sum_{1\le i<j\le k}\frac{ji(k-i)}{k^4}\\
&= \frac{(k+1)(k-1)(7k^2-8)}{60k^2},
\end{aligned}$$

where the first sum on first line above corresponds to cases where $\sigma_1\sigma_2\sigma_3$ is an increasing sequence or a decreasing sequence, the second sum corresponds to cases where $\sigma_1 = \sigma_3 \neq \sigma_2$, and the third sum corresponds to the rest of the cases. Hence, by the fact that $Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2)$ we get that

$$\begin{aligned}
Cov(X_1, X_2) &= E(X_1X_2) - E(X_1)E(X_2)\\
&= \frac{(k+1)(k-1)(7k^2-8)}{60k^2} - \frac{(k+1)^2(k-1)^2}{9k^2}\\
&= \frac{(k+1)(k-1)(k+2)(k-2)}{180k^2},
\end{aligned}$$

which implies that

$$\begin{aligned}
Var(u_{n,k}) &= (n-1)\tfrac{(k+1)(k-1)(k^2+2)}{18k^2} + 2(n-2)\tfrac{(k+1)(k-1)(k+2)(k-2)}{180k^2}\\
&= (n-1)\tfrac{(k-1)(k+1)(k^2+1)}{15k^2} - \tfrac{(k-1)(k-2)(k+2)(k+1)}{90k^2},
\end{aligned} \tag{3.3}$$

which agrees with Theorem 3.2.

To this end, let us find explicit formula for variance of the discrete random variable $v_{n,k} = \sum_{i=0}^{n-1} X_i$. Since the discrete random variables X_i and X_j are independent if $|i-j| > 1$, we get that the variance of $v_{n,k}$ is given by

$$\begin{aligned}
Var(v_{n,k}) &= Var(X_0) + (n-1)Var(X_1) + 2Cov(X_0, X_1) + 2(n-2)Cov(X_1, X_2)\\
&= Var(X_0) + 2Cov(X_0, X_1) + Var(u_{n,k})
\end{aligned}$$

with

$$\begin{aligned}
Var(X_0) &= E(X_0^2) - E(X_0)^2\\
&= \sum_{m=1}^{k} m^2 P(X_0 = m) - \frac{(k+1)^2}{4}\\
&= \sum_{m=1}^{k}\frac{m^2}{k} - \frac{(k+1)^2}{4}\\
&= \frac{(k-1)(k+1)}{12}.
\end{aligned}$$

and

$$\begin{aligned} Cov(X_0, X_1) &= E(X_0X_1) - E(X_0)E(X_1) \\ &= \frac{1}{k^2}\sum_{1\le i<j\le k}(j-i+i) + \frac{1}{k^2}\sum_{1\le j\le i\le k}(i-j+i) - \frac{k+1}{2}\cdot\frac{(k+1)(k-1)}{3k} \\ &= \frac{(k+1)^2(k-1)}{6k} - \frac{(k+1)^2(k-1)}{6k} \\ &= 0, \end{aligned}$$

where the sums on second line above corresponds to cases where $\sigma_1\sigma_2$ is an increasing sequence or a decreasing sequence. Therefore,

$$Var(v_{n,k}) = \frac{(k-1)(k+1)}{12} + Var(u_{n,k}) = \frac{(k-1)(k+1)(12n(k^2+1)+k^2-4)}{180k^2},$$

which completes the proof of Theorem 3.2.

Acknowledgements

The author would like to thank Augustine Munagi for reading previous versions of the present paper and for a number of helpful discussions. Also, the author thanks one of the referees that he suggests the probability proof of Theorem 1.3 as described in Section 3.2.

References

[1] D. Aldous and P. Diaconis, Longest increasing subsequences: from patience sorting to the Baik-Deift-Johansson theorem, *Bull. Amer. Math. Soc.* **36** (1999) 413–432.

[2] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy and D. Gouyou-Beauchamps, Generating functions for generating trees, *Discr. Math.* **246:1-3** (2002) 29–55.

[3] C.C. Chao and W.Q. Liang, Arranging n distinct numbers on a line or a circle to reach extreme total variations, *Europ. J. Combin.* **13** (1992) 325–334.

[4] G.L. Cohen and E. Tonkes, Dartboard arrangements, *Elect. J. Combin.* **8(2)** (2001) #R4.

[5] S.A. Curtis, Darts and hoopla board design, *Inform. Proc. Lett.* **92** (2004) 53–56.

[6] G. Firro and T. Mansour, Restricted k-ary words and functional equations, *Discr. Appl. Math.* **157:4** (2009) 602–616.

[7] P. Flajolet and R. Sedwick, Analytic Combinatorics, web edition; available at `http:algo.inria.fr/flajolet/Publications/books.html`, Cambridge University Press, 2008, pages xii+811.

[8] Q. Hou and T. Mansour, Kernel method and linear recurrence system, *J. Comput. Appl. Math.* **261:1** (2008) 227–242.

[9] Y.J. Liao, M.Z Shieh and S.C. Tsai, Arranging numbers on circles to reach maximum total variations, *Elect. J. Combin.* **14** (2007) #R47.

[10] B. F. Logan and L. A. Shepp, A variational problem for random Young tableaux, *Adv. Math.* **26** (1977) 206–222.

[11] T. Mansour, Longest alternating subsequences of k-ary words, *Discr. Appl. Math.* **156:1** (2008) 119–124.

[12] Th. Rivlin, Chebyshev polynomials. From approximation theory to algebra and number theory, John Wiley, New York, 1990.

[13] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, *http://www.research.att.com/~njas/sequences*.

[14] R.P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge University Press, Cambridge, 1999.

[15] R. Stanley, Longest alternating subsequences of permutations, arXiv: math.CO/0511419.

[16] A. M. Vershik and K. V. Kerov, Asymptotic behavior of the Plancherel measure ofthe symmetric group andthe limit form of Young tableaux (Russian), *Dokl. Akad. Nauk SSSR* **223** (1977) 1024–1027. English translation: *Soviet Math. Dokl.* **233** (1977) 527–531.

[17] ——, Wikipedia, `http://en.wikipedia.org/wiki/Tridiagonal_matrix_algorithm`.