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Extremal $K_{(s,t)}$-free bipartite graphs

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In this paper new exact values of the Zarankiewicz function $z(m, n; s, t)$ are obtained assuming certain requirements on the parameters. Moreover, all the corresponding extremal graphs are characterized. Finally, an extension of this problem to 3-partite graphs is studied.

Keywords: Zarankiewicz problem, extremal graph theory, forbidden subgraphs.

1 Introduction

Throughout this paper only undirected simple graphs (without loops or multiple edges) are considered. Unless stated otherwise, we follow the book by Bollobás [2] for undefined terminology and definitions.

Let $G = G(m, n) = G(X, Y)$ denote a bipartite graph with vertex classes X and Y such that $|X| = m$ and $|Y| = n$. A complete bipartite subgraph with s vertices in the class X and t vertices in the class Y is denoted by $K_{(s,t)}$. A bipartite graph $G = G(X, Y)$ contains $K_{(s,t)}$ as a subgraph if there exists an s-set S in X and a t-set T in Y such that the induced subgraph by $S \cup T$ in G, denoted $G[S \cup T]$, is a complete bipartite $K_{(s,t)}$. Notice that a bipartite graph G may contain $K_{(s,t)}$ as a subgraph but it may be actually free of $K_{(t,s)}$.

A question that arises naturally is: *what is the maximum number of edges, denoted by $z(m, n; s, t)$, that a $K_{(s,t)}$-free bipartite graph $G = G(m, n)$ can have?* Zarankiewicz [17] posed this problem for the particular case in which $m = n$ and $s = t = 3$, shortly denoted by $z(n; 3)$, when $n = 4, 5, 6$. This problem was solved by Sierpinski [15]. In the following years additional numerical values of the extremal function $z(n; 3)$ for $n \geq 7$ were provided by Brzezinski, see [16], Culik [4], Guy [11, 12, 13] and Guy and Znám [14]. Finally, the general problem of determining the exact value of the function $z(m, n; s, t)$ as well as the family of extremal graphs has also been known as *the Zarankiewicz problem*. The extremal family for this problem, denoted by $\mathcal{Z}(m, n; s, t)$, is the set of bipartite graphs on $m + n$ vertices with extremal size $z(m, n; s, t)$, such that they do not contain $K_{(s,t)}$ as a subgraph.

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The Zarankiewicz problem belongs to a wide class of extremal problems known as Turán type problems. The aim of this kind of problems is to compute the maximum number of edges, denoted by $ex\,(n;\mathcal{F})$, that a graph G on n vertices can have in such a way that if $F \in \mathcal{F}$ then $F \not\subseteq G$. If the problem involves bipartite graphs on $m+n$ vertices then the extremal function is denoted by $ex\,(m,n;\mathcal{F})$. Clearly $z(m,n;s,t) = ex\left(m,n;\{K_{(s,t)}\}\right)$. Another interesting extremal Turán type problem is the study of $ex\left(m,n;\{K_{(s,t)},K_{(t,s)}\}\right)$, usually denoted by $ex\,(m,n;K_{s,t})$.

Taking into account that $K_{s,t} \not\subseteq G$ implies both $K_{(s,t)} \not\subseteq G$ and $K_{(t,s)} \not\subseteq G$, it is clear that these extremal problems are related by

$$ex\,(m,n;K_{s,t}) \leq \min\{z(m,n;s,t), z(m,n;t,s)\}.$$

For a survey of well-known results regarding bounds on the Zarankiewicz problem we refer the reader to Section VI.2 of [2]. Most of the contributions are bounds for the function $z(m,n;s,t)$ when s,t are fixed and m,n are much larger than s,t (see, for example, [5, 6, 7]).

Our goal in this paper is to present exact values for the extremal function and to characterize the extremal graphs if possible. Concerning exact values for the Zarankiewicz function it is worth pointing out the work of Culik [4] who proved that if s,t and m are fixed then

$$z(m,n;s,t) = (s-1)n + (t-1)\binom{m}{s}, \quad \text{for all } n > (t-1)\binom{m}{s}. \tag{1}$$

Regarding the particular case in which $m = n$ and $s = t = 2$ it is well known [2] that $z(n;2) \leq (n + n\sqrt{4n-3})/2$ and equality holds when $n = q^2+q+1$ for a prime power q. Goddard et al. [8] found the exact values of $z(n;2)$ for $n \leq 20$ and showed that in some cases the family $\mathcal{Z}(n;2)$ is formed by only one extremal graph. Carnielli and Monte Carmelo [3] use $z(n;2)$ to investigate the bipartite Ramsey number $b(n) = b(K_{2,2};K_{1,n})$; see also [13].

There is another equivalent formulation of the Zarankiewicz problem in terms of matrices. Namely, it is easy to check that evaluating the extremal function $z(m,n;s,t)$ for fixed positive integers $s \leq m$ and $t \leq n$, is exactly the same as answering the question of *how many ones a $\{0,1\}$-matrix of dimension $m \times n$ can have in such a way that it is free of submatrices of dimension $s \times t$ whose elements are all ones?* Using this matrix terminology Griggs et al. [9, 10] studied the so-called "half-half" case, that is to say, they evaluated the extremal function $z(2s,2t;s,t)$ when $s \leq t$. In [10] the following inequalities are settled,

$$4st - (2t + 2s - gcd(s,t) + 1) \leq z(2s,2t;s,t) \leq 4st - (2t + s + 1),$$

where $gcd(s,t)$ denotes the greatest common divisor of s and t. Furthermore, it is proved that if $t = ks + r$, $0 \leq r < s$, such that either $r = 0$, or $r > 0$ and $s \leq k + r$ (which is always true for $t > (s-1)^2$) then

$$z(2s,2t;s,t) = 4st - (2t + s + 1). \tag{2}$$

In a more recent paper Griggs et al. [9] studied the still open case $t = ks+1$, evaluating $z(2s,2t;s,t)$ for large enough s and t, and obtaining the exact values of this function whenever $t \leq 20$ and $s \leq 7$.

Recently, the authors [1] solved completely the problem for all the values of the parameters such that

$\max\{m, n\} \leq s + t - 1$ proving the following result:

> Let m, n, s, t be positive integers with $2 \leq s < m$, $2 \leq t < n$ and such that $\max\{m, n\} \leq s + t - 1$. Then
> $z(m, n; s, t) = ex(m, n; K_{s,t}) = mn - (m + n - s - t + 1)$ and (3)
> $\mathcal{Z}(m, n; s, t) = \mathcal{EX}(m, n; K_{s,t}) = \{K_{(m,n)} - M\}$,
> where M is any matching in $K_{(m,n)}$ with cardinality $m + n - s - t + 1$.

In this paper we obtain the exact value of the extremal function $z(m, n; s, t)$ when the parameters m, n, s, t satisfy certain requirements. Moreover, all the corresponding extremal graphs are described. Finally, we compute the exact value of the extremal function for an extension of the Zarankiewicz problem to r-partite graphs. More precisely, we study the function $z_3(n; t)$, namely, *the maximum number of edges of a* 3*-partite graph with n vertices in each vertex class, free of a complete* 3*-partite subgraph with t vertices in each vertex class*. In this regard we give the exact value for the extremal function $z_3(n; t)$ when $t \leq n \leq 2t - 1$. In Section 2 we present our results and prove them in Section 3.

2 Main Results

In order to avoid trivial cases we only study the Zarankiewicz problem when $s \geq 2$ and $t \geq 2$. Notice that $z(m, n; s, t) = z(n, m; t, s)$ for all m, n, s and t.

In the first theorem we compute the exact value of the Zarankiewicz function assuming certain requirements on the parameters.

Theorem 2.1 *Let m, n, s, t be integers such that $2 \leq s \leq m$ and $2 \leq t \leq n - s$. Suppose that (i) $(m - s)(\lfloor (n - t)/s \rfloor + 1) \leq t - 1$; or (ii) s divides $n - t$ and $(m - s)n \leq mt$, being $m \geq 2s$ if $(m - s)n = mt$. Then*

$$z(m, n; s, t) = mn - (\lfloor (n - t)/s \rfloor (m - s) + m + n - s - t + 1)\,.$$

Let us point out that when $m = 2s$, $n = 2t$ and s divides t, Theorem 2.1 gives the same exact value for the extremal function $z(2s, 2t; s, t)$ provided in (2), see Corollary 2.1. Moreover, Theorem 2.1 permit us to compute $z(m, 2t; s, t)$, for every $m \leq 2s$, and to characterize all the corresponding extremal graphs. Therefore, this result improves the "half-half" case [9, 10] when s divides t. Furthermore, if $\max\{m, n\} \leq s + t - 1$ then $\lfloor (n - t)/s \rfloor = \lfloor (m - s)/t \rfloor = 0$ and the value of the extremal function given by Theorem 2.1 matches the one obtained in (3), which in turn means that this theorem extends the results obtained by the authors in [1]. However, the results proved in this paper do not overlap those obtained in [1] because Theorem 2.1 is written under the hypothesis $n \geq s + t$, whereas $n \leq s + t - 1$ is assumed in (3).

Thus, assuming without loss of generality $m \leq n$, we focus our attention in the case $\lfloor (n - t)/s \rfloor \geq 1$, hence $s + t \leq n$ whose solution is not given by (3). Broadly speaking, (3) gives us the exact value of the extremal number $z(m, n; s, t)$ when m and n are very close to s and t respectively. However, with Theorem 2.1 we contribute the solution for this extremal problem when $n \geq s + t$.

In order to describe the corresponding extremal graphs we introduce next three families of graphs. Let m, n, s, t be integers such that $m \leq n$, $2 \leq s \leq m$ and $2 \leq t \leq n - s$. Let $H = H_1 \cup H_2 \subseteq K_{(m,n)}$ be any subgraph of $K_{(m,n)}$, with H_1 and H_2 disjoint.

(1) $\mathcal{F}_1(m, n; s, t)$ is the family of subgraphs $H \subseteq K_{(m,n)}$ such that:

- s divides $n-t$;
- the subgraph H_1 consists of $s-1$ disjoint copies of $K_{(1,(n-t)/s)}$;
- the subgraph H_2 has girth at least $2(s+1)$ (may be infinite) and every vertex of the m-class has degree $(n-t)/s+1$.

(2) $\mathcal{F}_2(m,n;s,t)$ stands for the family of subgraphs $H \subseteq K_{(m,n)}$ such that:

- $3 \le s \le m$ and $n-t = s\lfloor (n-t)/s \rfloor + r$ with $0 \le r \le s-3$;
- there exists an integer k with $2 \le \lceil (n-t+s^2-r(s+1))/(n-t+s-r) \rceil \le k \le s-r-1$ such that:
  - the subgraph H_1 consists of the disjoint union of $K_{(1,d_i)}$, with $0 \le d_1 \le \cdots \le d_{k-1} \le \lfloor (n-t)/s \rfloor$ and $d_1 + \ldots + d_{k-1} = (k-1)\left(\lfloor (n-t)/s \rfloor + 1\right) - s + r + 1$;
  - the subgraph H_2 consists of the disjoint union of $m-k+1$ copies of the graph $K_{(1,\lfloor (n-t)/s \rfloor+1)}$.

(3) $\mathcal{F}_3(m,n;s,t)$ is the family of subgraphs $H \subseteq K_{(m,n)}$ such that:

- $n-t = s\lfloor (n-t)/s \rfloor + r$ with $0 < r < s$;
- the subgraph H_1 consists of $s-r-1$ disjoint copies of $K_{(1,\lfloor (n-t)/s \rfloor)}$;
- the subgraph H_2 consists of $m-s+r+1$ disjoint copies of $K_{(1,\lfloor (n-t)/s \rfloor+1)}$.

Observe that $\mathcal{F}_1(m,n;s,t)$, $\mathcal{F}_2(m,n;s,t)$ and $\mathcal{F}_3(m,n;s,t)$ are pairwise disjoint families of graphs, and let us point out for every $i \in \{1,2,3\}$ that if $H \in \mathcal{F}_i(m,n;s,t)$ then the number of edges of H is

$$\lfloor (n-t)/s \rfloor (m-s) + m + n - s - t + 1. \tag{4}$$

As an example, Figure 1 depicts three subgraphs of $K_{(m,n)}$ that belong, respectively, to the aforementioned families. In all examples H_1 is the subgraph induced by the white vertices.

The next result describes all the corresponding extremal bipartite graphs to the exact values of Zarankiewicz numbers obtained in the above Theorem 2.1.

Theorem 2.2 *Let m,n,s,t be integers such that $2 \le s \le m$ and $2 \le t \le n-s$. Suppose that (i) $(m-s)(\lfloor (n-t)/s \rfloor + 1) \le t-1$; or (ii) s divides $n-t$ and $(m-s)n \le mt$, being $m \ge 2s$ if $(m-s)n = mt$. Then if $r \equiv n-t (\mathrm{mod} s)$ with $0 \le r < s$ the following assertions hold:*

1. *If $r=0$ and $s=2$ then*
$$\mathcal{Z}(m,n;s,t) = \{K_{(m,n)} - E(H) : H \in \mathcal{F}_1(m,n;s,t)\}.$$

2. *If $r=0$ and $s \ge 3$ then*
$$\mathcal{Z}(m,n;s,t) = \{K_{(m,n)} - E(H) : H \in \mathcal{F}_1(m,n;s,t) \cup \mathcal{F}_2(m,n;s,t)\}.$$

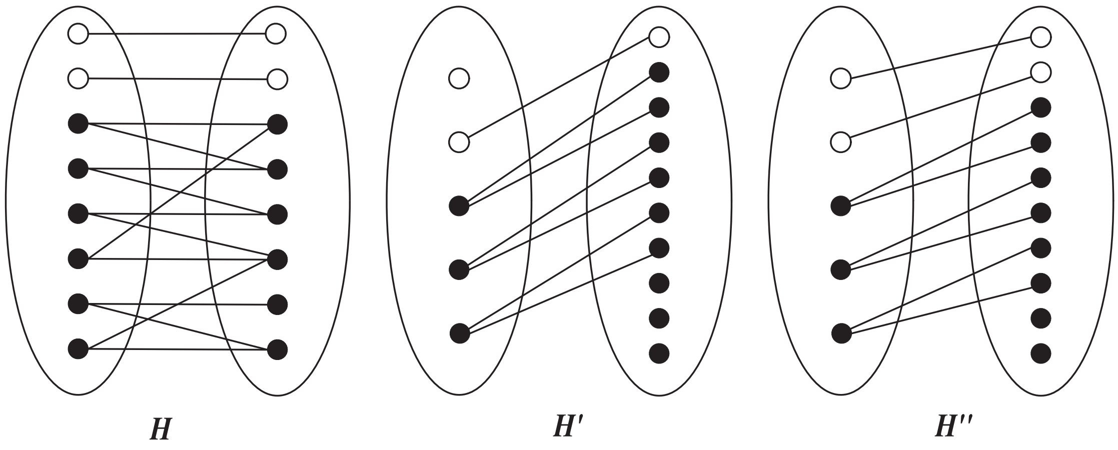

Fig. 1: $H \in \mathcal{F}_1(8,8;3,5)$; $H' \in \mathcal{F}_2(5,10;4,6)$ (with $k=3$); $H'' \in \mathcal{F}_3(5,10;4,5)$.

3. *If $r > 0$ and $s-2 \le r \le s-1$ then*

$$\mathcal{Z}(m,n;s,t) = \{K_{(m,n)} - E(H) : H \in \mathcal{F}_3(m,n;s,t)\}.$$

4. *If $0 < r \le s-3$ then*

$$\mathcal{Z}(m,n;s,t) = \{K_{(m,n)} - E(H) : H \in \mathcal{F}_2(m,n;s,t) \cup \mathcal{F}_3(m,n;s,t)\}.$$

Corollary 2.1 *Let $\lambda, s, t \ge 2$ be integers and suppose that $\lambda t/(\lambda-1)$ is an integer. Then if s divides $t/(\lambda-1)$ we have*

$$z\left(\lambda s, \frac{\lambda t}{\lambda-1}; s,t\right) = \frac{\lambda^2 st}{\lambda-1} - \left(\frac{\lambda t}{\lambda-1} + (\lambda-1)s + 1\right).$$

Moreover the following assertions hold:

1. *If $s=2$ then*

$$Z\left(2\lambda, \frac{\lambda t}{\lambda-1}; 2,t\right) = \left\{K_{\left(2\lambda, \frac{\lambda t}{\lambda-1}\right)} - E(H) : H \in \mathcal{F}_1\left(2\lambda, \frac{\lambda t}{\lambda-1}; 2,t\right)\right\}.$$

2. *If $s \ge 3$ then $Z\left(s\lambda, \lambda t/(\lambda-1); s,t\right)$ is the following set:*

$$\left\{K_{\left(2\lambda, \frac{\lambda t}{\lambda-1}\right)} - E(H) : H \in \mathcal{F}_1\left(s\lambda, \frac{\lambda t}{\lambda-1}; s,t\right) \cup \mathcal{F}_2\left(s\lambda, \frac{\lambda t}{\lambda-1}; s,t\right)\right\}.$$

Finally, let us denote by $K_3(t)$ the complete 3-partite graph with t vertices in each vertex class. We study the following problem: *w*hat is the maximum number of edges, denoted by $z_3(n;t)$, of a $K_3(t)$-free 3-partite graph $G = G(X,Y,Z)$ with n vertices in each class? In the next result we answer the question for $2 \le t \le n \le 2t-1$.

Theorem 2.3 *Let n, t be integers such that $2 \le t \le n \le 2t-1$. Then*

$$z_3(n;t) = 3n^2 - (2(n-t)+1)\,.$$

3 Proofs

The degree of a vertex w in a graph G is denoted by $d_G(w) = d(w)$, $N_G(w)$ is the set of adjacent vertices to w in G, and $N_G(T) = \left(\bigcup_{w\in T} N_G(w)\right) \setminus T$ denotes the neighborhood of a subset of vertices T. The bipartite complement of a bipartite graph $G = G(X,Y)$ with $|X| = m$ and $|Y| = n$ is $G^c = K_{(m,n)} - E(G)$, where $E(G)$ denotes the edge set of G. We also use $e(G)$ to denote the number of edges of G and $G[V']$ for the induced subgraph in G by the set of vertices $V' \subseteq V(G)$.

Remark 3.1 *Let m, n, s, t be integers such that $2 \le s \le m$ and $2 \le t \le n$. Let $G = G(X,Y)$ be a bipartite graph with $|X| = m$ and $|Y| = n$. Then G is $K_{(s,t)}$-free if and only if $|N_{G^c}(S)| \ge n-(t-1)$ for every s-subset S of the X class.*

Proof: If S is an s-subset of X such that $|N_{G^c}(S)| \le n-t$, then there exists a t-subset T of Y such that $N_{G^c}(x) \cap T = \emptyset$ for any $x \in S$. Then $G[S \cup T] = K_{(s,t)}$, hence $K_{(s,t)} \subseteq G$. Reciprocally, if $K_{(s,t)} \subseteq G$ then there exist both an s-subset S of X and a t-subset T of Y such that there is no edge joining a vertex in S with a vertex in T in the bipartite complement G^c of G. Hence $|N_{G^c}(S)| \le n-t$, finishing the proof. □

The following lemmas are the basis for proving Theorem 2.1 and Theorem 2.2.

Lemma 3.1 *Let m, n, s, t be integers such that $2 \le s \le m$ and $2 \le t \le n$. Then*

$$z(m,n;s,t) \le mn - (\alpha + m + n - s - t + 1)\,,$$

where $\alpha = \max\left\{ \lfloor (n-t)/s \rfloor (m-s), \lfloor (m-s)/t \rfloor (n-t) \right\}$.

Proof: Let $G = G(X,Y)$ be a graph belonging to $\mathcal{Z}(m,n;s,t)$. Let us order the vertices of $V(G)$ in such a way that $d_G(x_i) \ge d_G(x_{i+1})$, for $i \in \{1,\ldots,m-1\}$ and $d_G(y_j) \ge d_G(y_{j+1})$, for $j \in \{1,\ldots,n-1\}$. Let us consider the subsets $S = \{x_1,\ldots,x_s\}$ and $T = \{y_1,\ldots,y_t\}$.

First, since $K_{(s,t)} \not\subseteq G$ we have $K_{(s,t)} \not\subseteq G[S \cup Y]$ and therefore $e(G[S\cup Y]) \le z(s,n;s,t) = (s-1)n + t - 1$ because of (1). Suppose $d_G(x_s) \ge n - \lfloor (n-t)/s \rfloor$. Then $e(G[S \cup Y]) \ge s\Big(n - \lfloor (n-t)/s \rfloor\Big) \ge sn - n + t$ which is an absurdity. So $d_G(x_s) \le n - \lfloor (n-t)/s \rfloor - 1$ which implies $d_G(x_i) \le n - \lfloor (n-t)/s \rfloor - 1$ for all $i \in \{s+1,\ldots,m\}$, and hence,

$$\begin{aligned} z(m,n;s,t) = e(G) &= e(G[S\cup Y]) + e(G[(X\setminus S)\cup Y]) \\ &\le sn - n + t - 1 + (m-s)(n - \lfloor (n-t)/s \rfloor - 1) \\ &= mn - (\lfloor (n-t)/s \rfloor (m-s) + m + n - s - t + 1)\,. \end{aligned} \tag{5}$$

The same reasoning can be repeated for the t-subset T of the Y-class, so we obtain

$$z(m,n;s,t) \le mn - (\lfloor (m-s)/t \rfloor (n-t) + m + n - s - t + 1)\,,$$

and the result holds. □

Lemma 3.2 *Let m, n, s, t be integers such that $2 \leq s \leq m$ and $2 \leq t \leq n - s$. Assume that $\mathcal{F}_i(m, n; s, t) \neq \emptyset$, $i \in \{1, 2, 3\}$, and let $G = K_{(m,n)} - E(H_1 \cup H_2)$ be a bipartite graph with $H_1 \cup H_2 \in \mathcal{F}_i(m, n; s, t)$. Then G is $K_{(s,t)}$-free.*

Proof: Let $G = G(X, Y) = K_{(m,n)} - E(H_1 \cup H_2)$, with $H_1 \cup H_2 \in \mathcal{F}_i(m, n; s, t)$, $i \in \{1, 2, 3\}$. Let S be any s-subset of X. For each $i \in \{1, 2, 3\}$, notice that each vertex of X belongs to either $V(H_1)$ or $V(H_2)$, so S is formed by j vertices of $V(H_1)$ and $s - j$ vertices of $V(H_2)$, where $0 \leq j \leq s - 1$.

First, suppose that the subgraph $H_1 \cup H_2$ belongs to the family $\mathcal{F}_1(m, n; s, t)$, then $|V(H_1) \cap X| = s - 1$ and $|V(H_2) \cap X| = m - s + 1$. Let us denote by $W \subseteq H_2$ the subgraph of the bipartite complement G^c induced by the $s - j$ vertices in $S \cap V(H_2)$ and their neighbors. As the girth of H_2 is at least $2(s+1) \geq 6$ we deduce that W is acyclic and $|N_{G^c}(u) \cap N_{G^c}(u')| \leq 1$ for every distinct $u, u' \in V(W)$. Since the degree in G^c of every vertex in $V(H_2) \cap X$ is $(n - t)/s + 1$ it follows that $|V(W) \cap Y| \geq (s - j)(n - t)/s + 1$. Then $|N_{G^c}(S)| \geq j(n - t)/s + (s - j)(n - t)/s + 1 = n - (t - 1)$ which implies that $K_{(s,t)} \nsubseteq G$ according to Remark 3.1.

Second, suppose that $H_1 \cup H_2 \in \mathcal{F}_2(m, n; s, t)$, then $|V(H_1) \cap X| = k - 1$ and $|V(H_2) \cap X| = m - k + 1$ for some $2 \leq k \leq s - r - 1$, hence $0 \leq j \leq k - 1$. Let us denote by U the subgraph of G^c induced by the j vertices in $S \cap V(H_1)$ and their neighbors. Taking into account that $|V(H_1) \cap Y| = d_1 + \ldots + d_{k-1} = (k - 1)(\lfloor (n - t)/s \rfloor + 1) - s + r + 1$ and $d_{G^c}(x) \leq \lfloor (n - t)/s \rfloor$ for every vertex $x \in V(H_1)$, we have

$$\begin{aligned} |V(U) \cap Y| &= |V(H_1) \cap Y| - |(V(H_1) \setminus V(U)) \cap Y| \\ &\geq (k - 1)(\lfloor (n - t)/s \rfloor + 1) - s + r + 1 - (k - 1 - j)\lfloor (n - t)/s \rfloor \\ &= k - s + r + j\lfloor (n - t)/s \rfloor, \end{aligned}$$

and hence

$$\begin{aligned} |N_{G^c}(S)| &\geq k - s + r + j\lfloor (n - t)/s \rfloor + (s - j)(\lfloor (n - t)/s \rfloor + 1) \\ &= n - t + k - j \\ &\geq n - t + 1, \end{aligned}$$

which implies that $K_{(s,t)} \nsubseteq G$ due to Remark 3.1.

Finally, suppose that $H_1 \cup H_2 \in \mathcal{F}_3(m, n; s, t)$, then $0 \leq j \leq s - r - 1$ as $|V(H_1) \cap X| = s - r - 1$ and $|V(H_2) \cap X| = m - s + r + 1$. In this case $|N_{G^c}(S)| \geq j\lfloor (n - t)/s \rfloor + (s - j)(\lfloor (n - t)/s \rfloor + 1) = n - t - r + s - j \geq n - t + 1$. Once more, Remark 3.1 implies that $K_{(s,t)} \nsubseteq G$ and the proof is complete. □

Lemma 3.3 *Let m, n, s, t be integers such that $2 \leq s \leq m$ and $2 \leq t \leq n - s$. Suppose that (i) $(m - s)(\lfloor (n - t)/s \rfloor + 1) \leq t - 1$; or (ii) s divides $n - t$ and $(m - s)n \leq mt$, with $m \geq 2s$ if $(m - s)n = mt$. Then*

$$z(m, n; s, t) \geq mn - (\lfloor (n - t)/s \rfloor (m - s) + m + n - s - t + 1).$$

Proof: Observe that $m \leq n$ follows from the hypothesis.

First suppose that $(m - s)(\lfloor (n - t)/s \rfloor + 1) \leq t - 1$. If s divides $n - t$ then we can consider a graph $G = K_{(m,n)} - E(H)$ with $H = H_1 \cup H_2 \in \mathcal{F}_1(m, n; s, t)$, the girth of H_2 being infinite. Indeed the subgraphs H_1 and H_2 are well defined since the number of different vertices of $H_1 \cup H_2$ in the n-class is at most

$$(s - 1)\frac{n - t}{s} + (m - s + 1)\left(\frac{n - t}{s} + 1\right) = (m - s)\left(\frac{n - t}{s} + 1\right) + n - t + 1 \leq n.$$

By Lemma 3.2 we know that G is free of $K_{(s,t)}$. Moreover, from (4) we get

$$e(G) = mn - (\lfloor (n-t)/s \rfloor (m-s) + m + n - s - t + 1)\,,$$

so $z(m, n; s, t) \geq e(G)$ and the lemma holds.

If s does not divide $n-t$ then we consider a graph $G = K_{(m,n)} - E(H)$ with $H = H_1 \cup H_2 \in \mathcal{F}_3(m, n; s, t)$, and reasoning in the same way as before we are done.

From now on we can assume that $t \leq (m-s)((n-t)/s + 1)$.

Second suppose that s divides $n-t$ and $(m-s)n \leq mt$. As $t \leq (m-s)((n-t)/s + 1)$, we are restricting to $(m-s)n \leq mt \leq (m-s)(n+s)$. In this case we can consider a graph $G = K_{(m,n)} - E(H)$ with $H = H_1 \cup H_2 \in \mathcal{F}_1(m, n; s, t)$, the subgraph H_2 being taken as follows.

If $(m-s)n < mt$, which is equivalent to $m(n-t)/s < n$, then H_2 consists of a path of length $2(m-s+1)$ with end vertices in the n-class and such that it has $(n-t)/s - 1$ pendant edges attached to each vertex of the m-class. The graph G is well defined since the number of different vertices of $H_1 \cup H_2$ in the n-class is at most

$$(s-1)\frac{n-t}{s} + (m-s+1)\frac{n-t}{s} + 1 = m\frac{n-t}{s} + 1 \leq n.$$

If $(m-s)n = mt$, which is equivalent to $m(n-t)/s = n$, and $m \geq 2s$ then H_2 consists of a cycle of length $2(m-s+1)$ which has $(n-t)/s - 1$ pendant edges attached to each vertex of the m-class (notice that $m - s + 1 \geq s + 1$ is necessary because the girth of H_2 is at least $2(s+1)$, hence $m \geq 2s$ must hold). The graph G is well defined because the number of different vertices of $H_1 \cup H_2$ in the n-class is at most

$$(s-1)\frac{n-t}{s} + (m-s+1)\frac{n-t}{s} = m\frac{n-t}{s} = n.$$

Observe that $e(G) = mn - (\lfloor (n-t)/s \rfloor (m-s) + m + n - s - t + 1)$ from (4), the graph G being free of $K_{(s,t)}$ by Lemma 3.2. Then, we can state that $z(m, n; s, t) \geq e(G)$, ending the proof. $\square$

Reasoning as in the proof of Lemma 3.3, it is not difficult to see that

$$(m-s)(\lfloor (n-t)/s \rfloor + 1) \leq t - 1$$

is a necessary and sufficient condition for $\mathcal{F}_2(m, n; s, t) \neq \emptyset$ and $\mathcal{F}_3(m, n; s, t) \neq \emptyset$. As far as the family $\mathcal{F}_1(m, n; s, t)$ is concerned (where s must divide $n - t \geq 1$), the condition $(m-s)n \leq mt$ joint with $m \geq 2s$ suffice to guarantee $\mathcal{F}_1(m, n; s, t) \neq \emptyset$, by assuring the existence of a subgraph $H = H_1 \cup H_2 \in \mathcal{F}_1(m, n; s, t)$ where H_2 is a cycle of length $2(m-s+1)$ with $(n-t)/s - 1$ pendant edges attached to each vertex of the m-class; moreover, $\mathcal{F}_1(m, n; s, t)$ is also seen to be nonempty under the constraint $(m-s)n + 1 \leq mt$, in this case $H = H_1 \cup H_2 \in \mathcal{F}_1(m, n; s, t)$ is taken with H_2 being a path of length $2(m-s+1)$ with $(n-t)/s - 1$ pendant edges attached to each vertex of the m-class. Nevertheless, $(m-s)n \leq mt$ is far from being a necessary condition for $\mathcal{F}_1(m, n; s, t) \neq \emptyset$, as Figure 2 shows.

Now we are able to prove our main result regarding the exact values of Zarankiewicz function.

Proof: of Theorem 2.1. First of all, let us show that

$$\left\lfloor \frac{n-t}{s} \right\rfloor (m-s) \geq \left\lfloor \frac{m-s}{t} \right\rfloor (n-t) \tag{6}$$

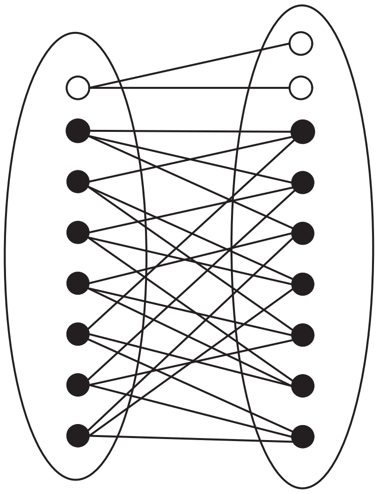

Fig. 2: A bipartite graph of $\mathcal{F}_1(8, 9; 2, 5)$.

follows from the hypothesis. If $(m-s)(\lfloor (n-t)/s \rfloor + 1) \leq t-1$ then it is easy to check that $0 \leq (m-s)\lfloor (n-t)/s \rfloor \leq s+t-1-m$ which implies $m \leq s+t-1$ and hence $\lfloor (m-s)/t \rfloor (n-t) = 0$. If s divides $n-t$ and $(m-s)n < mt$, which is equivalent to $m(n-t)/s < n$, then $m < n$. Hence $m \leq s+t-1$ for if not $0 \leq (m-s-t)n = (m-s)n - tn \leq (m-n)t$ implying that $m \geq n$. Therefore we have again that $\lfloor (m-s)/t \rfloor (n-t) = 0$. Finally, if s divides $n-t$, $m \geq 2s$ and $(m-s)n = mt$, which is equivalent to $m(n-t)/s = n$, then $m \leq n$. If $m \leq s+t-1$ then, reasoning as above we are done. If $m \geq s+t$ then $tn \leq (m-s)n = mt$ and hence $s+t = n = m \geq 2s$ which implies $t \geq s$. In this case we have

$$\left\lfloor \frac{n-t}{s} \right\rfloor (m-s) = t \geq s = \left\lfloor \frac{m-s}{t} \right\rfloor (n-t),$$

and the claimed inequality (6) holds.

Now, taking into account (6) we have

$$z(m, n; s, t) = mn - (\lfloor (n-t)/s \rfloor (m-s) + m + n - s - t + 1)$$

as a direct consequence of Lemmas 3.1 and 3.3. □

Proof: of Theorem 2.2. For the characterization of the family of extremal graphs with size $z(m, n; s, t)$, let us consider a graph $G = G(X, Y) \in \mathcal{Z}(m, n; s, t)$. Let us assume that the vertices of X are ordered as in the proof of Lemma 3.1, so that $d_{G^c}(x_1) \leq \cdots \leq d_{G^c}(x_m)$ in the bipartite complement of G. Let $S = \{x_1, \ldots, x_s\}$. Due to $e(G) = mn - (\lfloor (n-t)/s \rfloor (m-s) + m + n - s - t + 1)$, all inequalities (5) become equalities, hence $d_G(x_s) = \cdots = d_G(x_m) = n - \lfloor (n-t)/s \rfloor - 1$ and $e(G[S \cup Y]) = sn - n + t - 1$, which implies $e(G^c[S \cup Y]) = n - t + 1$. Let r be the integer $0 \leq r < s$ such that $n - t = s\lfloor (n-t)/s \rfloor + r$. Observe that $d_{G^c}(x_{s-r}) = \lfloor (n-t)/s \rfloor + 1$, for if not $e(G^c[S \cup Y]) \leq (s-r)\lfloor (n-t)/s \rfloor + r(\lfloor (n-$

$t)/s\rfloor + 1) = n - t$, which is not possible. So we have

$$\begin{array}{l} d_{G^c}(x_{s-r}) = \cdots = d_{G^c}(x_m) = \lfloor (n-t)/s \rfloor + 1; \text{ and} \\ d_{G^c}(x_i) \le \lfloor (n-t)/s \rfloor + 1 \text{ for every } i \in \{1, \dots, s-r-1\} \text{ if } r \le s-2. \end{array} \tag{7}$$

Moreover, notice that $d_{G^c}(x_1) = \lfloor (n-t)/s \rfloor + 1$ can only occur when $r = s-1$, because $e(G^c[S \cup Y]) = n-t+1$ equals $s(\lfloor (n-t)/s \rfloor + 1) = n-t-r+s$. It is also clear that $d_{G^c}(x_{s-1}) \ge 1$ because otherwise we have $d_{G^c}(x_i) = 0$ for all $i \in \{1, \dots, s-1\}$ and then $n-t+1 = e(G^c[S \cup Y]) = d_{G^c}(x_s)$, which is impossible since $d_{G^c}(x_s) = \lfloor (n-t)/s \rfloor + 1$ and $s \ge 2$.

Taking into account that $n-t+1 = e(G^c[S \cup Y]) \ge |N_{G^c}(S)|$ and by applying Remark 3.1 we have

$$e(G^c[S \cup Y]) = |N_{G^c}(S)| = n - t + 1. \tag{8}$$

From (8) it follows that $N_{G^c}(x) \cap N_{G^c}(x') = \emptyset$ for all distinct vertices $x, x' \in S$.

*C*laim 1: If $d_{G^c}(x_{s-1}) = \lfloor (n-t)/s \rfloor + 1$ then $N_{G^c}(x) \cap N_{G^c}(x') = \emptyset$ for any two distinct vertices $x, x' \in X$.

To prove this claim suppose $N_{G^c}(x) \cap N_{G^c}(x') \neq \emptyset$, hence we can assume $x' \in X \setminus S$. If x, x' share some neighbor, using (8) we obtain a contradiction with Remark 3.1 as follows:

$\star$ if $x \in S$, taking $x'' \in \{x_{s-1}, x_s\}$, $x'' \neq x$, it follows $|N_{G^c}((S \setminus \{x''\}) \cup \{x'\})| \le n - t$;

$\star$ if $x \in X \setminus S$ then $|N_{G^c}((S \setminus \{x_{s-1}, x_s\}) \cup \{x, x'\})| \le n - t$. $\Box$

As a consequence of Claim 1, if $r \ge 1$ then $N_{G^c}(x) \cap N_{G^c}(x') = \emptyset$ for any two distinct vertices $x, x' \in X$, which follows from (7) because in this case $d_{G^c}(x_{s-1}) = \lfloor (n-t)/s \rfloor + 1$.

Now assume that $r = s-1$. Then from (7) it follows that $d_{G^c}(x_i) = \lfloor (n-t)/s \rfloor + 1$ for all $i \in \{1, 2, \dots, m\}$, and by Claim 1 it turns out that $G = K_{(m,n)} - E(H)$ where H consists of the union of m disjoint copies of $K_{(1, \lfloor (n-t)/s \rfloor + 1)}$. Therefore $H \in \mathcal{F}_3(m, n; s, t)$ in this case. The proof continues for $0 \le r \le s-2$.

Assume now that $d_{G^c}(x_{s-r-1}) = \lfloor (n-t)/s \rfloor + 1$, which implies $0 \le r \le s-3$ (because $x_{s-r-1} = x_1$ for the case $r = s-2$, and $d_{G^c}(x_1) = \lfloor (n-t)/s \rfloor + 1$ if and only if $r = s-1$). Hence from (7) there exists some integer k with $2 \le k \le s-r-1$ such that $d_{G^c}(x_i) = \lfloor (n-t)/s \rfloor + 1$ for all $i \in \{k, \dots, m\}$ and $d_{G^c}(x_i) \le \lfloor (n-t)/s \rfloor$ for all $1 \le i \le k-1$. Hence $0 \le \sum_{i=1}^{k-1} d_{G^c}(x_i) = e(G^c[S \cup Y]) - (s-k+1)(\lfloor (n-t)/s \rfloor + 1) = (k-1)(\lfloor (n-t)/s \rfloor + 1) - (s-r) + 1$, yielding $k \ge (n-t+s^2-r(s+1))/(n-t+s-r)$. That is to say, $s-r-1 \ge k \ge \lceil (n-t+s^2-r(s+1))/(n-t+s-r) \rceil \ge 2$. Therefore by Claim 1 we conclude that $G = K_{(m,n)} - E(H_1 \cup H_2)$, the subgraphs H_1 and H_2 being disjoint, where H_1 is the union of $k-1$ disjoint subgraphs $K_{(1,d_1)}, \dots, K_{(1,d_{k-1})}$, with $0 \le d_1 \le \ldots \le d_{k-1} \le \lfloor (n-t)/s \rfloor$ and $d_1 + \ldots + d_{k-1} = (k-1)(\lfloor (n-t)/s \rfloor + 1) - (s-r) + 1$; and H_2 is the union of $m-k+1$ disjoint copies of the graph $K_{(1, \lfloor (n-t)/s \rfloor + 1)}$. Therefore $H_1 \cup H_2 \in \mathcal{F}_2(m, n; s, t)$.

Finally, assume that $d_{G^c}(x_{s-r-1}) \le \lfloor (n-t)/s \rfloor$. Then $d_{G^c}(x_i) = \lfloor (n-t)/s \rfloor$ for each $i \in \{1, \dots, s-r-1\}$, because otherwise

$$\begin{array}{rcl} e(G^c[S \cup Y]) & < & (s-r-1)\lfloor (n-t)/s \rfloor + (r+1)(\lfloor (n-t)/s \rfloor + 1) \\ & = & s\lfloor (n-t)/s \rfloor + r + 1 = n - t + 1, \end{array}$$

an absurdity. If $r \ge 1$ then $d_{G^c}(x_{s-1}) = \lfloor (n-t)/s \rfloor + 1$ by (7), so Claim 1 allows us to deduce that $G = K_{(m,n)} - E(H_1 \cup H_2)$, the subgraphs H_1 and H_2 being disjoint, where H_1 consists of the union of $s-r-1$ disjoint copies of $K_{(1,\lfloor (n-t)/s \rfloor)}$ and H_2 is the union of $m-s+r+1$ disjoint copies of the graph $K_{(1,\lfloor (n-t)/s \rfloor+1)}$. Then, $H_1 \cup H_2 \in \mathcal{F}_3(m,n;s,t)$.

If $r=0$, from (7) we have that all the vertices $x_s,\ldots,x_m$ have the same degree equal to $(n-t)/s+1$ in G^c, hence (8) applied to the s-set $S \setminus \{x_s\} \cup \{x_j\}$ yields $N_{G^c}(x_j) \cap N_{G^c}(S \setminus \{x_s\}) = \emptyset$ for every $j \in \{s,\ldots,m\}$. This means that the subgraphs $H_1 = G^c[(S\setminus\{x_s\}) \cup N_{G^c}(S\setminus\{x_s\})]$ and $H_2 = G^c[\{x_s,\ldots,x_m\} \cup N_{G^c}(\{x_s,\ldots,x_m\})]$ are disjoint, H_1 consisting of the union of $s-1$ disjoint subgraphs $K_{(1,(n-t)/s)}$. Notice that $|N_{G^c}(x) \cap N_{G^c}(x')| \le 1$ for every two distinct $x,x' \in \{x_s,\ldots,x_m\}$. Otherwise the set $S' = (S \setminus \{x_{s-1},x_s\}) \cup \{x,x'\}$ satisfies $|N_{G^c}(S')| \le |N_{G^c}(S\setminus\{x_{s-1},x_s\})| + |N_{G^c}(\{x,x'\})| \le (s-2)(n-t)/s + 2((n-t)/s+1) - 2 = n-t$, contradicting again Remark 3.1. Observe also that every cycle of H_2, if any, has length at least $2(s+1)$, because if C is such a cycle of length $2l$ with $2 \le l \le s$, then the s-subset S' of X consisting of l vertices of C and any $s-l$ vertices in $V(H_1) \cap X \subseteq \{x_1,\ldots,x_{s-1}\}$ satisfies $|N_{G^c}(S')| = (s-l)(n-t)/s + l(n-t)/s = n-t$, against Remark 3.1. As a consequence, H_2 is either acyclic, or each of its cycles has length at least $2(s+1)$. In any case all vertices of H_2 belonging to X have degree $(n-t)/s+1$ in G^c. Therefore $H_1 \cup H_2 \in \mathcal{F}_1(m,n;s,t)$. To finish, notice that for each of the cases 1 to 4 of the theorem there exists at least one suitable family of subgraphs $\mathcal{F}_i(m,n;s,t) \neq \emptyset$ such that $\mathcal{Z}(m,n;s,t)$ is guaranteed to be nonempty (see our comment after Lemma 3.3 and the hypotheses of the theorem). This finishes the proof. $\square$

Lemma 3.4 *Let n,t be integers such that $2 \le t \le n \le 2t-1$. Then the 3-partite graph $G = G(X,Y,Z) = K_3(n) - M$ does not contain $K_3(t)$ as a subgraph, where M is a matching of cardinality $2n-2t+1$ in the subgraph $G[X \cup Y]$.*

Proof: Let us consider the bipartite subgraph Γ induced by $X \cup Y$ in $K_3(n)-M$. Since $\Gamma = K_{(n,n)} - M$, then $\Gamma \in \mathcal{Z}(n;t)$ because M is a matching of cardinality $2n-2t+1$ (see Th. 1.2 of [1]), and therefore Γ is free of $K_{(t,t)}$ as a subgraph. So $K_3(n)-M$ is free of $K_3(t)$ and the result holds. $\square$

Lemma 3.5 *Let n,t be integers such that $2 \le t \le n$ and let $G = G(T,Y,Z)$ be a 3-partite graph with $|T| = t$, $|Y| = |Z| = n$ free of the complete 3-partite graph $K_3(t)$. Then*

$$e(G) \le n^2 + 2nt - (n-t+1)\,.$$

Proof: Assume that $e(G) \ge n^2 + 2nt - (n-t)$, hence $e(G^c) \le n-t$. If we denote by r_{TY}, r_{TZ} and r_{YZ} the number of edges of the subgraphs $G^c[T \cup Y]$, $G^c[T \cup Z]$ and $G^c[Y \cup Z]$, respectively, then we have $r_{TY} + r_{TZ} + r_{YZ} \le n-t$. Let us denote by Y' and Z', respectively, the subsets of Y and Z whose vertices have degree $t+n$ in G. It is clear that

$$|Y| = |Y'| + |Y \setminus Y'| \le |Y'| + (r_{TY} + r_{YZ}),$$
$$|Z| = |Z'| + |Z \setminus Z'| \le |Z'| + (r_{TZ} + r_{YZ}).$$

Hence $|Y'| \ge n - (r_{TY} + r_{YZ}) \ge t + r_{TZ} \ge t$ and $|Z'| \ge n - (r_{TZ} + r_{YZ}) \ge t + r_{TY} \ge t$. But then $G[T \cup Y' \cup Z']$ contains $K_3(t)$ and the result follows. $\square$

Proof: of Theorem 2.3. We know that $z_3(n) \geq 3n^2 - (2(n-t)+1)$ because of Lemma 3.4. To show the other inequality, let $G = G(X,Y,Z)$ be a 3-partite graph free of $K_3(t)$ as a subgraph. Let us order the vertices $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$ and $Z = \{z_1, \ldots, z_n\}$ so that $d_{G^c}(x_i) \leq d_{G^c}(x_{i+1})$, $d_{G^c}(y_i) \leq d_{G^c}(y_{i+1})$ and $d_{G^c}(z_i) \leq d_{G^c}(z_{i+1})$ for all $i \in \{1, \ldots, n-1\}$. At least one vertex v of the set $\{x_t, y_t, z_t\}$ has degree $d_{G^c}(v) \geq 1$, because if $d_{G^c}(x_t) = d_{G^c}(y_t) = d_{G^c}(z_t) = 0$ then $d_G(x_i) = d_G(y_i) = d_G(z_i) = 2n$ for all $i \in \{1, \ldots, t\}$ and hence $K_3(t) \subseteq G$ against our assumptions. Assume that $d_{G^c}(x_t) \geq 1$ and so $d_{G^c}(x_i) \geq 1$ for $i \in \{t, \ldots, n\}$. Then $G[T \cup Y \cup Z]$ must be free of $K_3(t)$ where $T = \{x_1, \ldots, x_t\}$. Hence, by applying Lemma 3.5, we have $e(G[T \cup Y \cup Z]) \leq n^2 + 2nt - (n-t+1)$. This implies that

$$\begin{aligned} e(G^c) &= e(G^c[T \cup Y \cup Z]) + e(G^c[(X \setminus T) \cup Y \cup Z]) \\ &\geq n - t + 1 + \textstyle\sum_{i=t+1}^{n} d_{G^c}(x_i) \\ &= n - t + 1 + n - t \\ &= 2(n-t) + 1, \end{aligned}$$

thus $e(G) \leq 3n^2 - (2(n-t)+1)$. This proves the theorem. □

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