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Number of connected spanning subgraphs on the Sierpinski gasket

Shu-Chiuan Chang[1] and Lung-Chi Chen[2]

[1] *Department of Physics, National Cheng Kung University, Tainan 70101, Taiwan*
[2] *Department of Mathematics, Fu Jen Catholic University, Taipei 24205, Taiwan*

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We study the number of connected spanning subgraphs $f_{d,b}(n)$ on the generalized Sierpinski gasket $SG_{d,b}(n)$ at stage n with dimension d equal to two, three and four for $b = 2$, and layer b equal to three and four for $d = 2$. The upper and lower bounds for the asymptotic growth constant, defined as $z_{SG_{d,b}} = \lim_{v\to\infty} \ln f_{d,b}(n)/v$ where v is the number of vertices, on $SG_{2,b}(n)$ with $b = 2, 3, 4$ are derived in terms of the results at a certain stage. The numerical values of $z_{SG_{d,b}}$ are obtained.

Keywords: Connected spanning subgraphs, Sierpinski gasket, recursion relations, asymptotic growth constant

1 Introduction

The enumeration of the number of connected spanning subgraphs $N_{CSSG}(G)$ on a graph G is a problem of interest in mathematics [Liu and Chow(1983)]. It is well known that the number of connected subgraphs is given by the Tutte polynomial $T(G, x, y)$ evaluated at $x = 1$, $y = 2$ [Welsh(1993)]. Alternatively, it corresponds to the partition function of the q-state Potts model in statistical mechanics with $q = 0$ and the temperature variable $v = e^{\beta J} - 1 = 1$, where J is the spin-spin coupling and $\beta = (k_B T)^{-1}$. Some recent studies on the enumeration of connected spanning subgraphs and the calculation of their asymptotic growth constants on regular lattices were carried out in Refs. [Shrock(2000), Chang and Shrock(2000), Chang and Shrock(2001)(a), Chang and Shrock(2001)(b), Chang and Shrock(2001)(c)]. It is of interest to consider connected spanning subgraphs on self-similar fractal lattices which have scaling invariance rather than translational invariance. Fractals are geometric structures of (generally noninteger) Hausdorff dimension realized by repeated construction of an elementary shape on progressively smaller length scales [Mandelbrot(1982), Falconer(2003)]. A well-known example of a fractal is the Sierpinski gasket. We shall derive the recursion relations for the numbers of connected spanning subgraphs on the Sierpinski gasket with dimension equal to two, three and four, and determine the asymptotic growth constants. We shall also consider the number of connected spanning subgraphs on the generalized Sierpinski gasket with dimension equal to two.

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2 Preliminaries

We first recall some relevant definitions for connected spanning subgraphs and the Sierpinski gasket in this section. A connected graph (without loops) $G = (V, E)$ is defined by its vertex (site) and edge (bond) sets V and E [Biggs(1993), Harary(1969)]. Let $v(G) = |V|$ be the number of vertices and $e(G) = |E|$ the number of edges in G. A spanning subgraph G' is a subgraph of G with the same vertex set V and an edge set $E' \subseteq E$. A connected spanning subgraph on G is a spanning subgraph of G that remains connected. In general, there can be cycles in a connected spanning subgraph. It is called a spanning tree when there is no cycles. The degree or coordination number k_i of a vertex $v_i \in V$ is the number of edges attached to it. A k-regular graph is a graph with the property that each of its vertices has the same degree k. In general, one can associate an edge weight x_{ij} to each edge connecting adjacent vertices v_i and v_j. For simplicity, all edge weights are set to one throughout this paper.

Consider an infinite sequence of graphs $\{G(n)\}$ with $v(G(n)) \to \infty$ as $n \to \infty$. If the number of connected spanning subgraphs $N_{CSSG}(G(n))$ grows exponentially with $v(G(n))$ when $n \to \infty$ as

$$N_{CSSG}(G(n)) = \exp\left[z_G v(G(n)) + o(v(G(n)))\right] , \tag{1}$$

where $\lim_{n\to\infty} o(v(G(n)))/v(G(n)) = 0$, then the constant z_G can describe this exponential growth:

$$z_G = \lim_{n\to\infty} \frac{\ln N_{CSSG}(G(n))}{v(G(n))} , \tag{2}$$

which is the entropy per site in statistical mechanics. Here G, when used as a subscript in this manner, implicitly refers to the thermodynamic limit.

The construction of the two-dimensional Sierpinski gasket $SG_2(n)$ at stage n is shown in Fig. 1. At stage $n = 0$, it is an equilateral triangle; while stage $(n + 1)$ is obtained by the juxtaposition of three n-stage structures. In general, the Sierpinski gaskets SG_d can be built in any Euclidean dimension d with fractal dimension $D = \ln(d + 1)/\ln 2$ [Gefen and Aharony(1981)]. For the Sierpinski gasket $SG_d(n)$, the numbers of edges and vertices are given by

$$e(SG_d(n)) = \binom{d+1}{2}(d+1)^n = \frac{d}{2}(d+1)^{n+1} , \tag{3}$$

$$v(SG_d(n)) = \frac{d+1}{2}[(d+1)^n + 1] . \tag{4}$$

Except the $(d+1)$ outmost vertices which have degree d, all other vertices of $SG_d(n)$ have degree $2d$. In the large n limit, SG_d is $2d$-regular.

The Sierpinski gasket can be generalized, denoted by $SG_{d,b}(n)$, by introducing the side length b which is an integer larger or equal to two [Hilfer and Blumen(1984)]. The generalized Sierpinski gasket at stage $(n + 1)$ is constructed from b layers of stage n hypertetrahedrons. The two-dimensional $SG_{2,b}(n)$ with $b = 3$ at stage $n = 1, 2$ and $b = 4$ at stage $n = 1$ are illustrated in Fig. 2. The ordinary Sierpinski gasket $SG_d(n)$ corresponds to the $b = 2$ case, where the index b is neglected for simplicity. The Hausdorff dimension for $SG_{d,b}$ is given by $D = \ln\binom{b+d-1}{d}/\ln b$ [Hilfer and Blumen(1984)]. Notice that $SG_{d,b}$ is not k-regular even in the thermodynamic limit.

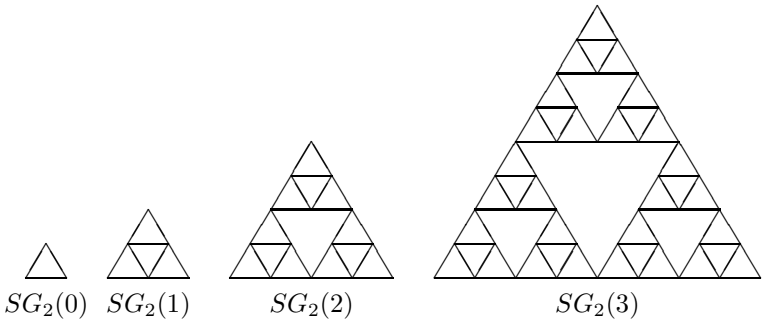

Fig. 1: The first four stages $n = 0, 1, 2, 3$ of the two-dimensional Sierpinski gasket $SG_2(n)$.

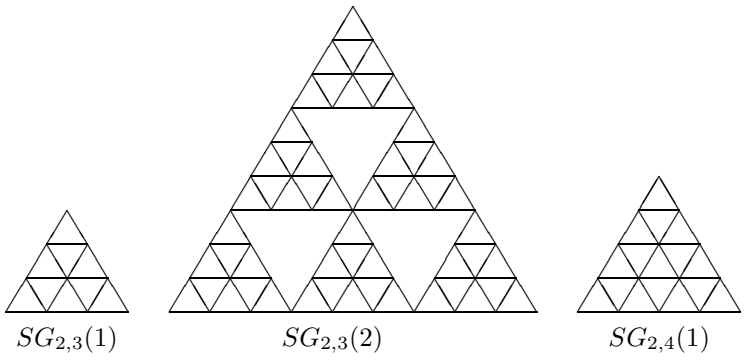

Fig. 2: The generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b = 3$ at stage $n = 1, 2$ and $b = 4$ at stage $n = 1$.

3 The number of connected spanning subgraphs on $SG_2(n)$

In this section we derive the asymptotic growth constant for the number of connected spanning subgraphs on the two-dimensional Sierpinski gasket $SG_2(n)$ in detail. Let us start with the definitions of the quantities to be used.

Definition 3.1 *Consider the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ at stage n.*

(i) Define $f_{2,b}(n) \equiv N_{CSSG}(SG_{2,b}(n))$ as the number of connected spanning subgraphs.

(ii) Define $g_{2,b}(n)$ as the number of spanning subgraphs with two connected components such that one certain outmost vertex, say the topmost vertex as illustrated in Fig. 3 for ordinary Sierpinski gasket, belongs to one component and the other two outmost vertices belong to another component.

(iii) Define $h_{2,b}(n)$ as the number of spanning subgraphs with three connected components such that each of the outmost vertices belongs to a different component.

Since we only consider the ordinary Sierpinski gasket in this section, we use the notations $f_2(n)$, $g_2(n)$ and $h_2(n)$ for simplicity. They are illustrated in Fig. 3, where only the outmost vertices are shown. Because of rotational symmetry, there are three possible $g_2(n)$. The initial values at stage zero are $f_2(0) = 4$, $g_2(0) = 1$ and $h_2(0) = 1$. The purpose of this section is to obtain the asymptotic behavior of $f_2(n)$ as follows. The three quantities $f_2(n)$, $g_2(n)$ and $h_2(n)$ satisfy recursion relations.

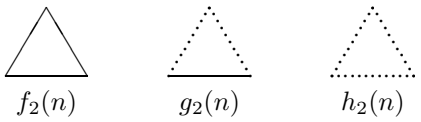

Fig. 3: Illustration for the connected spanning subgraphs $f_2(n)$, $g_2(n)$ and $h_2(n)$. The two outmost vertices at the ends of a solid line belong to one component, while the two outmost vertices at the ends of a dot line belong to separated components.

Lemma 3.1 *For any non-negative integer n,*

$$f_2(n+1) = f_2^3(n) + 6f_2^2(n)g_2(n) \,, \tag{5}$$

$$g_2(n+1) \;=\; f_2^2(n)g_2(n) + f_2^2(n)h_2(n) + 7f_2(n)g_2^2(n) \,, \tag{6}$$

$$h_2(n+1) \;=\; 3f_2(n)g_2^2(n) + 12f_2(n)g_2(n)h_2(n) + 14g_2^3(n) \,. \tag{7}$$

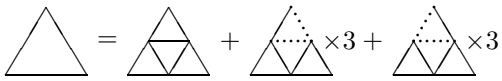

Fig. 4: Illustration for the expression of $f_2(n+1)$. The multiplication of three on the right-hand-side corresponds to the three possible orientations of $SG_2(n+1)$.

Proof: The Sierpinski gasket $SG_2(n+1)$ is composed of three $SG_2(n)$ with three pairs of vertices identified. The number $f_2(n+1)$ consists of one configuration where all three $SG_2(n)$ belong to the class that is enumerated by $f_2(n)$, and six configurations where one of the $SG_2(n)$ belongs to the class enumerated by $g_2(n)$ and the other two belong to the class enumerated by $f_2(n)$ as illustrated in Fig. 4. Eq. (5) is verified by adding these configurations.

Similarly, $g_2(n+1)$ and $h_2(n+1)$ for $SG_2(n+1)$ can be obtained with appropriate configurations of its three constituting $SG_2(n)$ as illustrated in Figs. 5 and 6 to verify Eqs. (6) and (7), respectively.

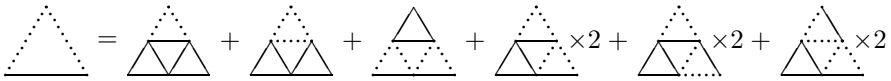

Fig. 5: Illustration for the expression of $g_2(n+1)$. The multiplication of two on the right-hand-side corresponds to the reflection symmetry with respect to the central vertical axis.

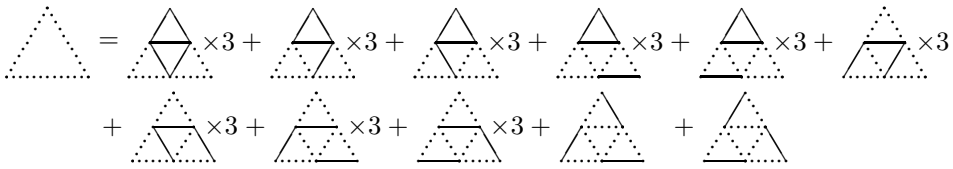

Fig. 6: Illustration for the expression of $h_2(n+1)$. The multiplication of three on the right-hand-side corresponds to the three possible orientations $SG_2(n+1)$.

□

The values of $f_2(n)$, $g_2(n)$, $h_2(n)$ for small n can be evaluated recursively by Eqs. (5), (6), (7) as listed in Table 1. These numbers grow exponentially, and do not have simple integer factorizations, in contrast to the corresponding results for the number of spanning trees [Chang et al.(2007)]. To estimate the value of the asymptotic growth constant defined in Eq. (2), we need the following lemmas. For the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$, define the ratios

$$\alpha_{2,b}(n) = \frac{f_{2,b}(n)}{g_{2,b}(n)}\,, \qquad \beta_{2,b}(n) = \frac{g_{2,b}(n)}{h_{2,b}(n)}\,, \tag{8}$$

where n is a non-negative integer. For the ordinary Sierpinski gasket in this section, they are simplified to be $\alpha_2(n)$, $\beta_2(n)$ and their values for small n are listed in Table 2.

Tab. 1: The first few values of $f_2(n)$, $g_2(n)$, $h_2(n)$.

n	0	1	2	3
$f_2(n)$	4	160	13,312,000	10,293,452,839,321,600,000,000
$g_2(n)$	1	60	7,462,400	8,864,355,990,896,640,000,000
$h_2(n)$	1	74	13,276,800	23,868,720,258,482,176,000,000

Tab. 2: The first few values of $\alpha_2(n)$, $\beta_2(n)$. The last digits given are rounded off.

n	0	1	2	3	4
$\alpha_2(n)$	4	2.66666666666667	1.78387650085763	1.16121835019855	0.736689163182441
$\beta_2(n)$	1	0.810810810810811	0.562063147746445	0.371379608747416	0.238302798822389

Lemma 3.2 *For any* $n \geq 0$*,*

$$3\beta_2(n) \leq \alpha_2(n) \leq 4\beta_2(n) \; . \tag{9}$$

The ratios $\alpha_2(n)$ *and* $\beta_2(n)$ *are both strictly decreasing sequences with the limits*

$$\lim_{n\to\infty} \alpha_2(n) = \lim_{n\to\infty} \beta_2(n) = 0 \; . \tag{10}$$

Proof: It is clear that $\alpha_2(n)$ and $\beta_2(n)$ cannot be negative. By Eqs. (5) - (7), we have

$$\frac{f_2(n+1)}{f_2^2(n)g_2(n)} = 6 + \alpha_2(n) \; , \tag{11}$$

$$\frac{g_2(n+1)}{f_2(n)g_2^2(n)} = \frac{\alpha_2(n)}{\beta_2(n)} + 7 + \alpha_2(n) \; , \tag{12}$$

$$\frac{h_2(n+1)}{g_2^3(n)} = 12\frac{\alpha_2(n)}{\beta_2(n)} + 14 + 3\alpha_2(n) \; . \tag{13}$$

Therefore,

$$\alpha_2(n+1) = \frac{\alpha_2(n)\beta_2(n)[6+\alpha_2(n)]}{\alpha_2(n)+7\beta_2(n)+\alpha_2(n)\beta_2(n)} = \alpha_2(n) - \frac{\alpha_2(n)[\alpha_2(n)+\beta_2(n)]}{\alpha_2(n)+7\beta_2(n)+\alpha_2(n)\beta_2(n)} \; , \tag{14}$$

which shows that $\alpha_2(n)$ is strictly decreasing. Similarly, we have

$$\beta_2(n+1) = \frac{\alpha_2(n)[\alpha_2(n) + 7\beta_2(n) + \alpha_2(n)\beta_2(n)]}{12\alpha_2(n) + 14\beta_2(n) + 3\alpha_2(n)\beta_2(n)} = \frac{\alpha_2(n)}{3} - \frac{\alpha_2(n)[3\alpha_2(n) - 7\beta_2(n)/3]}{12\alpha_2(n) + 14\beta_2(n) + 3\alpha_2(n)\beta_2(n)} . \tag{15}$$

With the initial values given in Table 2, $3\beta_2(n) \le \alpha_2(n)$ is proved by induction. By Eqs. (14) and (15),

$$\begin{aligned} & 4\beta_2(n+1) - \alpha_2(n+1) \\ = \; & \frac{\alpha_2(n)}{3} + \frac{\alpha_2(n)[\alpha_2(n) + \beta_2(n)]}{\alpha_2(n) + 7\beta_2(n) + \alpha_2(n)\beta_2(n)} - \frac{\alpha_2(n)[12\alpha_2(n) - 28\beta_2(n)/3]}{12\alpha_2(n) + 14\beta_2(n) + 3\alpha_2(n)\beta_2(n)} \\ = \; & \frac{\alpha_2(n)X(n)}{3[\alpha_2(n) + 7\beta_2(n) + \alpha_2(n)\beta_2(n)][12\alpha_2(n) + 14\beta_2(n) + 3\alpha_2(n)\beta_2(n)]} , \end{aligned} \tag{16}$$

where

$$X(n) = 12\alpha_2^2(n)+48\beta_2(n)[7\beta_2(n)-\alpha_2(n)]+12\alpha_2(n)\beta_2(n)[6\beta_2(n)-\alpha_2(n)]+3\alpha_2^2(n)\beta_2^2(n) \ge 0 \tag{17}$$

such that $\alpha_2(n) \le 4\beta_2(n)$ is proved again by induction. Eq. (15) can be rewritten as

$$\beta_2(n+1) = \beta_2(n) - \frac{\alpha_2(n)[4\beta_2(n) - \alpha_2(n)][1 + 3\beta_2(n)/4] + \alpha_2(n)\beta_2(n)[1 - \alpha_2(n)/4] + 14\beta_2^2(n)}{12\alpha_2(n) + 14\beta_2(n) + 3\alpha_2(n)\beta_2(n)} , \tag{18}$$

which shows that $\beta_2(n)$ is strictly decreasing since $\alpha_2(n)$ is less than four, *i*.e. its initial value, for all $n \ge 1$. Eq. (14) can be rewritten as

$$\alpha_2(n+1) = \frac{6\alpha_2(n)}{7}\Big[1 - \frac{\alpha_2(n)[1 - \beta_2(n)/6]}{\alpha_2(n) + 7\beta_2(n) + \alpha_2(n)\beta_2(n)}\Big] , \tag{19}$$

which is always less than $6\alpha_2(n)/7$ since $\beta_2(n)$ is less than one, *i*.e. its initial value, for all $n \ge 1$ such that $\lim_{n\to\infty} \alpha_2(n)$ is zero. Finally, since $\beta_2(n) \le \alpha_2(n)/3$, $\lim_{n\to\infty} \beta_2(n)$ is zero, and the proof is completed. □

We notice that the convergences of $\alpha_2(n)$ and $\beta_2(n)$ to zero as n increases are not rapid. The inequality $3\beta_2(n) \le \alpha_2(n)$ can be improved a bit, and we state it as the following lemma.

Lemma 3.3 *For any* $n \ge n_0$,

$$3\beta_2(n) + \frac{\alpha_2^2(n)}{27} \le \alpha_2(n) , \tag{20}$$

where $n_0 = inf\{n : \alpha_2(n) \le 3/4, \beta_2(n) \le 1/4\} = 4$.

Proof: By Eqs. (14) and (15), we have

$$\alpha_2(n+1) - 3\beta_2(n+1) - \frac{\alpha_2^2(n+1)}{27}$$

$$
\begin{aligned}
&= \frac{\alpha_2(n)[9\alpha_2(n)-7\beta_2(n)]}{12\alpha_2(n)+14\beta_2(n)+3\alpha_2(n)\beta_2(n)} - \frac{\alpha_2(n)[\alpha_2(n)+\beta_2(n)]}{\alpha_2(n)+7\beta_2(n)+\alpha_2(n)\beta_2(n)} \\
&\quad - \frac{\alpha_2^2(n)\beta_2^2(n)[6+\alpha_2(n)]^2}{27[\alpha_2(n)+7\beta_2(n)+\alpha_2(n)\beta_2(n)]^2} \\
&= \frac{\alpha_2(n)Y(n)}{[\alpha_2(n)+7\beta_2(n)+\alpha_2(n)\beta_2(n)]^2[12\alpha_2(n)+14\beta_2(n)+3\alpha_2(n)\beta_2(n)]} , \qquad (21)
\end{aligned}
$$

where

$$
\begin{aligned}
Y(n) &= [\alpha_2(n)+7\beta_2(n)+\alpha_2(n)\beta_2(n)]\Big\{3[\alpha_2(n)-3\beta_2(n)][4\beta_2(n)-\alpha_2(n)] \\
&\quad +3\beta_2(n)\Big[\alpha_2(n)-3\beta_2(n)-\frac{\alpha_2^2(n)}{27}\Big][3+2\alpha_2(n)]\Big\} \\
&\quad +\frac{\alpha_2(n)\beta_2(n)}{27}[9\alpha_2^2(n)+1008\beta_2^2(n)-153\alpha_2(n)\beta_2(n)+6\alpha_2^3(n)-93\alpha_2^2(n)\beta_2(n) \\
&\quad -60\alpha_2(n)\beta_2^2(n)-6\alpha_2^3(n)\beta_2(n)-50\alpha_2^2(n)\beta_2^2(n)-3\alpha_2^3(n)\beta_2^2(n)] \\
&\geq \frac{\alpha_2(n)\beta_2(n)}{27}\Big\{3\alpha_2(n)[\alpha_2(n)-3\beta_2(n)][3+2\alpha_2(n)]+252\beta_2(n)[4\beta_2(n)-\alpha_2(n)] \\
&\quad +\alpha_2(n)\beta_2(n)[126-75\alpha_2(n)-6\alpha_2^2(n)-60\beta_2(n)-50\alpha_2(n)\beta_2(n)-3\alpha_2^2(n)\beta_2(n)]\Big\} . \qquad (22)
\end{aligned}
$$

Because $Y(n)$ is positive whenever $\alpha_2(n) \leq 3/4$ and $\beta_2(n) \leq 1/4$, which is true for all $n \geq n_0 = 4$ by the previous lemma and Table 2, the inequality is established. □

We notice that although Eq. (20) is by no means optimum, it is enough for the following lemma.

Lemma 3.4 *The sequence of the ratio* $\{\alpha_2(n)/\beta_2(n)\}_{n=0}^{\infty}$ *decreases monotonically with the limit*

$$\lim_{n\to\infty} \alpha_2(n)/\beta_2(n) = 3 . \qquad (23)$$

Proof: The initial value of the ratio is $\alpha_2(0)/\beta_2(0) = 4$. It is clear from Eq. (15) that in the large n limit, the ratio $\alpha_2(n)/\beta_2(n)$ is equal to three. By Eqs. (14) and (15), we have

$$\frac{\alpha_2(n)}{\beta_2(n)} - \frac{\alpha_2(n+1)}{\beta_2(n+1)} = \frac{Z(n)}{\beta_2(n)[\alpha_2(n)+7\beta_2(n)+\alpha_2(n)\beta_2(n)]^2} , \qquad (24)$$

where

$$
\begin{aligned}
Z(n) &= \alpha_2(n)[\alpha_2(n)+7\beta_2(n)+\alpha_2(n)\beta_2(n)]^2 \\
&\quad -\beta_2^2(n)[6+\alpha_2(n)][12\alpha_2(n)+14\beta_2(n)+3\alpha_2(n)\beta_2(n)]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & [\alpha_2(n)-3\beta_2(n)]\Big[2\alpha_2^2(n)\beta_2(n)+\frac{32}{3}\alpha_2(n)\beta_2^2(n)+\alpha_2^2(n)\beta_2^2(n)\Big] \\
& +\Big[\alpha_2(n)-3\beta_2(n)-\frac{\alpha_2^2(n)}{27}\Big][\alpha_2^2(n)+17\alpha_2(n)\beta_2(n)+28\beta_2^2(n)] \\
& +\frac{\alpha_2^2(n)}{27}[\alpha_2^2(n)+17\alpha_2(n)\beta_2(n)+28\beta_2^2(n)]-\frac{8}{3}\alpha_2^2(n)\beta_2^2(n)\ . \qquad (25)
\end{aligned}
$$

With $\alpha_2(n) \geq 3\beta_2(n)$, $Z(n)$ is positive such that the sequence of the ratio decreases monotonically. □

The general expressions for $f_2(n)$ and $g_2(n)$ can be written as follows.

Lemma 3.5 *For a non-negative integer m and any positive integer $n > m$,*

$$
\begin{aligned}
f_2(n) \quad = \quad & f_2(m)^{\frac{3^{n-m}+1}{2}} g_2(m)^{\frac{3^{n-m}-1}{2}} \prod_{i=1}^{n-m}\Big[6+\alpha_2(n-i)\Big]^{\frac{3^{i-1}+1}{2}} \\
& \times \prod_{j=2}^{n-m}\Big[7+\alpha_2(n-j)+\frac{\alpha_2(n-j)}{\beta_2(n-j)}\Big]^{\frac{3^{j-1}-1}{2}}\ , \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
g_2(n) \quad = \quad & f_2(m)^{\frac{3^{n-m}-1}{2}} g_2(m)^{\frac{3^{n-m}+1}{2}} \prod_{i=2}^{n-m}\Big[6+\alpha_2(n-i)\Big]^{\frac{3^{i-1}-1}{2}} \\
& \times \prod_{j=1}^{n-m}\Big[7+\alpha_2(n-j)+\frac{\alpha_2(n-j)}{\beta_2(n-j)}\Big]^{\frac{3^{j-1}+1}{2}}\ . \qquad (27)
\end{aligned}
$$

Here when $n-m=1$, the products with lower limit two are defined to be one.

Proof: It is clear from Eqs. (11) and (12) that $f_2(m+1) = f_2^2(m)g_2(m)[6+\alpha_2(m)]$ and $g_2(m+1) = f_2(m)g_2^2(m)[7+\alpha_2(m)+\alpha_2(m)/\beta_2(m)]$. Consider Eqs. (26) and (27) hold for a certain positive integer $n=k$, then

$$
\begin{aligned}
f_2(k+1) \quad = \quad & f_2^2(k)g_2(k)[6+\alpha_2(k)] \\
= \quad & f_2(m)^{3^{k-m}+1} g_2(m)^{3^{k-m}-1} \prod_{i=1}^{k-m}\Big[6+\alpha_2(k-i)\Big]^{3^{i-1}+1} \\
& \times \prod_{j=2}^{k-m}\Big[7+\alpha_2(k-j)+\frac{\alpha_2(k-j)}{\beta_2(k-j)}\Big]^{3^{j-1}-1} \\
& \times f_2(m)^{\frac{3^{k-m}-1}{2}} g_2(m)^{\frac{3^{k-m}+1}{2}} \prod_{i=2}^{k-m}\Big[6+\alpha_2(k-i)\Big]^{\frac{3^{i-1}-1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
& \times \prod_{j=1}^{k-m} \Big[7+\alpha_2(k-j)+\frac{\alpha_2(k-j)}{\beta_2(k-j)}\Big]^{\frac{3^{j-1}+1}{2}} [6+\alpha_2(k)] \\
= \;& f_2(m)^{\frac{3^{k-m+1}+1}{2}} g_2(m)^{\frac{3^{k-m+1}-1}{2}} [6+\alpha_2(k)][6+\alpha_2(k-1)]^2 \\
& \times \prod_{i=2}^{k-m} \Big[6+\alpha_2(k-i)\Big]^{\frac{3^i+1}{2}} \Big[7+\alpha_2(k-1)+\frac{\alpha_2(k-1)}{\beta_2(k-1)}\Big] \\
& \times \prod_{j=2}^{k-m} \Big[7+\alpha_2(k-j)+\frac{\alpha_2(k-j)}{\beta_2(k-j)}\Big]^{\frac{3^j-1}{2}} \\
= \;& f_2(m)^{\frac{3^{k-m+1}+1}{2}} g_2(m)^{\frac{3^{k-m+1}-1}{2}} \prod_{i=1}^{k-m+1} \Big[6+\alpha_2(k+1-i)\Big]^{\frac{3^{i-1}+1}{2}} \\
& \prod_{j=2}^{k-m+1} \Big[7+\alpha_2(k+1-j)+\frac{\alpha_2(k+1-j)}{\beta_2(k+1-j)}\Big]^{\frac{3^{j-1}-1}{2}} \;, \qquad (28)
\end{aligned}
$$

and Eq. (26) is proved by induction. Eq. (27) can be established by the same procedure. □

From the above lemmas, we have the following bounds for the asymptotic growth constant.

Lemma 3.6 *The asymptotic growth constant for the number of connected spanning subgraphs on $SG_2(n)$ is bounded:*

$$
\frac{\ln[f_2(m)g_2(m)]+\frac{1}{2}\ln 60}{3^{m+1}} \le z_{SG_2} \le \frac{\ln[f_2(m)g_2(m)]+\frac{1}{2}\ln[6+\alpha_2(m)]\Big[7+\alpha_2(m)+\frac{\alpha_2(m)}{\beta_2(m)}\Big]}{3^{m+1}} \;, \qquad (29)
$$

where m is a positive integer.

Proof: By Lemma 3.5, we have

$$
\ln f_2(n) = \frac{3^{n-m}+1}{2}\ln f_2(m) + \frac{3^{n-m}-1}{2}\ln g_2(m) + \Delta(n,m) \;, \qquad (30)
$$

where

$$
\Delta(n,m) = \sum_{i=1}^{n-m} \frac{3^{i-1}+1}{2}\ln[6+\alpha_2(n-i)] + \sum_{j=2}^{n-m} \frac{3^{j-1}-1}{2}\ln\Big[7+\alpha_2(n-j)+\frac{\alpha_2(n-j)}{\beta_2(n-j)}\Big] \;. \qquad (31)
$$

We have shown that as m increases, $\alpha_2(m)$ decreases to zero in Lemma 3.2 and $\alpha_2(m)/\beta_2(m)$ decreases to three in Lemma 3.4 such that

$$
\Delta(n,m) \;\le\; \sum_{i=1}^{n-m} \frac{3^{i-1}+1}{2}\ln[6+\alpha_2(m)] + \sum_{j=2}^{n-m} \frac{3^{j-1}-1}{2}\ln\Big[7+\alpha_2(m)+\frac{\alpha_2(m)}{\beta_2(m)}\Big]
$$

$$\begin{aligned} &= \frac{1}{2}\Big(\frac{3^{n-m}-1}{2}+n-m\Big)\ln[6+\alpha_2(m)] \\ &\quad +\frac{1}{2}\Big(\frac{3^{n-m}-3}{2}-n+m+1\Big)\ln\Big[7+\alpha_2(m)+\frac{\alpha_2(m)}{\beta_2(m)}\Big] \end{aligned} \tag{32}$$

and

$$\begin{aligned} \Delta(n,m) &\geq \sum_{i=1}^{n-m}\frac{3^{i-1}+1}{2}\ln 6+\sum_{j=2}^{n-m}\frac{3^{j-1}-1}{2}\ln 10 \\ &= \frac{1}{2}\Big(\frac{3^{n-m}-1}{2}+n-m\Big)\ln 6+\frac{1}{2}\Big(\frac{3^{n-m}-3}{2}-n+m+1\Big)\ln 10\,. \end{aligned} \tag{33}$$

With the definition for z_{SG_2} given in Eq. (2) and the number of vertices of $SG_2(n)$ is $3(3^n+1)/2$ by Eq. (4), the proof is completed. □

As m increases, the difference between the upper and lower bounds in Eq. (29) is less than 3^{-m} and converges to zero. We calculate the number of connected spanning subgraphs $f_2(m)$ up to $m=15$, and we have the following proposition.

Proposition 3.1 *The asymptotic growth constant for the number of connected spanning subgraphs on the two-dimensional Sierpinski gasket $SG_2(n)$ in the large n limit exists with value $z_{SG_2}=1.276495930....$*

Without going into details, we state here without proof that the bounds can be improved. For a non-negative integer m and any positive integer $n>m$, the tighter bounds for $\alpha_2(n)$ are

$$d^{n-m}(m)\leq\alpha_2(n)\leq c^{n-m}(m)\,, \tag{34}$$

where

$$c(m)=\frac{6+\alpha_2(m)}{10+\alpha_2(m)}\,,\qquad d(m)=\frac{6\beta_2(m)}{\alpha_2(m)+7\beta_2(m)}\,. \tag{35}$$

It can be shown that

$$z_{SG_2}\leq\frac{1}{3^{m+1}}\Big\{\ln[f_2(m)g_2(m)]+\frac{1}{2}\ln\Big[42+6\frac{\alpha_2(m)}{\beta_2(m)}\Big]+\frac{\alpha_2(m)}{[3-c(m)]}\Big[\frac{1}{6}+\frac{\beta_2(m)}{7\beta_2(m)+\alpha_2(m)}\Big]\Big\} \tag{36}$$

and

$$z_{SG_2}\geq\frac{1}{3^{m+1}}\Big\{\ln[f_2(m)g_2(m)]+\frac{\ln 60}{2}+\frac{4\alpha_2(m)}{15[3-d(m)]}-\frac{17\alpha_2^2(m)}{900[3-d^2(m)]}\Big\}\,, \tag{37}$$

so that the asymptotic growth constant is $z_{SG_2}=1.27649593067....$ We notice that the numerical value can also be estimated by the standard technique given in [Aho and Sloane(1973)] with similar rate of convergence.

The asymptotic behaviors of $f_2(n)$, $g_2(n)$ and $h_2(n)$ can be derived as follows [Aho and Sloane(1973), Teufl and Wagner(2007)]. Rewriting Eqs. (14), (15) into

$$\begin{aligned} \frac{\alpha_2(n+1)}{\alpha_2(n)} &= \frac{6+\alpha_2(n)}{\frac{\alpha_2(n)}{\beta_2(n)}+7+\alpha_2(n)} , \\ \frac{\beta_2(n+1)}{\beta_2(n)} &= \frac{\frac{\alpha_2(n)}{\beta_2(n)}+7+\alpha_2(n)}{12+14\frac{\alpha_2(n)}{\beta_2(n)}+3\beta_2(n)} \end{aligned} \tag{38}$$

and using Lemmas 3.2 and 3.4, it is not hard to see that both $(5/3)^n\alpha_2(n)$ and $(5/3)^n\beta_2(n)$ tend to (positive, finite) limits. By employing Eq. (6) repeatedly, we have

$$\ln g_2(n) = 3^{n-m}\ln g_2(m) + \sum_{j=m+1}^{n} 3^{n-j}\ln\Big\{\alpha_2(j)\Big[\alpha_2(j)+\frac{\alpha_2(j)}{\beta_2(j)}+7\Big]\Big\} \tag{39}$$

for a certain m less than n. When both n and m are large, $\ln g_2(n)$ is given approximately by

$$\ln g_2(n) \sim 3^{n-m}\ln g_2(m) + \sum_{j=m+1}^{n} 3^{n-j}[a(j)+j\ln(3/5)] , \tag{40}$$

where $a(j)$ is close to a constant for all the j values. Carrying out the summation

$$\sum_{j=m+1}^{n} 3^{n-j}j = 3^{n-m}\Big(\frac{m}{2}+\frac{3}{4}\Big)-\frac{3}{4}-\frac{n}{2} , \tag{41}$$

it follows that

$$g_2(n) \sim C(g_2)\Big(\frac{5}{3}\Big)^{n/2}A_2^{3^n} , \tag{42}$$

where $C(g_2)$ and A_2 are constants with $z_{SG_2} = (2/3)\ln A_2$. Similarly, we have

$$f_2(n) \sim C(f_2)\Big(\frac{3}{5}\Big)^{n/2}A_2^{3^n} , \qquad h_2(n) \sim C(h_2)\Big(\frac{5}{3}\Big)^{3n/2}A_2^{3^n} , \tag{43}$$

where $C(f_2)$, $C(h_2)$ are constants.

4 The number of connected spanning subgraphs on $SG_{2,b}(n)$ with $b = 3, 4$

The method given in the previous section can be applied to the number of connected spanning subgraphs on $SG_{d,b}(n)$ with larger values of d and b. The number of configurations to be considered increases as d and b increase, and the recursion relations must be derived individually for each d and b. In this section, we consider the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with the number of layers b equal to three and four. For $SG_{2,3}(n)$, the numbers of edges and vertices are given by

$$e(SG_{2,3}(n)) = 3\times 6^n , \tag{44}$$

$$v(SG_{2,3}(n)) = \frac{7 \times 6^n + 8}{5} , \tag{45}$$

where the three outmost vertices have degree two. There are $(6^n - 1)/5$ vertices of $SG_{2,3}(n)$ with degree six and $6(6^n - 1)/5$ vertices with degree four. The initial values for the number of connected spanning subgraphs are the same as for SG_2: $f_{2,3}(0) = 4$, $g_{2,3}(0) = 1$ and $h_{2,3}(0) = 1$. By the method illustrated in the previous section, we obtain the following recursion relations for any non-negative integer n.

$$\begin{aligned} f_{2,3}(n+1) &= f_{2,3}^6(n) + 15f_{2,3}^5(n)g_{2,3}(n) + 3f_{2,3}^5(n)h_{2,3}(n) + 78f_{2,3}^4(n)g_{2,3}^2(n) \\ &\quad +18f_{2,3}^4(n)g_{2,3}(n)h_{2,3}(n) + 142f_{2,3}^3(n)g_{2,3}^3(n) , \end{aligned} \tag{46}$$

$$\begin{aligned} g_{2,3}(n+1) &= f_{2,3}^5(n)g_{2,3}(n) + f_{2,3}^5(n)h_{2,3}(n) + 16f_{2,3}^4(n)g_{2,3}^2(n) + 18f_{2,3}^4(n)g_{2,3}(n)h_{2,3}(n) \\ &\quad +89f_{2,3}^3(n)g_{2,3}^3(n) + 2f_{2,3}^4(n)h_{2,3}^2(n) + 77f_{2,3}^3(n)g_{2,3}^2(n)h_{2,3}(n) \\ &\quad +171f_{2,3}^2(n)g_{2,3}^4(n) , \end{aligned} \tag{47}$$

$$\begin{aligned} h_{2,3}(n+1) &= 3f_{2,3}^4(n)g_{2,3}^2(n) + 6f_{2,3}^4(n)g_{2,3}(n)h_{2,3}(n) + 51f_{2,3}^3(n)g_{2,3}^3(n) + 3f_{2,3}^4(n)h_{2,3}^2(n) \\ &\quad +129f_{2,3}^3(n)g_{2,3}^2(n)h_{2,3}(n) + 279f_{2,3}^2(n)g_{2,3}^4(n) + 60f_{2,3}^3(n)g_{2,3}(n)h_{2,3}^2(n) \\ &\quad +564f_{2,3}^2(n)g_{2,3}^3(n)h_{2,3}(n) + 468f_{2,3}(n)g_{2,3}^5(n) . \end{aligned} \tag{48}$$

The figures for these configurations are too many to be shown here. Some values of $f_{2,3}(n)$, $g_{2,3}(n)$, $h_{2,3}(n)$ are listed in Table 3. These numbers grow exponentially, and do not have simple integer factorizations.

Tab. 3: The first few values of $f_{2,3}(n)$, $g_{2,3}(n)$, $h_{2,3}(n)$.

n	0	1	2
$f_{2,3}(n)$	4	56,192	1,292,237,078,102,059,106,775,347,494,912
$g_{2,3}(n)$	1	24,624	1,015,755,670,321,368,497,188,308,516,864
$h_{2,3}(n)$	1	33,792	2,465,934,182,960,517,405,173,530,755,072

The sequences of the ratios $\{\alpha_{2,3}(n)\}_{n=1}^{\infty}$ and $\{\beta_{2,3}(n)\}_{n=1}^{\infty}$ defined in Eq. (8) again decrease monotonically with $\lim_{n\to\infty} \alpha_{2,3}(n) = \beta_{2,3}(n) = 0$. The ratio $\alpha_{2,3}(n)/\beta_{2,3}(n)$ decreases from four to three, the same as the results for $SG_2(n)$. The values of $\alpha_{2,3}(n)$, $\beta_{2,3}(n)$ for small n are listed in Table 4.

By the same method as in Lemma 3.5, we have the general expression for the number of connected spanning subgraphs.

$$f_{2,3}(n) = f_{2,3}(m)^{\frac{3}{5}(4\times 6^{n-m-1}+1)} g_{2,3}(m)^{\frac{3}{5}(6^{n-m}-1)} \prod_{i=1}^{n-m} P_{2,3}(n-j)^{\frac{3}{5}(4\times 6^{i-2}+1)}$$

Tab. 4: The first few values of $\alpha_{2,3}(n)$, $\beta_{2,3}(n)$. The last digits given are rounded off.

n	0	1	2	3	4
$\alpha_{2,3}(n)$	4	2.28200129954516	1.27219282732945	0.660858801678112	0.326587785819904
$\beta_{2,3}(n)$	1	0.728693181818182	0.411915158701392	0.215917449918629	0.107573237878269

$$\times \prod_{j=2}^{n-m} Q_{2,3}(n-j)^{\frac{3}{5}(6^{j-1}-1)} , \tag{49}$$

where

$$P_{2,3}(m) = 142 + 18\frac{\alpha_{2,3}(m)}{\beta_{2,3}(m)} + 78\alpha_{2,3}(m) + 3\frac{\alpha_{2,3}^2(m)}{\beta_{2,3}(m)} + 15\alpha_{2,3}^2(m) + \alpha_{2,3}^3(m) , \tag{50}$$

$$Q_{2,3}(m) = 171+77\frac{\alpha_{2,3}(m)}{\beta_{2,3}(m)}+2\frac{\alpha_{2,3}^2(m)}{\beta_{2,3}^2(m)}+89\alpha_{2,3}(m)+18\frac{\alpha_{2,3}^2(m)}{\beta_{2,3}(m)}+16\alpha_{2,3}^2(m)+\frac{\alpha_{2,3}^3(m)}{\beta_{2,3}(m)}+\alpha_{2,3}^3(m) . \tag{51}$$

By the same argument given in Lemma 3.6, we have the upper and lower bounds of the asymptotic growth constant for the number of connected spanning subgraphs on $SG_{2,3}(n)$:

$$\begin{aligned} &\frac{1}{7\times 6^m}[2\ln f_{2,3}(m) + 3\ln g_{2,3}(m)] + \frac{1}{35\times 6^m}[2\ln 196 + 3\ln 420] \le z_{SG_{2,3}} \\ &\le \frac{1}{7\times 6^m}[2\ln f_{2,3}(m) + 3\ln g_{2,3}(m)] + \frac{1}{35\times 6^m}[2\ln P_{2,3}(m) + 3\ln Q_{2,3}(m)] , \end{aligned} \tag{52}$$

with m a positive integer. We have the following proposition.

Proposition 4.1 *The asymptotic growth constant for the number of connected spanning subgraphs on the two-dimensional Sierpinski gasket $SG_{2,3}(n)$ in the large n limit exists with value $z_{SG_{2,3}} = 1.3972789680...$.*

Similar to the argument given at the end of Section 3, we find that both $(15/7)^n\alpha_{2,3}(n)$ and $(15/7)^n\beta_{2,3}(n)$ tend to (positive, finite) limits, such that

$$\begin{aligned} f_{2,3}(n) &\sim C(f_{2,3})\Big(\frac{7}{15}\Big)^{(3/5)n} A_{2,3}^{6^n} , \qquad g_{2,3}(n) \sim C(g_{2,3})\Big(\frac{15}{7}\Big)^{(2/5)n} A_{2,3}^{6^n} , \\ h_{2,3}(n) &\sim C(h_{2,3})\Big(\frac{15}{7}\Big)^{(7/5)n} A_{2,3}^{6^n} , \end{aligned} \tag{53}$$

where $C(f_{2,3})$, $C(g_{2,3})$, $C(h_{2,3})$ are constants with $z_{SG_{2,3}} = (5/7)\ln A_{2,3}$.

For $SG_{2,4}(n)$, the numbers of edges and vertices are given by

$$e(SG_{2,4}(n)) = 3\times 10^n , \tag{54}$$

$$v(SG_{2,4}(n)) = \frac{4 \times 10^n + 5}{3}\,, \tag{55}$$

where again the three outmost vertices have degree two. There are $(10^n - 1)/3$ vertices of $SG_{2,4}(n)$ with degree six, and $(10^n - 1)$ vertices with degree four. The initial values for the number of connected spanning subgraphs are the same as for SG_2: $f_{2,4}(0) = 4$, $g_{2,4}(0) = 1$ and $h_{2,4}(0) = 1$. We wrote a computer program to obtain the recursion relations for $SG_{2,4}(n)$. They are lengthy and given in the appendix. Some values of $f_{2,4}(n)$, $g_{2,4}(n)$, $h_{2,4}(n)$ are listed in Table 5. These numbers grow exponentially, and do not have simple integer factorizations.

Tab. 5: The first few values of $f_{2,4}(n)$, $g_{2,4}(n)$, $h_{2,4}(n)$.

n	1	2
$f_{2,4}(n)$	164,119,040	27,140,375,625,882,898,681,725,275,604,427,985,839,201,951,967,246,962,831,668,354,270,630,763,784,348,631,040,000
$g_{2,4}(n)$	77,622,016	25,675,411,803,142,714,297,950,351,525,972,498,833,548,895,007,181,465,816,231,861,389,426,797,930,493,247,488,000
$h_{2,4}(n)$	112,848,672	74,273,341,808,825,211,957,637,724,253,224,196,638,029,720,486,058,269,503,940,976,372,670,504,798,196,334,592,000

The sequences of the ratios $\{\alpha_{2,4}(n)\}_{n=1}^{\infty}$ and $\{\beta_{2,4}(n)\}_{n=1}^{\infty}$ defined in Eq. (8) again decrease monotonically with $\lim_{n\to\infty} \alpha_{2,4}(n) = \beta_{2,4}(n) = 0$. The ratio $\alpha_{2,4}(n)/\beta_{2,4}(n)$ decreases from four to three. The values of $\alpha_{2,4}(n)$, $\beta_{2,4}(n)$ for small n are listed in Table 6.

Tab. 6: The first few values of $\alpha_{2,4}(n)$, $\beta_{2,4}(n)$. The last digits given are rounded off.

n	0	1	2	3	4
$\alpha_{2,4}(n)$	4	2.11433622131123	1.05705707211134	0.475214294459902	0.199993476915309
$\beta_{2,4}(n)$	1	0.687841643364664	0.345688118749653	0.156412715166630	0.662275943767262

By the same method as in Lemma 3.5, we have the general expression for the number of connected spanning subgraphs.

$$f_{2,4}(n) = f_{2,4}(m)^{\frac{1}{3}(10^{n-m}+2)} g_{2,4}(m)^{\frac{2}{3}(10^{n-m}-1)} \prod_{i=1}^{n-m} P_{2,4}(n-j)^{\frac{1}{3}(10^{i-1}+2)} \prod_{j=2}^{n-m} Q_{2,4}(n-j)^{\frac{2}{3}(10^{j-1}-1)}\,, \tag{56}$$

where

$$P_{2,4}(m) \;=\; 11354 + 5856\frac{\alpha_{2,4}(m)}{\beta_{2,4}(m)} + 516\frac{\alpha_{2,4}^2(m)}{\beta_{2,4}^2(m)} + 2\frac{\alpha_{2,4}^3(m)}{\beta_{2,4}^3(m)} + 13626\alpha_{2,4}(m) + 4140\frac{\alpha_{2,4}^2(m)}{\beta_{2,4}(m)}$$

$$+174\frac{\alpha_{2,4}^3(m)}{\beta_{2,4}^2(m)}+6936\alpha_{2,4}^2(m)+1140\frac{\alpha_{2,4}^3(m)}{\beta_{2,4}(m)}+15\frac{\alpha_{2,4}^4(m)}{\beta_{2,4}^2(m)}+1928\alpha_{2,4}^3(m)$$
$$+144\frac{\alpha_{2,4}^4(m)}{\beta_{2,4}(m)}+309\alpha_{2,4}^4(m)+7\frac{\alpha_{2,4}^5(m)}{\beta_{2,4}(m)}+27\alpha_{2,4}^5(m)+\alpha_{2,4}^6(m)\ , \tag{57}$$

$$\begin{aligned} Q_{2,4}(m) &= 13732+14480\frac{\alpha_{2,4}(m)}{\beta_{2,4}(m)}+2786\frac{\alpha_{2,4}^2(m)}{\beta_{2,4}^2(m)}+82\frac{\alpha_{2,4}^3(m)}{\beta_{2,4}^3(m)}+16250\alpha_{2,4}(m) \\ &+10609\frac{\alpha_{2,4}^2(m)}{\beta_{2,4}(m)}+1095\frac{\alpha_{2,4}^3(m)}{\beta_{2,4}^2(m)}+12\frac{\alpha_{2,4}^4(m)}{\beta_{2,4}^3(m)}+8015\alpha_{2,4}^2(m)+3130\frac{\alpha_{2,4}^3(m)}{\beta_{2,4}(m)} \\ &+142\frac{\alpha_{2,4}^4(m)}{\beta_{2,4}^2(m)}+2148\alpha_{2,4}^3(m)+462\frac{\alpha_{2,4}^4(m)}{\beta_{2,4}(m)}+6\frac{\alpha_{2,4}^5(m)}{\beta_{2,4}^2(m)}+332\alpha_{2,4}^4(m) \\ &+34\frac{\alpha_{2,4}^5(m)}{\beta_{2,4}(m)}+28\alpha_{2,4}^5(m)+\frac{\alpha_{2,4}^6(m)}{\beta_{2,4}(m)}+\alpha_{2,4}^6(m)\ . \end{aligned} \tag{58}$$

By the same argument given in Lemma 3.6, we have the upper and lower bounds of the asymptotic growth constant for the number of connected spanning subgraphs on $SG_{2,4}(n)$:

$$\frac{1}{4\times 10^m}[\ln f_{2,4}(m)+2\ln g_{2,4}(m)]+\frac{1}{36\times 10^m}[\ln 33620+2\ln 84460]\le z_{SG_{2,4}}$$
$$\le \frac{1}{4\times 10^m}[\ln f_{2,4}(m)+2\ln g_{2,4}(m)]+\frac{1}{36\times 10^m}[\ln P_{2,4}(m)+2\ln Q_{2,4}(m)]\ , \tag{59}$$

with m a positive integer. We have the following proposition.

Proposition 4.2 *The asymptotic growth constant for the number of connected spanning subgraphs on the two-dimensional Sierpinski gasket $SG_{2,4}(n)$ in the large n limit exists with value $z_{SG_{2,4}}=1.484911260....$*

Similar to the argument given at the end of Section 3, we find that both $(103/41)^n\alpha_{2,4}(n)$ and $(103/41)^n\beta_{2,4}(n)$ tend to (positive, finite) limits, such that

$$\begin{aligned} f_{2,4}(n) &\sim C(f_{2,4})\Big(\frac{41}{103}\Big)^{(2/3)n}A_{2,4}^{10^n}\ , \qquad g_{2,4}(n)\sim C(g_{2,4})\Big(\frac{103}{41}\Big)^{n/3}A_{2,4}^{10^n}\ , \\ h_{2,4}(n) &\sim C(h_{2,4})\Big(\frac{103}{41}\Big)^{(4/3)n}A_{2,4}^{10^n}\ , \end{aligned} \tag{60}$$

where $C(f_{2,4})$, $C(g_{2,4})$, $C(h_{2,4})$ are constants with $z_{SG_{2,4}}=(3/4)\ln A_{2,4}$.

5 The number of connected spanning subgraphs on $SG_d(n)$ with $d=3,4$

In this section, we derive the asymptotic growth constant of connected spanning subgraphs on $SG_d(n)$ with $d=3,4$. For the three-dimensional Sierpinski gasket $SG_3(n)$, we use the following definitions.

Definition 5.1 *Consider the three-dimensional Sierpinski gasket $SG_3(n)$ at stage n.*

(i) Define $f_3(n) \equiv N_{CSSG}(SG_3(n))$ as the number of connected spanning subgraphs.

(ii) Define $g_3(n)$ as the number of spanning subgraphs with two components such that one certain outmost vertices belongs to one component and the other three outmost vertices belong to another component.

(iii) Define $h_3(n)$ as the number of spanning subgraphs with two components such that two certain outmost vertices belong to one component and the other two outmost vertices belong to another component.

(iv) Define $r_3(n)$ as the number of spanning subgraphs with three components such that two certain outmost vertices belong to one component and the other two outmost vertices separately belong to other components.

(v) Define $s_3(n)$ as the number of spanning subgraphs with four components such that each of the outmost vertices belongs to a different component.

The quantities $f_3(n)$, $g_3(n)$, $h_3(n)$, $r_3(n)$ and $s_3(n)$ are illustrated in Fig. 7, where only the outmost vertices are shown. There are four different classes of connected subgraphs enumerated by $g_3(n)$, three classes enumerated by $h_3(n)$, and six classes enumerated by $r_3(n)$. The initial values at stage zero are $f_3(0) = 38$, $g_3(0) = 4$, $h_3(0) = 1$, $r_3(0) = 1$ and $s_3(0) = 1$.

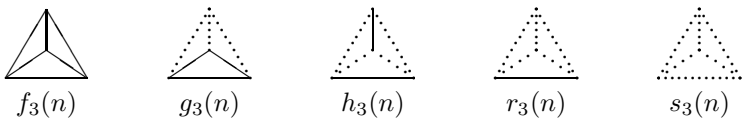

Fig. 7: Illustration for the spanning subgraphs $f_3(n)$, $g_3(n)$, $h_3(n)$, $r_3(n)$ and $s_3(n)$. The two outmost vertices at the ends of a solid line belong to one component, while the two outmost vertices at the ends of a dot line belong to separated components.

The recursion relations are lengthy and given in the appendix. Some values of $f_3(n)$, $g_3(n)$, $h_3(n)$, $r_3(n)$, $s_3(n)$ are listed in Table 7. These numbers grow exponentially, and do not have simple integer factorizations.

We find it is difficult to derive the bounds of the asymptotic growth constant for the number of connected spanning subgraphs on $SG_3(n)$. We calculate $f_3(m)$ up to $m = 10$, and fit the numerical value of the asymptotic growth constant to have the following proposition.

Proposition 5.1 *The asymptotic growth constant for the number of connected spanning subgraphs on the three-dimensional Sierpinski gasket $SG_3(n)$ in the large n limit exists with value $z_{SG_3} = 2.0637052....$*

Tab. 7: The first few values of $f_3(n)$, $g_3(n)$, $h_3(n)$, $r_3(n)$, $s_3(n)$.

n	0	1	2
$f_3(n)$	38	8,554,560	25,988,410,915,610,195,960,527,441,920
$g_3(n)$	4	1,271,416	4,544,490,996,892,396,578,747,598,336
$h_3(n)$	1	39,502	73,629,059,909,730,939,289,401,696
$r_3(n)$	1	254,462	917,115,147,969,863,922,701,973,216
$s_3(n)$	1	153,824	637,427,406,318,067,141,227,862,784

For the four-dimensional Sierpinski gasket $SG_4(n)$, we use the following definitions.

Definition 5.2 *Consider the four-dimensional Sierpinski gasket $SG_4(n)$ at stage n.*

(i) Define $f_4(n) \equiv N_{CSSG}(SG_4(n))$ as the number of connected spanning subgraphs.

(ii) Define $g_4(n)$ as the number of spanning subgraphs with two components such that two certain outmost vertices belong to one component and the other three outmost vertices belong to another component.

(iii) Define $g'_4(n)$ as the number of spanning subgraphs with two components such that one certain outmost vertices belong to one component and the other four outmost vertices belong to another component.

(iv) Define $h_4(n)$ as the number of spanning subgraphs with three components such that one certain outmost vertices belong to one component, two certain other outmost vertices belong to another component and the remaining two outmost vertices belong to a third component.

(v) Define $h'_4(n)$ as the number of spanning subgraphs with three components such that three certain outmost vertices belong to one component and the other two outmost vertices separately belong to other components.

(vi) Define $r_4(n)$ as the number of spanning subgraphs with four components such that two certain outmost vertices belong to one component and the other three outmost vertices separately belong to other components.

(vii) Define $s_4(n)$ as the number of spanning subgraphs with five components such that each of the outmost vertices belongs to a different component.

The quantities $f_4(n)$, $g_4(n)$, $g'_4(n)$, $h_4(n)$, $h'_4(n)$, $r_4(n)$ and $s_4(n)$ are illustrated in Fig. 8, where only the outmost vertices are shown. There are ten different classes of connected subgraphs enumerated by $g_4(n)$, five classes enumerated by $g'_4(n)$, fifteen classes enumerated by $h_4(n)$, ten classes enumerated by $h'_4(n)$ and ten classes enumerated by $r_4(n)$. The initial values at stage zero are $f_4(0) = 728$, $g_4(0) = 4$, $g'_4(0) = 38$, $h_4(0) = 1$, $h'_4(0) = 4$, $r_4(0) = 1$ and $s_4(0) = 1$.

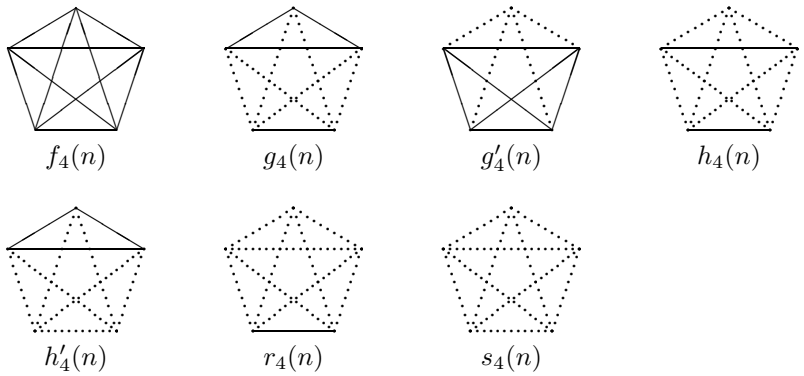

Fig. 8: Illustration for the spanning subgraphs $f_4(n)$, $g_4(n)$, $g'_4(n)$, $h_4(n)$, $h'_4(n)$, $r_4(n)$ and $s_4(n)$. The two outmost vertices at the ends of a solid line belong to one component, while the two outmost vertices at the ends of a dot line belong to separated components.

We wrote a computer program to obtain the recursion relations for $SG_4(n)$. They are too lengthy to be included here, and are available from the authors on request. Some values of $f_4(n)$, $g_4(n)$, $g'_4(n)$, $h_4(n)$, $h'_4(n)$, $r_4(n)$, $s_4(n)$ are listed in Table 8. These numbers grow exponentially, and do not have simple integer factorizations.

Tab. 8: The first few values of $f_4(n)$, $g_4(n)$, $g'_4(n)$, $h_4(n)$, $h'_4(n)$, $r_4(n)$, $s_4(n)$.

n	1	2
$f_4(n)$	778,626,762,895,872	1,024,406,418,765,003,907,906,096,145,114,250,200,136,082,865,744,856,739,402,552,777,712,697,606,144
$g_4(n)$	88,489,486,528	3,197,766,124,028,071,576,597,031,293,816,293,624,011,891,902,039,451,422,566,691,192,700,928
$g'_4(n)$	52,683,007,497,792	69,863,645,008,967,428,965,504,302,095,638,435,727,081,373,061,446,549,243,263,724,682,869,735,424
$h_4(n)$	15,626,116,736	482,200,982,250,980,661,780,613,757,386,400,225,524,299,614,671,798,975,981,818,835,107,840
$h'_4(n)$	3,629,303,504,832	4,765,691,494,696,738,414,627,738,223,389,422,884,987,040,021,355,690,863,179,839,765,575,892,992
$r_4(n)$	258,767,297,696	325,229,810,040,355,155,302,761,176,752,191,820,409,762,202,792,611,492,002,077,131,210,227,712
$s_4(n)$	94,459,269,024	110,974,534,976,153,854,286,043,758,382,592,092,762,465,813,295,695,669,459,295,951,908,765,696

It is even more difficult to derive the bounds of the asymptotic growth constant for the number of connected spanning subgraphs on $SG_4(n)$. We calculate $f_4(m)$ up to $m = 6$, and we are satisfied with numerical fitting of the asymptotic growth constant to have the following proposition.

Proposition 5.2 *The asymptotic growth constant for the number of connected spanning subgraphs on the four-dimensional Sierpinski gasket $SG_4(n)$ in the large n limit exists with value $z_{SG_4} = 2.7686025...$.*

Tab. 9: Numerical values of $\underline{z}_{SG_{d,b}}$, $\tilde{z}_{SG_{d,b}}$, $z_{SG_{d,b}}$, and their ratios. The last digits given are rounded off.

d	b	D	$\underline{z}_{SG_{d,b}}$	$\tilde{z}_{SG_{d,b}}$	$z_{SG_{d,b}}$	$\underline{z}_{SG_{d,b}}/z_{SG_{d,b}}$	$\tilde{z}_{SG_{d,b}}/z_{SG_{d,b}}$
2	2	1.585	1.048594857	1.247337199	1.276495931	0.8214635326	0.9771572077
2	3	1.631	1.133231895	1.312357559	1.397278968	0.8110276625	0.9392237265
2	4	1.661	1.194401490	1.360516466	1.484911260	0.8043588344	0.9162274555
3	2	2	1.569396409	1.666806281	2.063705	0.7604751	0.8076765
4	2	2.322	1.914853265	1.981017076	2.768603	0.6916317	0.7155296

6 Discussion

Compared with the asymptotic growth constant for the number of spanning forests N_{SF} on the Sierpinski gasket, defined as

$$\tilde{z}_G = \lim_{v(G)\to\infty} \frac{\ln N_{SF}(G)}{v(G)} , \tag{61}$$

in Ref. [Chang and Chen(2008)], we find that N_{CSSG} is larger than N_{SF} for all the considered cases. We conjecture that this inequality holds for all the generalized Sierpinski gasket. We list the first few values of $\tilde{z}_{SG_{d,b}}$, $z_{SG_{d,b}}$, and their ratios in Table 9.

As the spanning tree is a special case of connected spanning subgraphs where there is no cycles allowed, the number of spanning trees $N_{ST}(G)$ is always less than $N_{CSSG}(G)$. Define

$$\underline{z}_G = \lim_{v(G)\to\infty} \frac{\ln N_{ST}(G)}{v(G)} , \tag{62}$$

then $\underline{z}_G < z_G$. We have obtained such asymptotic growth constants exactly for the number of spanning trees on the Sierpinski gasket SG_d for general d and $SG_{2,b}$ with $b = 3, 4$ in Ref. [Chang et al.(2007)]. They serve as the lower bounds for our current consideration for the connected spanning subgraphs. We list the first few values of $\underline{z}_{SG_{d,b}}$, $z_{SG_{d,b}}$, and their ratios in Table 9. Notice that lower bound $\underline{z}_{SG_{d,b}}$ is closer to the exact value $z_{SG_{d,b}}$ when d or b is small, in contrast to the results for the spanning forests given in [Chang and Chen(2008)] that $\underline{z}_{SG_{d,b}}$ is relatively closer to $\tilde{z}_{SG_{d,b}}$ when d or b is large.

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A Recursion relations for $SG_{2,4}(n)$

We give the recursion relations for the generalized two-dimensional Sierpinski gasket $SG_{2,4}(n)$ here. Since the subscript is $(d, b) = (2, 4)$ for all the quantities throughout this section, we will use the simplified notation f_{n+1} to denote $f_{2,4}(n+1)$ and similar notations for other quantities. For any non-negative integer n, we have

$$\begin{aligned} f_{n+1} &= f_n^{10} + 27f_n^9 g_n + 7f_n^9 h_n + 309f_n^8 g_n^2 + 144f_n^8 g_n h_n + 1928f_n^7 g_n^3 + 15f_n^8 h_n^2 + 1140f_n^7 g_n^2 h_n \\ &\quad + 6936f_n^6 g_n^4 + 174f_n^7 g_n h_n^2 + 4140f_n^6 g_n^3 h_n + 13626f_n^5 g_n^5 + 2f_n^7 h_n^3 + 516f_n^6 g_n^2 h_n^2 \\ &\quad + 5856f_n^5 g_n^4 h_n + 11354f_n^4 g_n^6 \,, \end{aligned} \tag{63}$$

$$\begin{aligned} g_{n+1} &= f_n^9 g_n + f_n^9 h_n + 28f_n^8 g_n^2 + 34f_n^8 g_n h_n + 332f_n^7 g_n^3 + 6f_n^8 h_n^2 + 462f_n^7 g_n^2 h_n + 2148f_n^6 g_n^4 \\ &\quad + 142f_n^7 g_n h_n^2 + 3130f_n^6 g_n^3 h_n + 8015f_n^5 g_n^5 + 12f_n^7 h_n^3 + 1095f_n^6 g_n^2 h_n^2 + 10609f_n^5 g_n^4 h_n \\ &\quad + 16250f_n^4 g_n^6 + 82f_n^6 g_n h_n^3 + 2786f_n^5 g_n^3 h_n^2 + 14480f_n^4 g_n^5 h_n + 13732f_n^3 g_n^7 \,, \end{aligned} \tag{64}$$

$$\begin{aligned} h_{n+1} &= 3f_n^8 g_n^2 + 6f_n^8 g_n h_n + 87f_n^7 g_n^3 + 3f_n^8 h_n^2 + 189f_n^7 g_n^2 h_n + 1068f_n^6 g_n^4 + 117f_n^7 g_n h_n^2 \\ &\quad + 2558f_n^6 g_n^3 h_n + 7113f_n^5 g_n^5 + 15f_n^7 h_n^3 + 1869f_n^6 g_n^2 h_n^2 + 17763f_n^5 g_n^4 h_n + 26934f_n^4 g_n^6 \\ &\quad + 444f_n^6 g_n h_n^3 + 12756f_n^5 g_n^3 h_n^2 + 61422f_n^4 g_n^5 h_n + 53826f_n^3 g_n^7 + 20f_n^6 h_n^4 + 2388f_n^5 g_n^2 h_n^3 \\ &\quad + 30948f_n^4 g_n^4 h_n^2 + 83234f_n^3 g_n^6 h_n + 42210f_n^2 g_n^8 \,. \end{aligned} \tag{65}$$

B Recursion relations for $SG_3(n)$

We give the recursion relations for the three-dimensional Sierpinski gasket $SG_3(n)$ here. Since the subscript is $d = 3$ for all the quantities throughout this section, we will use the simplified notation f_{n+1} to denote $f_3(n + 1)$ and similar notations for other quantities. For any non-negative integer n, we have

$$\begin{aligned} f_{n+1} &= f_n^4 + 12f_n^3 g_n + 12f_n^3 h_n + 12f_n^3 r_n + 48f_n^2 g_n^2 + 96f_n^2 g_n h_n + 48f_n^2 h_n^2 + 72f_n^2 g_n r_n \\ &\quad + 72f_n^2 h_n r_n + 56f_n g_n^3 + 168f_n g_n^2 h_n + 168f_n g_n h_n^2 + 56f_n h_n^3 \,, \end{aligned} \tag{66}$$

$$\begin{aligned} g_{n+1} &= f_n^3 g_n + 3f_n^3 r_n + 9f_n^2 g_n^2 + 12f_n^2 g_n h_n + f_n^3 s_n + 36f_n^2 g_n r_n + 30f_n^2 h_n r_n + 28f_n g_n^3 \\ &\quad + 66f_n g_n^2 h_n + 54f_n g_n h_n^2 + 6f_n^2 g_n s_n + 6f_n^2 h_n s_n + 24f_n^2 r_n^2 + 108f_n g_n^2 r_n \\ &\quad + 192f_n g_n h_n r_n + 84f_n h_n^2 r_n + 20g_n^4 + 72g_n^3 h_n + 96g_n^2 h_n^2 + 56g_n h_n^3 \,, \end{aligned} \tag{67}$$

$$h_{n+1} = 2f_n^2 h_n^2 + 4f_n^2 h_n r_n + 12f_n g_n^2 h_n + 12f_n g_n h_n^2 + 16f_n h_n^3 + 2f_n^2 r_n^2 + 12f_n g_n^2 r_n$$

$$+48f_ng_nh_nr_n+36f_nh_n^2r_n+2g_n^4+16g_n^3h_n+36g_n^2h_n^2+32g_nh_n^3+22h_n^4\,, \qquad (68)$$

$$\begin{aligned} r_{n+1} &= f_n^2g_n^2+2f_n^2h_n^2+6f_n^2g_nr_n+6f_n^2h_nr_n+6f_ng_n^3+22f_ng_n^2h_n+14f_ng_nh_n^2+16f_nh_n^3 \\ &\quad +2f_n^2g_ns_n+2f_n^2h_ns_n+12f_n^2r_n^2+60f_ng_n^2r_n+132f_ng_nh_nr_n+66f_nh_n^2r_n+12g_n^4 \\ &\quad +52g_n^3h_n+78g_n^2h_n^2+48g_nh_n^3+22h_n^4+6f_n^2r_ns_n+14f_ng_n^2s_n+28f_ng_nh_ns_n \\ &\quad +14f_nh_n^2s_n+120f_ng_nr_n^2+120f_nh_nr_n^2+88g_n^3r_n+264g_n^2h_nr_n+264g_nh_n^2r_n \\ &\quad +88h_n^3r_n\,, \end{aligned} \qquad (69)$$

$$\begin{aligned} s_{n+1} &= 4f_ng_n^3+36f_ng_n^2r_n+24f_ng_nh_nr_n+12g_n^4+24g_n^3h_n+12f_ng_n^2s_n+24f_ng_nh_ns_n \\ &\quad +12f_nh_n^2s_n+144f_ng_nr_n^2+120f_nh_nr_n^2+144g_n^3r_n+360g_n^2h_nr_n+216g_nh_n^2r_n \\ &\quad +144f_ng_nr_ns_n+144f_nh_nr_ns_n+56g_n^3s_n+168g_n^2h_ns_n+168g_nh_n^2s_n+56h_n^3s_n \\ &\quad +208f_nr_n^3+720g_n^2r_n^2+1440g_nh_nr_n^2+720h_n^2r_n^2\,. \end{aligned} \qquad (70)$$

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