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Refinements of the Littlewood-Richardson rule

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Abstract.

We refine the classical Littlewood-Richardson rule in several different settings. We begin with a combinatorial rule for the product of a Demazure atom and a Schur function. Building on this, we also describe the product of a quasisymmetric Schur function and a Schur function as a positive sum of quasisymmetric Schur functions. Finally, we provide a combinatorial formula for the product of a Demazure character and a Schur function as a positive sum of Demazure characters. This last rule implies the classical Littlewood-Richardson rule for the multiplication of two Schur functions.

Résumé.

Nous décrivons trois nouvelles règles de Littlewood-Richardson, et chaque nouvelle règle partage la vielle règle de Littlewood-Richardson. La première règle multiplie un atome de Demazure et un fonction de Schur. Le deuxième multiplie un fonction de quasisymmetric-Schur et un fonction de Schur. Le troisième multiplie un caractère de Demazure et un fonction de Schur. Cette dernière règle est une description de la vielle règle de Littlewood-Richardson.

Keywords: key polynomials, nonsymmetric Macdonald polynomials, Littlewood-Richardson rule, quasisymmetric functions, Schur functions

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1 Introduction

In (5), Haglund, Haiman, and Loehr obtained a new combinatorial formula for the type A nonsymmetric Macdonald polynomial $E_\alpha(x_1,\ldots,x_n;q,t)$ first introduced by Macdonald (8), where α is a (weak) composition into n nonnegative parts. This formula involves *inversion triples*, a combinatorial construct introduced in the study of symmetric Macdonald polynomials (3), (4). By letting $q=t=0$, we obtain a new combinatorial formula for the $E_\alpha(X;0;0)$, which are known (13) to be certain B-module characters studied by Demazure, now commonly referred to as Demazure characters. Marshall (9) has shown that many of the nice analytic properties of type A symmetric Macdonald polynomials, such as their occurrence in a generalization of Selberg's Integral, are shared by type A nonsymmetric Macdonald polynomials as well. In his work he used a modified version obtained from the E_α by replacing q,t by $1/q,1/t$, reversing the order of the x_i-variables, and reversing the order of the parts of α. The combinatorics of the case $q=t=0$ of these polynomials, i.e. $E_{\alpha_n,\ldots,\alpha_1}(x_n,\ldots,x_1;\infty,\infty)$, was investigated in (10), (11), including a direct combinatorial proof that they are in fact the same as polynomials introduced by Lascoux and Schützenberger (7) in connection with the study of Schubert polynomials, which they called standard bases, and which equal the characters of quotients of Demazure modules. The Demazure characters are sometimes called key polynomials (12) and in prior work as well as in this article the standard bases of Lascoux and Schützenberger are referred to as Demazure atoms. It is known that the Demazure character is a positive sum of Demazure atoms, and that the Schubert polynomial is a positive sum of Demazure characters.

Schur functions are special cases of both Demazure characters and Schubert polynomials, and the decomposition of a Schur function as a positive sum of Demazure atoms was proved directly in (10) using an extension of the RSK algorithm. It is well-known (2) that the Schur function $s_\lambda(x_1,\ldots,x_n)$ is a sum of Gessel's fundamental quasisymmetric functions, one for each standard Young tableau of shape λ. It is natural to investigate how this decomposition correlates with the expansion of s_λ into Demazure atoms. In the predecessor to this article (6), the authors introduced a new basis for the ring of quasisymmetric functions called "quasisymmetric Schur functions" denoted $\mathcal{S}_\beta(x_1,\ldots,x_n)$, where β is a (strong) composition. They defined $\mathcal{S}_\beta$ as a sum of certain Demazure atoms, and it follows immediately from the results in (10) and the decomposition of Schur functions into atoms that $s_\lambda=\sum_\beta \mathcal{S}_\beta$, where the sum is over all multiset permutations β of the parts of λ. Note that if the Ferrers shape of λ is a rectangle, then the sum has only one term and $s_\lambda=\mathcal{S}_\lambda$. The authors showed that if you multiply a quasisymmetrc Schur function by an elementary symmetric function $e_k(=s_{1^k})$ or a complete homogeneous symmetric function $h_k(=s_k)$ then this result can be expressed in a simple combinatorial way as a positive sum of quasisymmetric Schur functions. From this rule the classical Pieri rule for multiplying a Schur function by an e_k or h_k can be easily derived.

In this article we generalize the result by showing that the product of a Schur function with a quasisymmetric Schur function (respectively Demazure character, Demazure atom) expands positively into quasisymmetric Schur functions (respectively Demazure characters, atoms), and we give a simple combinatorial rule for the coefficients in this expansion. The description of the rule contains many elements in common with the classical Littlewood-Richardson (LR) rule for the multiplication of two Schur functions, and the authors' proof is essentially a refinement of the proof of the LR-rule in Fulton's book on Young Tableaux (1) involving the combinatorial constructs (such as inversion triples) occurring in the new combinatorial formulas for quasisymmetric Schur functions, Demazure characters, and Demazure atoms. One can obtain the classical LR-rule from the rule for Demazure atoms by careful bookkeeping combined with the decomposition of Schur functions into atoms.

1.1 Sequences and words

A *strong (resp. weak) composition* is a finite sequence of positive (resp. nonnegative) integers. A *partition* is a multiset of nonnegative integers, which we usually present as a weakly decreasing sequence. By $\widetilde{\gamma}$ we denote the underlying partition of γ, and by γ^+ the strong composition underlying γ i.e. γ with its zero parts removed. When necessary, any of these may be considered to be an infinite sequence of integers in which all but a finite number of entries is zero. By γ^* we denote the sequence that contains the parts of γ in reverse order.

For any finite sequence $\alpha=(\alpha_1,\ldots,\alpha_r)$ we denote $\ell(\alpha):=r$ and $|\alpha|:=\sum_i \alpha_i$. Given two (possibly weak) compositions $\alpha=(\alpha_1,\ldots,\alpha_r)$ and $\beta=(\beta_1,\ldots,\beta_r)$ of the same length, we say that α is *contained* in β, denoted $\alpha\subseteq\beta$, if $\alpha_i\leq\beta_i$ for all $1\leq i\leq r$.

A finite sequence w of positive integers is called a *lattice word* if for every positive integer i, every prefix of w contains at least as many i's as $(i+1)$'s. A finite sequence w of positive integers is called a *reverse lattice word* (or *Yamanouchi word*) if for every positive integer i, every suffix of w contains at least as many i's as $(i+1)$'s. The *weight* of a word w is the sequence $\lambda=(\lambda_1,\ldots,\lambda_r)$ such that for every $1\leq i\leq r$, w contains exactly λ_i elements of value i. Note that if w is a lattice or reverse lattice word, then its weight will be a partition. Define the function $\phi_n : \mathbb{Z} \to \mathbb{Z}$ by $\phi_n(k) = n + 1 - k$. Let $w = (w_1, \ldots, w_t)$ be a sequence of integers with largest element r. Define $\Phi(w) = (\phi_r(w_1), \ldots, \phi_r(w_t))$. Then we will say that a word w is a *contre-lattice* word if $\Phi(w)$ is a lattice word and the weight of $\Phi(w)$ is a partition of length r, i.e. the weight of w is λ^* for some partition λ. Similarly, we will say that w is a *reverse contre-lattice* (or *contre-Yamanouchi*) word if w^* is a contre-lattice word.

1.2 Diagrams and tableaux

Given any partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, we have a *partition diagram*, denoted $dg(\lambda)$, consisting of the usual left-justified arrangement of rows of cells (sometimes called squares or boxes), one row for each part of λ, the part λ_i giving the number of cells in row i. We use the English convention for our diagrams, so that the longest row (of length λ_1) at the top of the diagram. We index the cells of a diagram by $(row, column)$ pairs of positive integers.

A *semistandard Young tableau* (SSYT) is a partition diagram filled in such a way that the entries within each row increase weakly left-to-right and the entries within each column increase strictly top-to-bottom. A *standard* Young tableau (SYT) is a SSYT in which the set of entries is exactly $[n] = \{1, \ldots, n\}$, where the diagram has n cells altogether. We use the word *tableau* without modifiers to refer to an SSYT unless otherwise indicated. Given partitions $\mu \subseteq \lambda$, the diagram of *skew shape* λ/μ consists of those cells of $dg(\lambda)$ that are not in $dg(\mu)$. Skew tableaux, both standard and semistandard, are defined analogously in the obvious way.

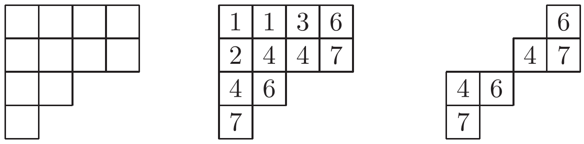


diagram *tableau* *skew tableau on* $(4, 4, 2, 1)/(3, 2)$

Fig. 1: Diagram and tableau examples

The *content* of a tableau T is the weak composition γ for which γ_i is the number of entries of T with value i, for all positive i. For example, in Figure 1, the first tableau has content $(2, 1, 1, 3, 0, 2, 2)$. Given a set of variables $X = \{x_1, x_2, x_3, \ldots\}$ indexed by positive integers, the *(monomial) weight* of T, denoted $\mathbf{x}^T$, is the monomial for which the exponent of x_i is γ_i, the number of entries of T with value i. For the aforementioned example, the monomial weight is $x_1^2 x_2 x_3 x_4^3 x_6^2 x_7^2$.

The *row reading order* is a total ordering of the cells of a (possibly skew) diagram where $(i, j) <_{row} (i', j')$ if either $i > i'$ or ($i = i'$ and $j < j'$). That is, the row reading order reads the cells from left-to-right in each row, starting with the bottommost row and proceeding upwards to the top row. The *row reading word* of a tableau T, denoted $w_{row}(T)$ is the sequence of integers formed by the entries of T taken in row reading order. A *Littlewood-Richardson skew tableau* is a skew tableau whose row reading word is a reverse lattice word.

We will also make use of a slightly different reading order on diagrams, which we will refer to as the *column reading order*. In the column reading order, we have $(i, j) <_{col} (i', j')$ if either $j > j'$ or ($j = j'$ and $i < i'$). That is, the column reading order reads the cells from top-to-bottom within each column, starting with the rightmost column and working leftwards. The *column reading word* of a tableau T, denoted $w_{col}(T)$, is the sequence of integers formed by the entries of T taken in column reading order.

1.3 The Littlewood-Richardson rule

The graded algebra of symmetric functions

$$\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots \subseteq \mathbb{Z}[[x_1, x_2, x_3, \ldots]]$$

has each graded piece given by

$$\Lambda_n = \text{ span } \{s_\lambda | \lambda \text{ is a partition of } n\}$$

where $\Lambda_0 = \{1\}$. The *Schur function* s_λ can in turn be defined as

$$s_\lambda = \sum_{\substack{T \in SSYT \\ shape(T)=\lambda}} \mathbf{x}^T.$$

The product of two Schur functions expands as a sum of Schur functions whose coefficients are nonnegative. The coefficients are called Littlewood-Richardson coefficients and also arise in representation theory and algebraic geometry. The rule for the computing the coefficients is called the Littlewood-Richardson rule, and can be described as follows.

Theorem 1 (Littlewood-Richardson rule) *Let λ, μ, ν be partitions. In the expansion*

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu,$$

the Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ is the number of Littlewood-Richardson skew tableaux of shape ν/λ with content μ.

1.4 Skyline and composition diagrams

Earlier papers (5; 10; 11) introduce *column diagrams*, or *skyline diagrams*, similar to partition diagrams, whose shapes are indexed by weak compositions. The parts of the composition specify the number of cells in the respective *columns* of the diagram. These diagrams are usually augmented by a *basement*, an extra row on the bottom (row 0) whose entries contain positive integers. For example, for the augmented diagram indexed by the weak composition $(2, 0, 3, 1, 2)$ with increasing basement (left-to-right) is shown in the leftmost diagram of Figure 2.

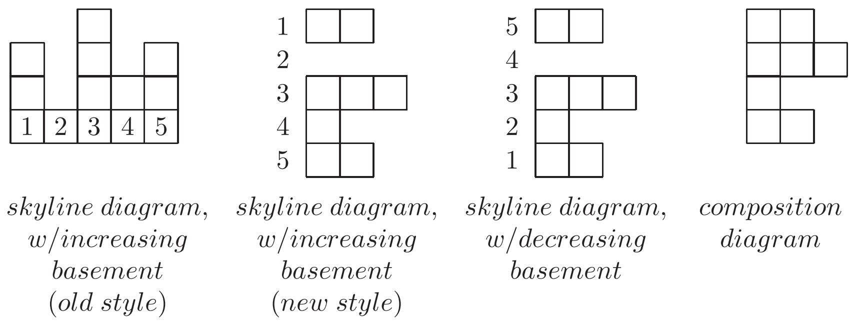
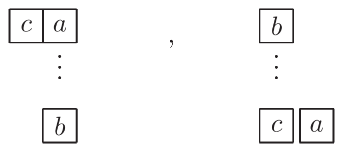


Fig. 2: Skyline and composition diagram examples

However, for consistency with other diagrams, including tableaux, in this paper we will draw our skyline diagrams in sideways fashion, the columns then becoming rows. As with partition diagrams, we number the rows in the English style with row 1 at the top. We will also introduce the related notion of *composition diagrams*, which are indexed by strong compositions, and for which we typically do not indicate a basement. In Figure 2, the skyline diagrams are indexed by the weak composition $(2, 0, 3, 1, 2)$, while the composition diagram is indexed by the composition $(2, 3, 1, 2)$.

Just as we fill partition diagrams according to certain restrictions to obtain tableaux, we define fillings for composition diagrams subject to certain rules. To state the rules, we first define the notion of a *triple* of cells, of which there are two types.



Type A $\gamma_i \geq \gamma_j$ Type B $\gamma_i < \gamma_j$

A *type A triple* of a diagram of shape γ is a set of three cells a, b, c of the form $(i, k), (j, k), (i, k-1)$ for some pair of rows $i < j$ of the diagram and some column $k > 0$, where row i is at least as long as row j, i.e. $\gamma_i \geq \gamma_j$. A *type B triple* is a set of three cells a, b, c of the form $(j, k+1), (i, k), (j, k)$ for some pair of rows $i < j$ of the diagram and some column $k \geq 0$, where row i is strictly shorter than row j, i.e. $\gamma_i < \gamma_j$. Note that basement cells can be elements of triples. We say that a triple of either type is an *inversion* triple if the relative order of the entries is either $b < a \leq c$ or $a \leq c < b$.

We say that a skyline diagram filling is *semistandard* if

(i) each row is weakly decreasing left-to-right (including the basement), and

(ii) all triples (including triples with cells in the basement) are inversion triples.

We refer to such a filled skyline diagram as a *semistandard augmented filling* (SSAF), or simply as a *skyline*.

We say that a composition diagram filling is *semistandard* if

(i) the first column is strictly increasing, top-to-bottom,

(ii) each row is weakly decreasing left-to-right, and

(iii) all triples are inversion triples.

We refer to such a filled composition diagram as a *semistandard composition tableau* (SSCT), or simply as a *composition tableau*.

1.5 Combinatorial formulas for formal power series

We have already mentioned the well-known combinatorial formula for Schur functions, which restricts to *Schur polynomials* over the variables $X = \{x_1, \ldots, x_n\}$:

$$s_\lambda(X) = \sum_{\substack{T \in SSYT(n),\\ shape(T)=\lambda}} \mathbf{x}^T, \tag{1.1}$$

where $SSYT(n)$ is the set of all semistandard tableaux with entries in $[n]$. The formula for the Schur function over the infinite variable set $X = \{x_1, x_2, x_3, \ldots\}$ is the same except that the tableau entries are taken over $\mathbb{Z}_{>0}$ instead of $[n]$. Earlier works (10; 11) have similarly provided combinatorial formulas for *Demazure atoms* and *Demazure characters*, respectively, in terms of skylines:

$$\mathcal{A}_\gamma(X) = \sum_{\substack{Y \in SSAFI(n),\\ shape(Y)=\gamma}} \mathbf{x}^Y, \tag{1.2}$$

$$\kappa_\gamma(X) = \sum_{\substack{Y \in SSAFD(n),\\ shape(Y)=\gamma^*}} \mathbf{x}^Y, \tag{1.3}$$

where $SSAFI(n)$ is the set of all semistandard augmented fillings with increasing basement with entries in $[n]$, and $SSAFD(n)$ is the set of all semistandard augmented fillings with decreasing basement with entries in $[n]$. The analogous formula for *quasisymmetric Schur functions* over the variable set $X = \{x_1, x_2, x_3, \ldots\}$ is given by

$$\mathcal{S}_\alpha = \sum_{\substack{T \in SSCT,\\ shape(T)=\alpha}} \mathbf{x}^T, \tag{1.4}$$

where $SSCT$ is the set of all semistandard composition tableaux. We are now ready to provide Littlewood-Richardson rules for these latter three formal power series.

2 Littlewood-Richardson rule for Demazure atoms

Given a weak composition γ with $\ell(\gamma) = n$, we say that an *extended basement* of shape γ is a skyline in which the basement entries (in the 0-th column) are distinct integers, each of which is strictly greater than n, and the entries in each row of the diagram are equal to the column 0 entry for the respective row. The *standard decreasing (respectively increasing) extended basement* of shape γ is the extended basement of shape γ skyline in which the basement entries (in the 0-th column) are decreasing (respectively increasing) , beginning with $2n$ in the first row and ending with $n+1$ in the last row (respectively $n+1$ to $2n$).

Given weak compositions $\gamma \subset \delta$, where we assume $\ell(\gamma) = \ell(\delta)$, the *skew skyline diagram* (or simply *skew diagram*) of shape δ/γ is the set of those cells of the skyline diagram of shape δ that are not in the skyline diagram of shape γ. For this section, we consider the skew diagram to be "resting on" a standard decreasing extended basement of shape γ. Finally, a *Littlewood-Richardson skew skyline* (LRS) of shape δ/γ will be a filling $\sigma : S \to [n]$ of the empty cells of a skew diagram S resting on a standard decreasing extended basement of shape γ, where $n = \ell(\delta)$, that satisfies the rules for skyline diagram fillings and for which the column reading word (excluding extended basement entries) is a contre-lattice word. Figure 3 shows an example of a standard decreasing extended basement of shape $\gamma = (2, 0, 3, 1, 2)$, a skew diagram of shape δ/γ where $\delta = (3, 1, 4, 2, 5)$, and an LRS of the same skew shape with column reading word 3231321, which is contre-lattice of weight $(3, 2, 2)^*$. We can now state our Littlewood-Richardson rule for the product of Schur polynomials and Demazure atoms.

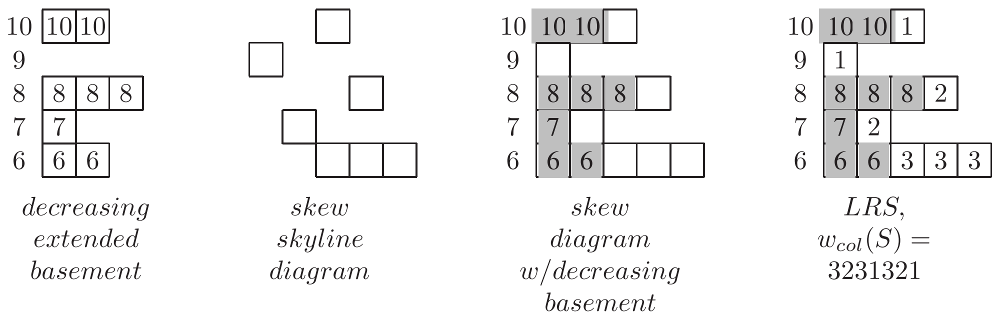


Fig. 3: Skyline extended basement and LRS

Theorem 2 *Let λ be a partition and γ, δ be weak compositions. In the expansion*

$$\mathcal{A}_\gamma \cdot s_\lambda = \sum_\delta a^\delta_{\gamma\lambda} \mathcal{A}_\delta, \tag{2.1}$$

the coefficient $a^\delta_{\gamma\lambda}$ is the number of LRS of shape δ/γ with content λ^*.

3 Littlewood-Richardson rule for quasisymmetric Schur functions

The method of proof for Theorem 2 leads easily to a corresponding Littlewood-Richardson rule for the product of a Schur function and a quasisymmetric Schur function. To state it we define the analogue of an LRS for composition diagrams.

Given a strong composition β and a weak composition γ such that $\ell(\gamma) = \ell(\beta)$ and $\gamma \subset \beta$, we say that the *skew composition diagram of shape β/γ* is the set of cells of the composition diagram of β that are not in the diagram of γ. We naturally associate to this skew composition diagram a skew skyline diagram with decreasing extended basement, where all of the of the entries in the basement are larger than the number of empty cells in the skew shape β/γ. Now to every LRS filling of this skew skyline diagram we associate the corresponding *Littlewood-Richardson skew composition tableau* (LRC), which is simply the filling of the cells of the skew composition diagram with the corresponding (nonbasement) entries from the LRS. An example is given in Figure 4. We can now state our Littlewood-Richardson rule for the product of Schur functions and quasisymmetric Schur functions.

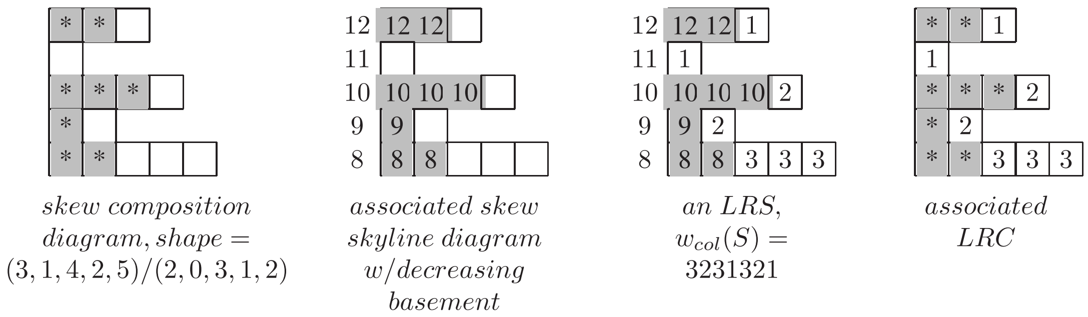


Fig. 4: Composition skew diagram and LRC

Theorem 3 *Let λ be a partition and α, β be strong compositions. In the expansion*

$$\mathcal{S}_\alpha \cdot s_\lambda = \sum_\beta C^\beta_{\alpha\lambda} \mathcal{S}_\beta, \tag{3.1}$$

the coefficient $C^\beta_{\alpha\lambda}$ is the number of LRC of shape β/γ with content λ^* and $\gamma^+ = \alpha$.

4 Littlewood-Richardson rule for Demazure characters

Given a standard increasing extended basement of shape α, with $\ell(\alpha) = n$, any weak composition β such that $\ell(\beta) = n$ and $\alpha \subset \beta$, we define a *Littlewood-Richardson key skyline* (LRK) of shape β/α to be a filling $\sigma : S \to [n]$ of the empty cells of a skew diagram S that satisfies the skyline diagram filling rules and for which the column reading word (excluding extended basement entries) is a contre-lattice word. Figure 5 provides an example of a standard increasing extended basement of shape $\alpha = (2, 0, 1, 2, 3)$, a skew diagram of shape β/α where $\beta = (5, 1, 3, 2, 4)$, and an LRK of the same shape with column reading word 3323121, which is contre-lattice of weight $(3, 2, 2)^*$. We are now ready to state our Littlewood-Richardson rule for the product of Schur polynomials and Demazure characters.

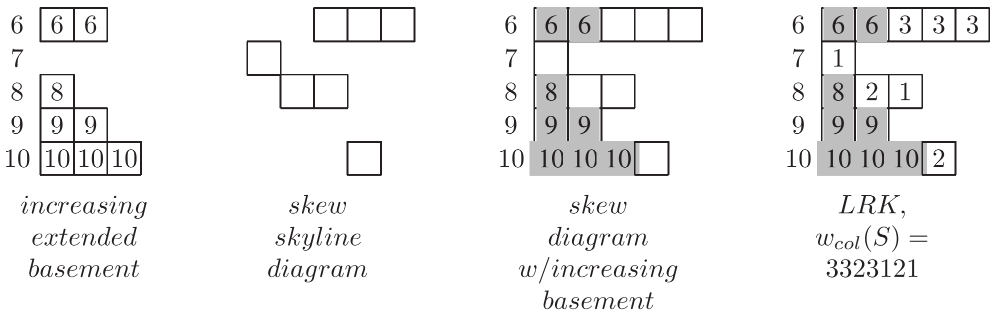


Fig. 5: Skyline extended basement and LRK

Theorem 4 *Let λ be a partition and γ, δ be weak compositions. In the expansion*

$$\kappa_\gamma \cdot s_\lambda = \sum_\delta b^\delta_{\gamma\lambda} \kappa_\delta, \tag{4.1}$$

the coefficient $b^\delta_{\gamma\lambda}$ is the number of LRK of shape δ^*/γ^* with content λ^*.

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