Asymptotics of Decomposable Combinatorial Structures of Alg-Log Type With Positive Log Exponent

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We consider the multiset construction of decomposable structures with component generating function $C(z)$ of alg-log type, *i.e.*,

$$C(z) = (1-z)^{-\alpha}\left(\log\frac{1}{1-z}\right)^{\beta}.$$

We provide asymptotic results for the number of labeled objects of size n in the case when α is positive and β is positive and in the case $\alpha = 0$ and $\beta \geq 2$. The case $0 < -\alpha < 1$ and any β and the case $\alpha > 0$ and $\beta = 0$ have been treated in previous papers. Our results extend previous work of Wright.

Keywords: decomposable structures, generating functions, alg-log type, analytic combinatorics, saddle point method

1 Introduction

Certain combinatorial structures can be decomposed into smaller and simpler objects called *components*. For example, permutations decompose into cycles, and forests can be decomposed into trees. If a structure can be so decomposed, it is called a *decomposable combinatorial structure.* We focus on the multiset construction [8]. In the case of labeled objects, we are interested in the coefficients of the generating $F(z) = \exp(C(z))$, where $C(z)$ is the generating function for components.

A component generating function $C(z)$ is of alg-log type if, near the singularity ρ, $C(z)$ behaves like

$$c + d(1-z/\rho)^{-\alpha}\left(\log\frac{1}{1-z/\rho}\right)^{\beta}(1+o(1)).$$

The exponents α and β are called the *algebraic exponent* and the *logarithmic exponent*, respectively. This type of component generating function belongs to the class of alg-log component generating functions, which was introduced by Flajolet and Soria [10] in their study of the number of components in combinatorial structures.

In the particular case $\alpha = 0$ and $\beta = 1$, the components are said to be of *logarithmic type*, and the combinatorial structures are said to be in the *exp-log* class. The exp-log class was also introduced by

Flajolet and Soria [9] who studied the behaviour of the number of components for structures of this class. There has also been much work done on this class with regard to both the rth largest component (Gourdon [11]) and the rth smallest component (Panario and Richmond [14]). Finally, Dong, Gao and Panario [3, 4] studied objects in the exp-log class with a restricted pattern combined with information on the smallest components' sizes.

For the exp-alglog class, Dong, Gao, Panario, and Richmond [5] considered the cases $0 < -\alpha < 1$ with any β, and $\alpha > 0$ and $\beta = 0$. Their asymptotic analysis used two different approaches: the first case was done with singularity analysis, and the second was done using the saddle point method. They proved results about the number of objects with restricted patterns in both the labeled and unlabeled cases and results regarding their respective probability distributions.

Both the exp-alglog class and the exp-log class are extensions of the problem of finding the coefficients of the power series expansion of $\exp(a/(1-z))$, which was initially considered by Wright [15]. Wright also investigated generalizations of his initial problem [16], and those results provide further motivation for our work.

In this paper we continue the study initiated in [5] and assume that the component generating function $C(z)$ is defined by

$$C(z) = (1-z)^{-\alpha} \left(\log \frac{1}{1-z} \right)^{\beta}.$$

In Section 2 we use the saddle point method to find an asymptotic result for the number of labeled objects of size n whose generating function is $\exp(C(z))$ with $\alpha > 0$ and $\beta > 0$ (Theorem 1). In Section 3 we prove the analogous result for the labeled case $\alpha = 0$ and $\beta \geq 2$ (Theorem 2). This is the main result of the paper. The study of the particular important case when $\alpha = 0, \beta = 2$ was left as an open problem in [7], p. 98. This case is also related to the study of partitions into parts of size powers of 2 [1, 6]. Examples are given in Section 4. We close in Section 5 with some further directions of study.

2 Alg-log components with negative algebraic exponent and positive logarithmic exponent

Let $\mathcal{F}$ be a set of labeled combinatorial structures whose components are enumerated by the component generating function $C(z) = (1-z)^{-\alpha}(\log \frac{1}{1-z})^{\beta}$, and let $n!f_n$ be the number of structures of size n in $\mathcal{F}$. Then,

$$f_n = [z^n] \exp\left((1-z)^{-\alpha} \left(\log \frac{1}{1-z} \right)^{\beta} \right).$$

In this paper we use the saddle point method (Hayman's method [12]) to find an asymptotic expression for the Cauchy integral

$$f_n = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\exp(C(z))}{z^{n+1}} dz, \tag{1}$$

where $\mathcal{C}$ is a counterclockwise circle of radius less than one which is centred at the origin. While it may be that $\exp(C(z))$ is Hayman-admissible (as defined in [12]), we choose to evaluate the integral in (1) without using admissibility. We do so because it will allow us, in the future, to easily consider objects with restricted patterns as Dong *et al.* do in [5].

Let $r = r(n)$ be defined by

$$rC'(r) = n, 0 < r < 1, \tag{2}$$

and let $z = re^{i\theta}$, where $|\theta| \le \pi$. Equation (2) is called the *saddle point equation*. Then

$$f_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{\exp(C(z))}{z^{n+1}}dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\exp(C(re^{i\theta})}{r^n e^{in\theta}}d\theta = \frac{r^{-n}}{2\pi}\int_{-\pi}^{\pi} \exp(C(re^{i\theta}) - in\theta)d\theta. \tag{3}$$

2.1 Solution to the saddle point equation

We cannot solve (2) directly, so we use the bootstrap method [2] to find an asymptotic expression for r. From (2), we have

$$\frac{r}{(1-r)^{\alpha+1}}\left(\log\frac{1}{1-r}\right)^{\beta-1}\left(\alpha\log\frac{1}{1-r}+\beta\right) = n. \tag{4}$$

If we let $t = 1 - r$, then (4) becomes

$$\frac{n}{\alpha} = \frac{1-t}{t^{\alpha+1}}\left(\log\frac{1}{t}\right)^{\beta}\left(1-\frac{\beta}{\alpha\log t}\right),$$

which we rearrange below to find a recursive expression for t:

$$\begin{aligned} t^{\alpha+1} &= \frac{\alpha}{n}(1-t)\left(\log\frac{1}{t}\right)^{\beta}\left(1+\frac{\beta}{\alpha\log\frac{1}{t}}\right) \\ t &= \left(\frac{\alpha}{n}\right)^{\frac{1}{1+\alpha}}(1-t)^{\frac{1}{1+\alpha}}\left(\log\frac{1}{t}\right)^{\frac{\beta}{1+\alpha}}\left(1+\frac{\beta}{\alpha\log\frac{1}{t}}\right)^{\frac{1}{1+\alpha}}. \end{aligned} \tag{5}$$

It is clear from (4) that as $n \to \infty$ the value r approaches 1 from the left. Therefore, as $n \to \infty$, $t \to 0$ from the right. From this fact and from (5) we deduce that

$$t \sim \left(\frac{\alpha}{n}\right)^{\frac{1}{1+\alpha}}\left(\log\frac{1}{t}\right)^{\frac{\beta}{1+\alpha}}, \text{ and } \frac{1}{t} \sim \left(\frac{n}{\alpha}\right)^{\frac{1}{1+\alpha}}\left(\log\frac{1}{t}\right)^{\frac{-\beta}{1+\alpha}}.$$

Therefore

$$\log\frac{1}{t} \sim \frac{1}{1+\alpha}\log\frac{n}{\alpha}, \text{ and } t \sim \left(\frac{\alpha}{n}\right)^{\frac{1}{1+\alpha}}\left(\frac{1}{1+\alpha}\log\frac{n}{\alpha}\right)^{\frac{\beta}{1+\alpha}}. \tag{6}$$

We next combine (5) and (6) to determine an estimate for $\log\frac{1}{t}$ which includes an error term:

$$\begin{aligned} \log\frac{1}{t} &= \frac{1}{1+\alpha}\left(\log\frac{n}{\alpha}+\log\frac{1}{1-t}-\beta\log\log\frac{1}{t}+\log\left(\frac{1}{1+\frac{\beta}{\alpha\log\frac{1}{t}}}\right)\right) \\ &= \frac{1}{1+\alpha}\log n + O(\log\log n) = \frac{1}{1+\alpha}\log n\left(1+O\left(\frac{\log\log n}{\log n}\right)\right). \end{aligned} \tag{7}$$

Finally, we use (5), (6), and (7) to find an asymptotic estimate for t:

$$\begin{aligned}t &= \left(\frac{\alpha}{n}\right)^{\frac{1}{1+\alpha}} \left(\frac{1}{1+\alpha}\right)^{\frac{\beta}{1+\alpha}} (\log n)^{\frac{\beta}{1+\alpha}} \left(1+O\left(\frac{1}{\log n}\right)\right)^{\frac{1}{1+\alpha}} \left(1+O\left(\frac{\log\log n}{\log n}\right)\right)^{\frac{\beta}{1+\alpha}}\\ &= \left(\frac{\alpha}{n}\right)^{\frac{1}{1+\alpha}} \left(\frac{\log n}{1+\alpha}\right)^{\frac{\beta}{1+\alpha}} \left(1+O\left(\frac{\log\log n}{\log n}\right)\right).\end{aligned}$$

2.2 Angle of interest

In order to use the saddle point method we must first find an angle θ_0 such that

$$r^2\theta_0^2 C''(r) \to \infty \quad \text{and} \quad r^3\theta_0^3 C^{(3)}(r) \to 0 \tag{8}$$

as $n \to \infty$. If we can find such a θ_0, then for $|\theta| < \theta_0$ the behaviour of the Taylor series for the expression $C(re^{i\theta}) - in\theta$ is dominated by the term $C''(r)\theta_0^2$. First we determine that

$$\begin{aligned}C''(r) &= (\alpha^2+\alpha)(1-r)^{-\alpha-2}\left(\log\frac{1}{1-r}\right)^{\beta}\left(1+O\left(\left(\log\frac{1}{1-r}\right)^{-1}\right)\right)\\ &= (\alpha^2+\alpha)t^{-\alpha-2}\left(\log\frac{1}{t}\right)^{\beta}\left(1+O\left(\frac{1}{\log t}\right)\right)\\ &= (\alpha^2+\alpha)\left(\frac{n}{\alpha}\right)^{\frac{\alpha+2}{\alpha+1}}\left(\frac{1+\alpha}{\log n}\right)^{\beta\frac{\alpha+2}{\alpha+1}}(\log n)^{\beta}\left(1+O\left(\frac{1}{\log n}\right)\right) \qquad (9)\\ &\geq K n^{\frac{\alpha+2}{\alpha+1}}(\log n)^{\frac{-\beta}{1+\alpha}}, \text{ and}\\ C^{(3)}(r) &= (\alpha^3+3\alpha^2+2\alpha)t^{-\alpha-3}\left(\log\frac{1}{t}\right)^{\beta}\left(1+O\left(\frac{1}{\log t}\right)\right)\\ &= O\left(n^{\frac{\alpha+3}{\alpha+1}}(\log n)^{-\frac{2\beta}{1+\alpha}}\right),\end{aligned}$$

where K is some positive constant. Let $\theta_0 = n^k$, and note that $r = O(1)$. Then in order to satisfy (8), we may choose any constant k satisfying

$$2k + \frac{\alpha+2}{\alpha+1} > 0 \quad \text{and} \quad 3k + \frac{\alpha+3}{\alpha+1} < 0.$$

In particular, we may choose $k = -\frac{5\alpha+12}{12(\alpha+1)}$, *i.e.*, $\theta_0 = n^{-\frac{5\alpha+12}{12(\alpha+1)}}$.

2.3 Central approximation

For $|\theta| \leq \theta_0$, we have, from the Taylor series of $C(z)$ and e^z,

$$\begin{aligned}C(re^{i\theta}) &= C(r) + rC'(r)(e^{i\theta}-1) + \frac{r^2}{2}C''(r)(e^{i\theta}-1)^2 + O(r^3C'''(r)\theta^3)\\ &= C(r) + i\theta rC'(r) - \frac{r\theta^2C'(r)}{2} - \frac{r^2\theta^2C''(r)}{2} + O(r^3C'''(r)\theta^3)\\ &= C(r) + i\theta rC'(r) - \frac{b(r)}{2}\theta^2 + o(1), \qquad (10)\end{aligned}$$

where $b(r) = rC'(r) + r^2 C''(r)$. Let $\gamma = \alpha^{-\frac{1}{1+\alpha}}(1+\alpha)^{1+\frac{\beta(\alpha+2)}{\alpha+1}}$. Then from (9) we have

$$b(r) \sim \gamma n^{\frac{\alpha+2}{\alpha+1}} (\log n)^{\frac{-\beta}{1+\alpha}}.$$

We next divide the integral from (3) into two parts:

$$I_1 = \int_{-\theta_0}^{\theta_0} \exp(C(re^{i\theta}) - in\theta) d\theta$$

and

$$I_2 = \int_{|\theta| \geq \theta_0} \exp(C(re^{i\theta}) - in\theta) d\theta.$$

From (4) and (10) we have

$$I_1 = \int_{-\theta_0}^{\theta_0} \exp\left(C(r) - \frac{\theta^2}{2} b(r) + o(1)\right) d\theta.$$

Let $t = \theta\sqrt{b(r)}$. Then $dt = \sqrt{b(r)}d\theta$, and

$$I_1 = \frac{e^{C(r)}}{\sqrt{b(r)}} \int_{-\sqrt{b(r)}\theta_0}^{\sqrt{b(r)}\theta_0} \exp\left(-\frac{t^2}{2} + o(1)\right) dt,$$

which is asymptotic to a Gaussian integral, since $\sqrt{b(r)}\theta_0 \sim \sqrt{\gamma} n^{\frac{\alpha}{12(\alpha+1)}} (\log n)^{\frac{-\beta}{2(1+\alpha)}}$ as $n \to \infty$. Therefore

$$I_1 \sim e^{C(r)} \sqrt{\frac{2\pi}{b(r)}} \sim e^{C(r)} \sqrt{\frac{2\pi}{\gamma} n^{-\frac{\alpha+2}{\alpha+1}} (\log n)^{\frac{\beta}{1+\alpha}}}.$$

2.4 Tails pruning and completion

First we examine the function $1 - re^{i\theta}$. As θ ranges from $-\pi$ to π, the function $1 - re^{i\theta}$ traces out a circle of radius r centred at $z = 1$. This circle begins at the point $1 - r$ and proceeds clockwise. The function $|1 - re^{i\theta}|$ is unimodal for $-\pi \leq \theta \leq \pi$ and achieves a unique minimum of $1 - r$ at $\theta = 0$. Therefore $|1 - re^{i\theta}|^{-\alpha}$ has a unique maximum at $\theta = 0$ and a unique minimum at $\theta = \pi$. Furthermore, for $|\theta| \geq \theta_0$ we have

$$|1 - re^{i\theta}|^{-\alpha} \leq |1 - re^{i\theta_0}|^{-\alpha} = |1 - 2r\cos\theta_0 + r^2|^{-\alpha/2}.$$

We next apply the Taylor series expansion for $\cos\theta_0$ to obtain

$$|1 - re^{i\theta}|^{-\alpha} \leq (1-r)^{-\alpha} - \frac{\alpha}{2} r(1-r)^{-\alpha-2}\theta_0^2 + O((1-r)^{-\alpha-4}\theta_0^4).$$

From our choice of θ_0,

$$\frac{\alpha}{2} r(1-r)^{-\alpha-2}\theta_0^2 = \frac{\alpha}{2}\left[\left(\frac{\alpha}{n}\right)^{\frac{1}{1+\alpha}} \left(\frac{\log n}{1+\alpha}\right)^{\frac{\beta}{1+\alpha}} \left(1 + O\left(\frac{\log\log n}{\log n}\right)\right)\right]^{-\alpha-2} n^{-\frac{5\alpha+12}{6(\alpha+1)}},$$

which implies that

$$|1-re^{i\theta}|^{-\alpha} \leq (1-r)^{-\alpha} - Kn^{\frac{\alpha}{6(\alpha+1)}}(\log n)^{-\frac{\beta(\alpha+2)}{(1+\alpha)}},$$

where K is some positive constant. It follows that, for $\theta_0 \leq |\theta| \leq \pi$,

$$\begin{aligned} |C(re^{i\theta})| &\leq |1-re^{i\theta}|^{-\alpha}\left|\left(\log\frac{1}{1-re^{i\theta}}\right)^{\beta}\right| \\ &\leq \left((1-r)^{-\alpha} - Kn^{\frac{\alpha}{6(\alpha+1)}}(\log n)^{-\frac{\beta(\alpha+2)}{(1+\alpha)}}\right)\left(\log\frac{1}{1-r}\right)^{\beta} \\ &\leq C(r) - K'n^{\frac{\alpha}{6(\alpha+1)}}(\log n)^{-\frac{\beta}{(1+\alpha)}} \end{aligned}$$

for some positive constant K'. Thus

$$|\exp(C(re^{i\theta}))| \leq \exp\left(C(r) - K'n^{\frac{\alpha}{6(\alpha+1)}}(\log n)^{-\frac{\beta}{(1+\alpha)}}\right), \tag{11}$$

uniformly as $n \to \infty$ for $\theta_0 \leq |\theta| \leq \pi$, where K' is a positive constant.

From (11) we immediately obtain

$$\begin{aligned} \frac{|I_2|}{|I_1|} &= \frac{O\left(\exp\left(C(r) - \left(K'n^{\alpha/6}\log^{-\beta} n\right)^{\frac{1}{1+\alpha}}\right)\right)}{\sqrt{2\pi}\exp(C(r))/\sqrt{b(r)}} \\ &= O\left(\exp\left(-\left(K'n^{\alpha/6}\log^{-\beta} n\right)^{\frac{1}{1+\alpha}}\right)n^{\frac{\alpha+2}{2(\alpha+1)}}(\log n)^{\frac{-\beta}{2(\alpha+1)}}\right), \end{aligned}$$

which implies that

$$\lim_{n\to\infty}\frac{|I_2|}{|I_1|} = 0.$$

Therefore,

$$|I_2| = o\left(\frac{\exp(C(r))}{\sqrt{b(r)}}\right),$$

so the contribution of I_2 to (3) is negligible compared to that of I_1. We are now able to state the main theorem of this section.

Theorem 1 *Let α and β be positive numbers, and let $C(z) = (1-z)^{-\alpha}\left(\log\frac{1}{1-z}\right)^{\beta}$. Then as $n \to \infty$,*

$$[z^n]\exp(C(z)) \sim e^{C(r)}r^{-n}\sqrt{\frac{1}{2\pi\gamma}n^{-\frac{\alpha+2}{\alpha+1}}(\log n)^{\frac{\beta}{1+\alpha}}},$$

where $\gamma = \alpha^{-\frac{1}{1+\alpha}}(1+\alpha)^{1+\frac{\beta(\alpha+2)}{\alpha+1}}$, and $r \sim 1 - \left(\frac{\alpha}{n}\right)^{\frac{1}{1+\alpha}}\left(\frac{\log n}{1+\alpha}\right)^{\frac{\beta}{1+\alpha}}$. Here r is defined by the saddle point equation (4).

Unfortunately we are unable to find an asymptotic expression for either $e^{C(r)}$ or r^{-n}. This is because the error terms for $C(r)$ and r contribute infinitely to $e^{C(r)}$ and r^{-n}, respectively. A similar situation arises in the case of the Bell numbers [13]. However, we are able to determine an asymptotic expression for $\log\left(e^{C(r)}r^{-n}\right) = C(r) - n\log r$.

First, recall that $r = 1-t$, where

$$\begin{aligned} t &= \left(\frac{\alpha}{n}\right)^{\frac{1}{1+\alpha}}\left(\frac{\log n}{1+\alpha}\right)^{\frac{\beta}{1+\alpha}}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right), \qquad \text{and} \\ \log\frac{1}{t} &= \frac{\log n}{1+\alpha}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right). \end{aligned}$$

Then

$$\begin{aligned} C(r) - n\log r &= (1-r)^{-\alpha}\left(\log\frac{1}{1-r}\right)^{\beta} - n\log r = t^{-\alpha}\left(\log\frac{1}{t}\right)^{\beta} - n\log(1-t) \\ &= t^{-\alpha}\left(\log\frac{1}{t}\right)^{\beta} + nt(1+O(t)) \\ &= \left(\left(\frac{n}{\alpha}\right)^{\frac{\alpha}{1+\alpha}}\left(\frac{\log n}{1+\alpha}\right)^{\frac{\beta}{1+\alpha}} + n^{\frac{\alpha}{1+\alpha}}\alpha^{\frac{1}{1+\alpha}}\left(\frac{\log n}{1+\alpha}\right)^{\frac{\beta}{1+\alpha}}\right)\left(1+O\left(\frac{\log\log n}{\log n}\right)\right) \\ &= (1+\alpha)\left(\frac{n}{\alpha}\right)^{\frac{\alpha}{1+\alpha}}\left(\frac{\log n}{1+\alpha}\right)^{\frac{\beta}{1+\alpha}}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right). \end{aligned}$$

3 Alg-log components with $\alpha = 0$ and $\beta \geq 2$

Let $\alpha = 0$, *i.e.*, consider the case where $C(z)$ has only a logarithmic component. At first glance, it may seem like this case is redundant in light of Section 2. However, the behavior of $\exp(C(z))$ near $z = 1$ in the case $\alpha > 0$ is so dominated by the algebraic term in $C(z)$ that the case $\alpha = 0$ requires a separate analysis.

As in Section 2, let r be defined by $rC'(r) = n$. Then the saddle point equation becomes

$$\frac{r\beta\left(\log\frac{1}{1-r}\right)^{\beta-1}}{1-r} = n. \tag{12}$$

First we note that as $n \to \infty$, $r \to 1$, and let $r = 1-t$. We rewrite (12) using $x = \beta/n$:

$$t = x\left(\log\frac{1}{t}\right)^{\beta-1}(1-t) \sim x\left(\log\frac{1}{t}\right)^{\beta-1}, \tag{13}$$

$$\log\frac{1}{t} = \log\frac{1}{x} - (\beta-1)\log\log\frac{1}{t} + \log\left(\frac{1}{1-t}\right) \sim \log\frac{1}{x}. \tag{14}$$

By substituting (14) into the right hand side of (13), we find that $t \sim x\left(\log\frac{1}{x}\right)^{\beta-1}$.

In order to find an expression for t in terms of n alone, we make the following calculations:

$$\begin{aligned} t &= x\left(\log\frac{1}{t}\right)^{\beta-1}(1-t) = x\left(\log\frac{1}{t}\right)^{\beta-1}\left(1-O\left(x\left(\log\frac{1}{x}\right)^{\beta-1}\right)\right), \\ \log\frac{1}{t} &= \log\frac{1}{x} + (1-\beta)\log\log\frac{1}{t} + \log\frac{1}{1-t}, \\ \log\log\frac{1}{t} &= \log\log\frac{1}{x} + O\left(\frac{\log\log\frac{1}{x}}{\log\frac{1}{x}}\right). \end{aligned}$$

Thus

$$t = \frac{\beta}{n}(\log n)^{\beta-1}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right).$$

3.1 Angle of interest

In order to utilize the saddle point method, we must again find an angle θ_0 such that $r^2\theta_0^2C''(r) \to \infty$ and $r^3\theta_0^3C'''(r) \to 0$ as $n \to \infty$. First, we have

$$r^2C''(r) \sim \frac{\beta\left(\log\frac{1}{1-r}\right)^{\beta-1}}{(1-r)^2} \sim \frac{\beta\left(\log\frac{1}{t}\right)^{\beta-1}}{t^2} \sim \frac{n^2}{\beta(\log n)^{\beta-1}},$$

and

$$r^3C'''(r) = O\left(\frac{r^3\left(\log\frac{1}{1-r}\right)^{\beta-1}}{(1-r)^3}\right) = O\left(\frac{\left(\log\frac{1}{t}\right)^{\beta-1}}{t^3}\right) = O\left(\frac{n^3}{(\log n)^{2(\beta-1)}}\right).$$

In order to satisfy the required conditions, we choose $\theta_0 = \frac{1}{n}(\log n)^j$ such that

$$\frac{\beta-1}{2} < j < \frac{2(\beta-1)}{3}.$$

3.2 Central approximation

As in Section 2.3, we have, for $|\theta| \le \theta_0$,

$$\begin{aligned} C(re^{i\theta}) &= C(r) + rC'(r)(e^{i\theta}-1) + \frac{r^2}{2}C''(r)(e^{i\theta}-1)^2 + O(r^3C'''(r)\theta^3) \\ &= C(r) + i\theta rC'(r) - \frac{r\theta^2C'(r)}{2} - \frac{r^2\theta^2C''(r)}{2} + O(r^3C'''(r)\theta^3) \\ &= C(r) + i\theta rC'(r) - \frac{b(r)}{2}\theta^2 + o(1), \end{aligned} \tag{15}$$

where

$$b(r) = rC'(r) + r^2C''(r) \sim \frac{1}{\beta}n^2(\log n)^{1-\beta}. \tag{16}$$

From (12) and (15) we have

$$I_1 = \int_{-\theta_0}^{\theta_0} \exp\left(C(r) - \frac{\theta^2}{2} b(r) + o(1)\right) d\theta.$$

Let $t = \theta\sqrt{b(r)}$. Then $dt = \sqrt{b(r)}d\theta$, and

$$I_1 = \frac{e^{C(r)}}{\sqrt{b(r)}} \int_{-\sqrt{b(r)}\theta_0}^{\sqrt{b(r)}\theta_0} \exp\left(-t^2 + o(1)\right) dt,$$

which is asymptotic to a Gaussian integral, since $\sqrt{b(r)}\theta_0 \sim \sqrt{\frac{1}{\beta}}(\log n)^{\frac{2j-\beta+1}{2}} \to \infty$ as $n \to \infty$. Therefore

$$I_1 \sim e^{C(r)}\sqrt{\frac{2\pi}{b(r)}} \sim \frac{e^{C(r)}}{n}\sqrt{2\pi\beta(\log n)^{\beta-1}}.$$

3.3 Tails pruning and completion

For convenience we write

$$\frac{1}{1 - re^{i\theta}} = \rho e^{i\phi}, \text{ where } |\phi| \le \pi, \text{ and } \rho = \frac{1}{|1 - re^{i\theta}|}.$$

To show that the saddle point method works for all $\beta \ge 2$, we further divide the tail region into the following two subregions $\theta_0 \le |\theta| \le \theta_1$ and $\theta_1 \le |\theta| \le \pi$, where

$$\theta_1 = \frac{1}{n}(\log n)^{\beta}.$$

We also use I_2' and I_2'' to denote the contributions to the Cauchy integral from these two subregions, respectively.

Recall that for $|\theta| \le \pi$ the function $|1 - re^{i\theta}|$ is unimodal with its minimum at $\theta = 0$. For $|\theta| \ge \theta_1$, we have

$$\Re\left(\left(\log\frac{1}{1 - re^{i\theta}}\right)^{\beta}\right) \le \left|\log\frac{1}{1 - re^{i\theta}}\right|^{\beta} \le (\log\rho + \pi)^{\beta} \le \left(\log\frac{1}{|1 - re^{i\theta_1}|} + \pi\right)^{\beta},$$

where the last inequality is due to the unimodality of $|1 - re^{i\theta}|$. Because θ_1 is a very small angle, we may express $\cos(\theta_1)$ in terms of its Taylor series, *i.e.*, $\cos(\theta_1) = 1 - \frac{\theta_1^2}{2} + O(\theta_1^4)$. Therefore $\cos(\theta_1) \le 1 - \frac{\theta_1^2}{4}$, which implies that

$$\begin{aligned}
\Re\left(\left(\log\frac{1}{1 - re^{i\theta}}\right)^{\beta}\right) &\le \left(\frac{1}{2}\log\frac{1}{1 + r^2 - 2r\cos(\theta_1)} + \pi\right)^{\beta} \\
&\le \left(\frac{1}{2}\log\frac{1}{1 + r^2 - 2r(1 - \theta_1^2/4)} + \pi\right)^{\beta} \\
&\le \left(\log\frac{1}{1 - r} + \frac{1}{2}\log\left(\frac{1}{1 + \frac{r\theta_1^2}{2(1-r)^2}}\right) + \pi\right)^{\beta}.
\end{aligned}$$

By our choice of θ_1 and our expression of the value r, it is clear that $\frac{r\theta_1^2}{2t^2} \to \infty$ as $n \to \infty$. Furthermore, $\theta_1 \geq \sqrt{2}e^9(1-r)$ for sufficiently large n. Therefore

$$\begin{aligned}
\frac{1}{1+\frac{r\theta_1^2}{2(1-r)^2}} &\leq \frac{2(1-r)^2}{r\theta_1^2}, \text{ and} \\
\frac{1}{2}\log\frac{1}{1+\frac{r\theta_1^2}{2(1-r)^2}} &\leq \frac{1}{2}\log\frac{2(1-r)^2}{r\theta_1^2} \leq \frac{1}{2}\log\frac{2(1-r)^2}{2e^{18}(1-r)^2} = -9.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Re\left(\left(\log\frac{1}{1-re^{i\theta}}\right)^\beta\right) &\leq \left(\log\frac{1}{1-r} - 9 + \pi\right)^\beta \leq \left(\log\frac{1}{1-r}\right)^\beta - 5\left(\log\frac{1}{1-r}\right)^{\beta-1} \\
&\leq \left(\log\frac{1}{1-r}\right)^\beta - 4\log n,
\end{aligned}$$

where the last inequality arises from the fact that $\beta \geq 2$. This implies that

$$\frac{|I_2''|}{|I_1|} = \frac{O\left(\exp\left(\left(\log\frac{1}{1-r}\right)^\beta - 4\log n\right)\right)}{\frac{e^{C(r)}}{n}\sqrt{2\pi\beta(\log n)^{\beta-1}}} = O(n^{-3}(\log n)^{(1-\beta)/2}),$$

i.e., $I_2'' = o(I_1)$.

For $\theta_0 \leq |\theta| \leq \theta_1$, we note $\log\rho \to \infty$, and hence

$$0 < \Re\left(1 + \frac{i\phi}{\log\rho}\right)^\beta \leq 1.$$

Therefore

$$\begin{aligned}
\Re\left(\left(\log\frac{1}{1-re^{i\theta}}\right)^\beta\right) &= \Re\left((\log\rho + i\phi)^\beta\right) = \Re\left((\log\rho)^\beta\left(1+\frac{i\phi}{\log\rho}\right)^\beta\right) \\
&\leq \left(\log\frac{1}{|1-re^{i\theta_0}|}\right)^\beta \\
&= \left(\log\frac{1}{1-r} - \left(\frac{r}{2}\frac{\theta_0}{1-r}\right)^2 + O\left(\frac{\theta_0}{1-r}\right)^4\right)^\beta \\
&= \left(\log\frac{1}{1-r}\right)^\beta\left(1 - \frac{1}{\log\frac{1}{1-r}}\left(\frac{r}{2}\frac{\theta_0}{1-r}\right)^2 + O\left((\log n)^{4j-4\beta+3}\right)\right)^\beta \\
&\leq \left(\log\frac{1}{1-r}\right)^\beta - \frac{1}{2}(\log n)^{2j-\beta+1}.
\end{aligned}$$

It follows that

$$\begin{aligned} |I_2'| &\leq \theta_1 \exp\left(\left(\log \frac{1}{1-r}\right)^{\beta} - \frac{1}{2}(\log n)^{2j-\beta+1}\right) \\ &= \frac{1}{n}(\log n)^{\beta} \exp\left(\left(\log \frac{1}{1-r}\right)^{\beta} - \frac{1}{2}(\log n)^{2j-\beta+1}\right) \\ &= o(I_1). \end{aligned}$$

Therefore we conclude with the main theorem of this section.

Theorem 2 *Let $\beta \geq 2$, and let $C(z) = \left(\log \frac{1}{1-z}\right)^{\beta}$. Then as $n \to \infty$,*

$$[z^n]\exp(C(z)) \sim \frac{e^{C(r)}r^{-n}}{n}\sqrt{\frac{\beta}{2\pi}(\log n)^{\beta-1}},$$

where

$$r \sim 1 - \frac{\beta}{n}(\log n)^{\beta-1}.$$

Here r is defined by the saddle point equation $rC'(r) = n$.

As in Section 2, we are unable to find an asymptotic expansion for either $\exp(C(r))$ or r^{-n}, but we again find an asymptotic expansion for $\log(\exp(C(r))r^{-n}) = C(r) - n\log r$. We begin by recalling that $r = 1-t$, where

$$t = \frac{\beta}{n}(\log n)^{\beta-1}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right), \qquad \text{and}$$
$$\log\frac{1}{t} = \log n\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right).$$

Then

$$\begin{aligned} C(r) - n\log r &= \left(\log\frac{1}{t}\right)^{\beta} - n\log(1-t) = \left(\log\frac{1}{t}\right)^{\beta} + nt(1+O(t)) \\ &= (\log n)^{\beta}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right) + \beta(\log n)^{\beta-1}(1 + O(n^{-1}(\log n)^{\beta-1})) \\ &= (\log n)^{\beta}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right). \end{aligned}$$

4 Examples

We now give a simple example. Let $\mathcal{C}$ be the class of ordered pairs of non-empty cycles that we call *bicycles*. In the cases where $n = 2$ and $n = 3$, we have

$$\begin{aligned} n = 2: \qquad \mathcal{C} &= \{((1),(2)),((2),(1))\} \\ n = 3: \qquad \mathcal{C} &= \{((12),(3)),((13),(2)),((23),(1)),((1),(23)),((2),(13)),((3),(12))\}. \end{aligned}$$

Let $C(z)$ be the exponential generating function of $\mathcal{C}$. Then $C(z) = \left(\log \frac{1}{1-z}\right)^2$. Therefore, $C(z)$ is of alg-log type with $\alpha = 0$ and $\beta = 2$. Let $\mathcal{F}$ be the class of bicycles (sets of ordered pairs of non-empty cycles). Then the exponential generating function of $\mathcal{F}$, which we denote by $F(z)$, is given by $F(z) = \exp(C(z))$ and Theorem 2 applies. This result generalizes to three or more ordered non-empty cycles.

It is interesting to note that the bicycles are also connected to partitions such that the size of each part is a power of 2 (see [1] for the original results and [6] for generalizations).

5 Conclusions

So far we have covered the labeled case where $C(z) = (1-z)^{-\alpha}(\log \frac{1}{1-z})^{\beta}$, for $\alpha > 0$, $\beta > 0$ and for $\alpha = 0$, $\beta \geq 2$. We plan to complete the range $\alpha = 0$, $\beta < 2$; this may require other asymptotic techniques. We are also extending the results to unlabelled structures. In this latter case, as usual, we assume that the singularity ρ satisfies $\rho < 1$. For the case where $C(z)$ behaves like $c + d\left(\ln \frac{1}{1-z/\rho}\right)^{\beta}$, it may be possible to relax the condition that $\beta \geq 2$ in the labeled case and determine analogous results for the unlabeled case.

Furthermore, we plan to extend the work of Dong et al. [5] to find asymptotic results for combinatorial objects with a *restricted pattern*. This should also include probability results for the expected size of the *smallest component* in an object with a restricted pattern.

Another interesting possible line of research is the study of the moments of the number of components in $\exp\left(u\left(\log \frac{1}{1-z}\right)^2\right)$, or for the generalized bicycles when the power of the combinatorial logarithm is bigger than 2.

Finally, it will be interesting to derive similar results for constructions different than the multiset one, *i.e.*, cycle constructions and sequence constructions.

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