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Linear Systems on Tropical Curves

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Abstract. A tropical curve Γ is a metric graph with possibly unbounded edges, and tropical rational functions are continuous piecewise linear functions with integer slopes. We define the complete linear system $|D|$ of a divisor D on a tropical curve Γ analogously to the classical counterpart. We investigate the structure of $|D|$ as a cell complex and show that linear systems are quotients of tropical modules, finitely generated by vertices of the cell complex. Using a finite set of generators, $|D|$ defines a map from Γ to a tropical projective space, and the image can be modified to a tropical curve of degree equal to $\deg(D)$. The tropical convex hull of the image realizes the linear system $|D|$ as a polyhedral complex.

Résumé. Une courbe tropicale Γ est un graphe métrique pouvant contenir des arêtes infinies, et une fonction rationnelle tropicale est une fonction continue linéaire par morceaux à pentes entières. Le système linéaire complet $|D|$ d'un diviseur D sur une courbe tropicale Γ est défini de façon analogue au cas classique. Nous étudions la structure de $|D|$ en tant que complexe cellulaire et montrons que les systèmes linéaires sont des quotients de modules tropicaux engendrés par un nombre fini de sommets du complexe. Etant donné un ensemble fini de générateurs, $|D|$ définit une application de Γ vers un espace projectif tropical, dont l'image peut être modifiée en une courbe tropicale de degré égal à $\deg(D)$. L'enveloppe convexe tropicale de l'image réalise le système linéaire $|D|$ en tant que complexe polyédral.

Keywords: tropical curves, divisors, linear systems, canonical embedding, chip-firing games, tropical convexity

1 Introduction

An abstract tropical curve Γ is a connected metric graph with possibly unbounded edges. A *divisor* D on Γ is a formal finite $\mathbb{Z}$-linear combination $D = \sum_{x\in\Gamma} D(x)\cdot x$ of points of Γ. The *degree* of a divisor is the sum of the coefficients, $\sum_x D(x)$. The divisor is *effective* if $D(x) \geq 0$ for all $x \in \Gamma$; in this case we write $D \geq 0$. We call $\operatorname{supp}(D) = \{x \in \Gamma \,:\, D(x) \neq 0\}$ the support of the divisor D.

A *(tropical) rational function* f on Γ is a continuous function $f : \Gamma \to \mathbb{R}$ that is piecewise-linear on each edge with finitely many pieces and integral slopes. The *order* $\operatorname{ord}_x(f)$ of f at a point $x \in \Gamma$ is the sum of outgoing slopes at x. The *principal divisor* associated to f is

$$(f) := \sum_{x\in\Gamma} \operatorname{ord}_x(f)\cdot x.$$

A point $x \in \Gamma$ is called a *zero* of f if $\operatorname{ord}_x(f) > 0$ and a *pole* of f if $\operatorname{ord}_x(f) < 0$. We call two divisors D and D' linearly equivalent and write $D \sim D'$ if $D - D' = (f)$ for some f. For any divisor D on Γ, let $R(D)$ be the set of all rational functions f on Γ such that the divisor $D + (f)$ is effective, and $|D| = \{D' \geq 0 : D' \sim D\}$, the *linear system* of D. Let $\mathbb{1}$ denote the set of constant functions on Γ.

The set $R(D)$ is naturally embedded in the set $\mathbb{R}^\Gamma$ of all real-valued functions on Γ, and $|D|$ is a subset of the d^{th} symmetric product of Γ where $d = \deg(D)$. The map $R(D)/\mathbb{1} \to |D|$ given by $f \mapsto D + (f)$

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is a homeomorphism from $R(D)/\mathbb{1}$ to $|D|$. It was shown in [GK08, MZ06] that $|D|$ is a cell complex, so is $R(D)/\mathbb{1}$. Our aim is to study the combinatorial and algebraic structure of this object $R(D)$.

In Section 2 we give definitions and state linear equivalence in terms of weighted chip firing moves, which are continuous analogues of the chip firing games on finite graphs. In Section 3 we show that $R(D)$ is a finitely generated tropical semi-module and describe a generating set. In Section 4, we study the cell complex structure of $|D|$. We show that the vertex set of $|D|$ coincides with the generating set of $R(D)$ described in Section 3. We give a triangulation of the link of each cell as the order complex of a poset of possible weighted chip firing moves.

Any finite set $\mathcal{F}$ of linearly equivalent divisors induces a map $\phi_{\mathcal{F}}$ from the abstract curve to a tropical projective space. This map is described in Section 5. If $\mathcal{F}$ generates $R(D)$, we show that the tropical convex hull of the image of this map is homeomorphic to $|D|$. The image of this map $\phi_{\mathcal{F}}$ can be naturally modified to an embedded tropical curve.

2 Metric graphs, rational functions, and chip-firing

A *metric graph* Γ is a complete connected metric space such that each point $x \in \Gamma$ has a neighborhood U_x isometric to a star-shaped set of valence $\text{val}(x) \geq 1$ endowed with the path metric. To be precise, a star-shaped set of valence v is a set of the form

$$S(v,r) = \{z \in \mathbb{C} : z = te^{2\pi ik/v} \text{ for some } 0 \leq t < r \text{ and } k \in \mathbb{Z}\}.$$

The points $x \in \Gamma$ with valence different from 2 are precisely those where Γ fails to look locally like an open interval. Accordingly, we refer to a point of valence 2 as a *smooth* point.

Let $V(\Gamma)$ be any finite nonempty subset of Γ such that $V(\Gamma)$ contains all of the points with $\text{val}(x) \neq 2$. Then $\Gamma \setminus V(\Gamma)$ is a finite disjoint union of open intervals. For a metric graph Γ, we say that a choice of such $V(\Gamma)$ gives rise to a *model* $G(\Gamma)$ for Γ. Each edge has a nonzero length inherited from the metric space Γ.

Let $V_0(\Gamma) = \{x \in \Gamma : \text{val}(x) \neq 2\}$, where val denotes the valence of a vertex of $V(\Gamma)$. Unless Γ is a circle, $V_0(\Gamma)$ gives a model. For some of our applications, we may choose a model whose vertex set is strictly bigger than $V_0(\Gamma)$. However unless otherwise specified, the reader may assume that $G(\Gamma)$ denotes the coarsest model and that a *vertex* is an element of $V_0(\Gamma)$.

A *tropical curve* is a metric graph in which the leaf edges may have length ∞. A leaf edge is an edge adjacent to a one-valent vertex. Note that we add a "point at infinity" for each unbounded edge. A tropical rational function on a tropical curve may attain values $\pm\infty$ at points at infinity.

We will use the term *subgraph* in a topological sense, that is, as a compact subset of a tropical curve Γ with a finite number of connected components. For a subgraph $\Gamma' \subset \Gamma$ and a positive real number l, the *chip firing move* $\text{CF}(\Gamma', l)$ by a (not necessarily connected) subgraph is the tropical rational function $\text{CF}(\Gamma', l)(x) = -\min(l, \text{dist}(x, \Gamma'))$. It is constant 0 on Γ', has slope -1 in the l-neighborhood of Γ' directed away from Γ', and it is constant $-l$ on the rest of the graph. We will sometimes refer to an effective divisor D as a *chip configuration*. For example, for $D = c_1 \cdot x_1 + \cdots + c_n \cdot x_n$, we say that there are c_i chips at the point $x_i \in \Gamma$. The total number of chips is the *degree* of the divisor. We say that a subgraph $\Gamma' \subset \Gamma$ can *fire* if for each boundary point of Γ' there are at least as many chips as the number of edges pointing out of Γ'. In other words, Γ' can fire if the divisor $D + (\text{CF}(\Gamma', l))$ is effective for some positive real number l. The chip configuration $D + (\text{CF}(\Gamma', l))$ is then obtained from D by moving one chip from the boundary of Γ' along each edge out of Γ' by distance l. Here we assume that l was chosen to be small enough so that the chips do not pass through each other or pass through a non-smooth point.

We will now show that these chip firing moves are enough to move between linearly equivalent divisors (Proposition 3 below). To this end, call a tropical rational function f a *weighted chip firing move* if there are two disjoint (not necessarily connected) proper closed subgraphs Γ_1 and Γ_2 such that the complement $\Gamma \setminus (\Gamma_1 \cup \Gamma_2)$ consists only of open line segments and such that f is constant on Γ_1 and Γ_2 and linear (smooth) with integer slopes on the complement.

A weighted chip firing move f can also be thought of as a combinatorial transformation that acts on chip configurations. Such transformations move chips from the boundary of Γ_2 along the open line segments in the complement $\Gamma \setminus (\Gamma_1 \cup \Gamma_2)$. (Here we assume w.l.o.g. that $f(\Gamma_2) > f(\Gamma_1)$.) During this process, a law of conservation of momentum holds so that a stack of m chips that move together will only move a distance of l/m. The numbers l and m can be different on each component of the complement. Note that a (simple) chip firing move $\mathrm{CF}(\Gamma', l)$ with small l is a special case of a weighted chip firing move when all the slopes are 0 or ± 1. The following two lemmas make the connection between $R(D)$ and chip firing games.

Lemma 1. *A weighted chip firing move is an (ordinary) sum of chip firing moves (plus a constant).*

Lemma 2. *Every tropical rational function is an (ordinary) sum of chip firing moves (plus a constant).*

Note that even if we start with a tropical rational function $f \in R(D)$, the sequence of weighted chip firing moves $f_1, \ldots, f_n$ for which $f = f_1 + \cdots + f_n$ may not be in $R(D)$, i.e. the divisors $D + (f_i)$ may not be effective although $D + (f)$ is. The following proposition follows easily from the two previous lemmas.

Proposition 3. *Two divisors are linearly equivalent if and only if one can be attained from the other using chip firing moves.*

Studying linear equivalence of divisors is partially motivated by a certain rank function satisfying tropical Riemann-Roch. In particular, the *rank* $r(D)$ of a divisor D is the maximum integer r such that $|D - E| \neq \emptyset$ for all degree-r divisors E. The Riemann-Roch Theorem [GK08, MZ06] (based on work of [BN07]) , which is the same for classical and tropical geometry, says that

$$r(D) - r(K - D) = \deg D + 1 - g, \tag{RR}$$

where g is the genus of tropical curve Γ, and the canonical divisor of Γ, K, is defined in Section 4.2.

3 Extremals and Generators of $R(D)$

The *tropical semiring* $(\mathbb{R}, \oplus, \odot)$ is the set of real numbers $\mathbb{R}$ with two tropical operations:

$$a \oplus b = \max(a, b), \text{ and } a \odot b = a + b.$$

The space $R(D)$ is naturally a subset of the space $\mathbb{R}^\Gamma$ of real-valued functions on Γ. For $f, g \in \mathbb{R}^\Gamma$, and $a \in \mathbb{R}$, the functions $f \oplus g$ and $a \odot f$ are defined by taking tropical sums and tropical products pointwise.

Lemma 4. *The space $R(D)$ is a tropical semi-module, i.e. it is closed under tropical addition and tropical scalar multiplication.*

Tropical semi-modules in $\mathbb{R}^n$ are also called *tropically convex sets* [DS04]. Since $R(D+(f)) = R(D)+f$, the tropical algebraic structure of $R(D)$ does not depend on the choice of the representative D. An element $f \in R(D)$ is called *extremal* if for any $g_1, g_2 \in R(D)$, $f = g_1 \oplus g_2 \implies f = g_1$ or $f = g_2$. Any generating set of $R(D)$ must contain all extremals up to tropical scalar multiplication.

Lemma 5. *A tropical rational function f is an extremal of $R(D)$ if and only if there are not two proper subgraphs Γ_1 and Γ_2 covering Γ (i.e. $\Gamma_1 \cup \Gamma_2 = \Gamma$) such that each can fire on $D + (f)$.*

A *cut set* of a graph Γ is a set of points $A \subset \Gamma$ such that $\Gamma \backslash A$ is not connected. A *smooth cut set* is a cut set consisting of smooth points (2-valent points). Note that being a smooth cut set depends only on the topology of Γ and is not affected by the choice of model $G(\Gamma)$.

Theorem 6. *Let $\mathcal{S}$ be the set of rational functions $f \in R(D)$ such that the support of $D + (f)$ does not contain a smooth cut set. Then*

(a) $\mathcal{S}$ contains all the extremals of $R(D)$,

(b) $\mathcal{S}$ is finite modulo tropical scaling, and

(c) $\mathcal{S}$ generates $R(D)$ as a tropical semi-module.

For the proof of (b) we need a boundedness lemma that improves the bound in [GK08, Lemma 1.8].

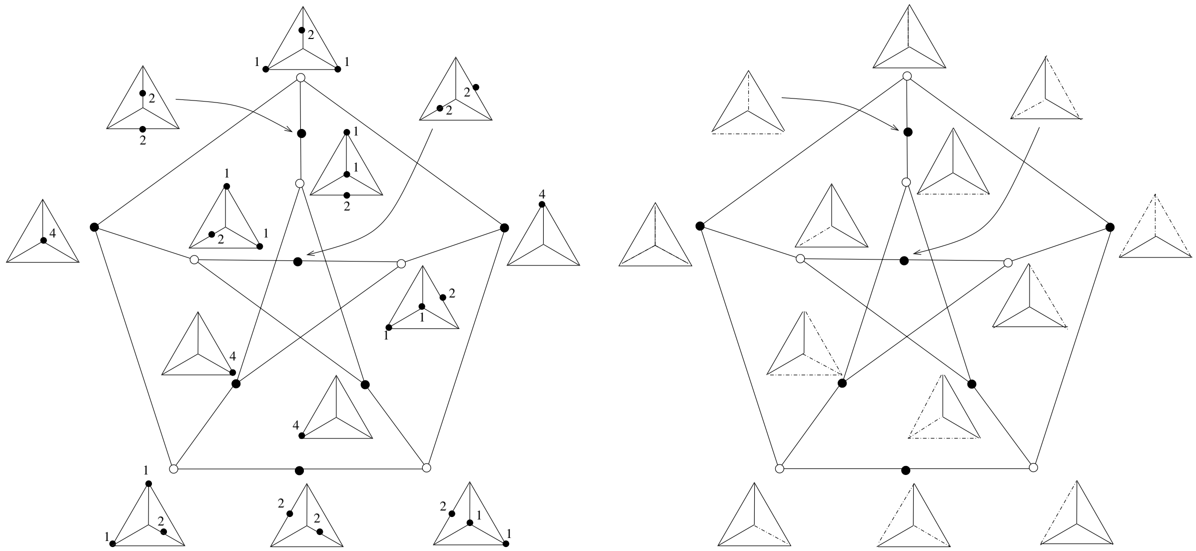

Fig. 1: (Left): The linear system $|K|$ for the tropical curve $\Gamma = K_4$, the complete graph on four vertices with edges of equal length, and the canonical divisor K. The 13 divisors shown here, together with K, correspond to the elements of $\mathcal{S}$ that generate $R(K)$, from Theorem 6. The seven black dots correspond to the extremals. See Example 10.

(Right): The link of the canonical divisor in the canonical class, where Γ is the complete graph on four vertices, with arbitrary edge lengths. This graph is also the order complex of the firing poset. The firing subgraphs in Γ are shown by solid lines. See Example 26. Compare with Figure 2 in [AK06].

Lemma 7. *For $D \geq 0$ every slope of $f \in R(D)$ is bounded by* $\deg D$.

of Theorem 6. (a) Suppose $f \notin \mathcal{S}$, then $D + (f)$ splits Γ into two subgraphs Γ_1 and Γ_2. Both of these graphs can fire, and the union of their closures is the entire Γ, so by Lemma 5, f is not an extremal.

(b) Let $f \in \mathcal{S}$. The support of $D + (f)$ meets the interior of each edge in at most one point, because two points on the same edge form a smooth cut set. Removing the set of edges meeting the support of $D + (f)$ does not disconnect Γ, and so the remaining edges contain a spanning tree of Γ. There are finitely many spanning trees in a graph and finitely many possible slopes for each edge in this spanning tree because of Lemma 7. Therefore, the number of possible values of f on vertices of Γ is finite modulo tropical scaling. (Here, vertices are non-smooth points. If Γ is a circle, then fix any point as a vertex.) On each non-tree edge, knowing the values and the slopes of f at the two end points uniquely determines f since all the chips of $D + (f)$ must fall on the same point of a given edge. We conclude that $\mathcal{S}$ is finite modulo tropical scaling.

(c) Let f be an arbitrary function in $R(D)$. We need to show that f can be written as a finite tropical sum of elements of $\mathcal{S}$. Let $N(f)$ be the number of smooth points in $\mathrm{supp}(D + (f))$. If f is not already in $\mathcal{S}$, then there is a smooth cut set A and two components Γ_1 and Γ_2. Let g_1 and g_2 be the weighted chip firing moves that fire all chips on their boundaries as far as possible. Then $f = (f + g_1) \oplus (f + g_2)$. Repeating this decomposition terminates after a finite number of steps because $0 \leq N(f + g_i) < N(f)$ for each $i = 1, 2$. □

Proposition 8. *Any finitely generated tropical sub-semimodule M of $\mathbb{R}^\Gamma$ is generated by the extremals.*

Corollary 9. *The tropical semimodule $R(D)$ is generated by the extremals. This generating set is minimal and unique up to tropical scalar multiplication.*

The set of extremals can be obtained from $\mathcal{S}$ by removing the elements not satisfying the condition in Lemma 5.

Example 10. Let Γ be a tropical curve with the complete graph on 4 vertices with equal edge lengths as a model. Consider the canonical divisor K, that is the divisor with value 1 on the four vertices and zero elsewhere. The canonical divisor is defined in general in Section 4.2. Then the set $\mathcal{S}$ from Theorem 6 consists of 14 elements, 7 of which are extremals. The linear system $|K|$ is the cone over the Petersen graph.

If the edge lengths of the complete graph are not all equal, then the set $\mathcal{S}$ may be different from this. We describe the local cell complex structure of $R(K)$ near K in the next section, in Example 20. See Figure 1.

4 Cell complex structure of $|D|$

As seen in the previous section, $R(D) \subset \mathbb{R}^\Gamma$ is finitely generated as a tropical semi-module or a tropical polytope. However, it is not a polyhedral complex in the ordinary sense. For example, let Γ be the line segment $[0, 1]$, and D be the point 1. Then $R(D)$ is the tropical convex hull of $f, g \in \mathbb{R}^\Gamma$ where $f(x) = x$ and $g(x) = 0$. Although $R(D)$ is one-dimensional, it does not contain the usual line segment between any two points in it. Letting $\mathbb{1}$ denote the constant function taking the value 1 at all points, we consider functions in $R(D)$ modulo addition of $\mathbb{1}$, i.e. translation.

Lemma 11. *The set $R(D)/\mathbb{1}$ does not contain any nontrivial ordinary convex sets.*

Recall that $R(D)/\mathbb{1}$, i.e. $R(D)$ modulo tropical scaling can be identified with the linear system $|D| := \{D + (f) : f \in R(D)\}$ via the map $f \mapsto D + (f)$. In what follows, elements of $|D|$ and elements of projectivized $R(D)$, i.e. $R(D)/\mathbb{1}$, will be used interchangeably.

A choice of model $G(\Gamma)$ induces a polytopal cell decomposition of $\operatorname{Sym}^d \Gamma$, the d^{th} symmetric product of Γ. Andreas Gathmann and Michael Kerber [GK08] as well as Grigory Mikhalkin and Ilia Zharkov [MZ06] describe $|D|$ as a cell complex $|D|_{G(\Gamma)} \subset \operatorname{Sym}^d \Gamma$. Let us coordinatize this construction.

We identify each open edge $e \in E$ with the interval $(0, \ell(e))$ thereby giving the edge a direction, and we identify $\operatorname{Sym}^k e$ with the open simplex $\{x \in \mathbb{R}^k : 0 < x_1 < \ldots < x_k < \ell(e)\}$. A cell of $|D|$ is indexed by the following discrete data:

- $d_v \in \mathbb{Z}$ for every vertex $v \in V$,
- a composition (i.e. an ordered partition) $d_e = d_e^{(1)} + \cdots + d_e^{(r_e)}$ for every edge e of Γ, and
- an integer m_e for every edge e of Γ.

Then, a divisor D' belongs to that cell if

- $d_v = D'(v)$ for all $v \in V$,
- D' is given on e by $\sum_i d_e^{(i)} x_i$ for $0 < x_1 < \ldots < x_{r_e} < \ell(e)$, and
- the slope of f at the start of edge e is m_e, where f is such that $(f) + D = D'$.

The intersection of $|D|$ with an open cell of $\operatorname{Sym}^d \Gamma$ is a union of cells of $|D|$.

This cell complex structure depends on the choice of the model $G(\Gamma)$, but not on the choice of representative divisor D in the linear system $|D|$. In particular, choosing a finer model amounts to subdividing the cell complex $|D|$, and choosing a different divisor $D' = D + (g)$ amounts to changing the integer slopes at the starting points on the edges by the slopes of g, but this does not change the cells. Whenever we talk about a cell complex structure of $|D|$, we are impliciting assuming a model $G(\Gamma)$. Unless Γ is a circle, there is a unique coarsest model with the least number of vertices.

Example 12. Let Γ be a circle (for example a single vertex v with a loop edge e attached). Consider D to be the divisor $3v$. As we analyze in Example 17, $|D|$ contains two 2-cells in this case. The elements of both cells are divisors $D' = x + y + z$ with distinct points x, y, and z on the interior of e. However the two 2-cells differ from one another by the slope of the function f (defined by $D' = D + (f)$) at v. The outgoing slopes of f at v are given by $[-2, -1]$ for one 2-cell and by $[-1, -2]$ for the other. This example shows that the combinatorial type of the divisor D' – the cell of $\operatorname{Sym}^d \Gamma$ containing D' – does not determine the cell of $|D|$ containing D'. The different cells of $|D|$ in one cell of $\operatorname{Sym}^d \Gamma$ are indexed by the slopes of f.

Proposition 13. *For $D' \in |D|$, let $I_{D'}$ be the set of points in the support of D' that lie in the interior of edges. Then the dimension of the carrier of D' is one less than the number of connected components of $\Gamma \backslash I_{D'}$.*

Here, the carrier of D' is the cell containing D' in its interior. Recall that Γ is connected, and note that being in the interior of an edge depends on the model $G(\Gamma)$.

Theorem 14. *Let G be a model for Γ, and let $\mathcal{S}_G$ be the set of functions $f \in R(D)$ such that the support of $D + (f)$ does not contain an interior cut set (i.e. a cut set consisting of points in interior of edges in the model G). Then*

(a) $\mathcal{S}_G$ contains the set $\mathcal{S}$ from Theorem 6,

(b) $\mathcal{S}_G$ is finite modulo tropical scaling, and

(c) $\mathcal{S}_G = \{f \in R(D) : D + (f)$ is a vertex of $|D|\}$.

Proof. The statement (a) follows from definitions since points in the interior of edges (for any model) are smooth, and the statement (b) can be shown in the exact same way as Theorem 6(b). By the previous proposition, any element of $\mathcal{S}_G$ has dimension 0. This shows (c). □

This shows in particular that the cell complex $|D|$ has finitely many vertices. If the model G is the coarsest one, i.e. the vertices of G are non-smooth points of Γ, then $\mathcal{S}_G = \mathcal{S}$. If Γ is a circle, then there is no unique coarsest model.

Proposition 15. *Each closed cell in the cell complex is finitely-generated as a tropical semi-module by its vertices. In particular, it is tropically convex.*

Example 16. (Line Segment) Any tree is a genus zero tropical curve. Like genus zero algebraic curves, two divisors on a tree are linearly equivalent if and only if they have the same degree d. The simplest tree is a line segment consisting of an edge between two vertices, v_1 and v_2. In this case, $|D|$ is a d-simplex. The vertices of $|D|$ correspond to ordered pairs $[d_1, d_2]$ summing to d associated to the chip configuration at v_1 and v_2.

Example 17. (Circle) A circle is the only tropical curve where the canonical divisor K is 0. Let Γ be homeomorphic to a circle and let D be of degree 3. Then $D \sim 3x$ for some point $x \in \Gamma$. The coarsest cell structure of $R(D)$ is a triangle, but it is not realized by any model on Γ because Γ does not have a unique coarsest model. If the model contains only one vertex v and $D \sim 3v$, then $R(D)$ is a triangle subdivided by a median; see Figure 2. In particular $|D|$ contains four 0-cells, five 1-cells, and two 2-cells. If the model $G(\Gamma)$ consists of a vertex u such that $D \not\sim 3u$, then the cell complex structure would be different. If the model $G(\Gamma)$ consists of 3 equally spaced vertices v_1, v_2, v_3, and $D \sim 3v_1$, then $R(D)$ is isomorphic as a polyhedral complex to the barycentric subdivision of a triangle.

Example 18. (Circle with higher degree divisor) Let Γ be a circle graph with only a single vertex v and a single edge e, a loop based at v. Let $D = dv$; then the linear system $|D|$ is a cone over a cell complex, which we denote as $P_d(circle)$, which has an f-vector given by the following:

$$\text{The number of } i\text{−cells of } P_d(circle) = f_i = (i+1)\binom{d}{i+2}.$$

Consequently, the f-vector for $|D|$ is given by

$$\begin{cases} \binom{d}{2} + 1 \text{ if } i = 0 \\ (i+1)\binom{d}{i+2} + i\binom{d}{i+1} \text{ if } i \geq 1 \end{cases} .$$

To see how to get these f-vectors, we note that a divisor $D' \sim dv$ corresponds to a tropical rational function f such that $dv + (f) = D'$. One such f is the zero function, this corresponds to the cone point. Each other tropical rational function is parameterized by an increasing sequence of integer slopes $(a_1, \ldots, a_{i+2})$ such that $a_1 < 0$, $a_{i+2} > 0$, and $a_{i+2} - a_1 \leq d$. The first slope must be negative and the last slope must be

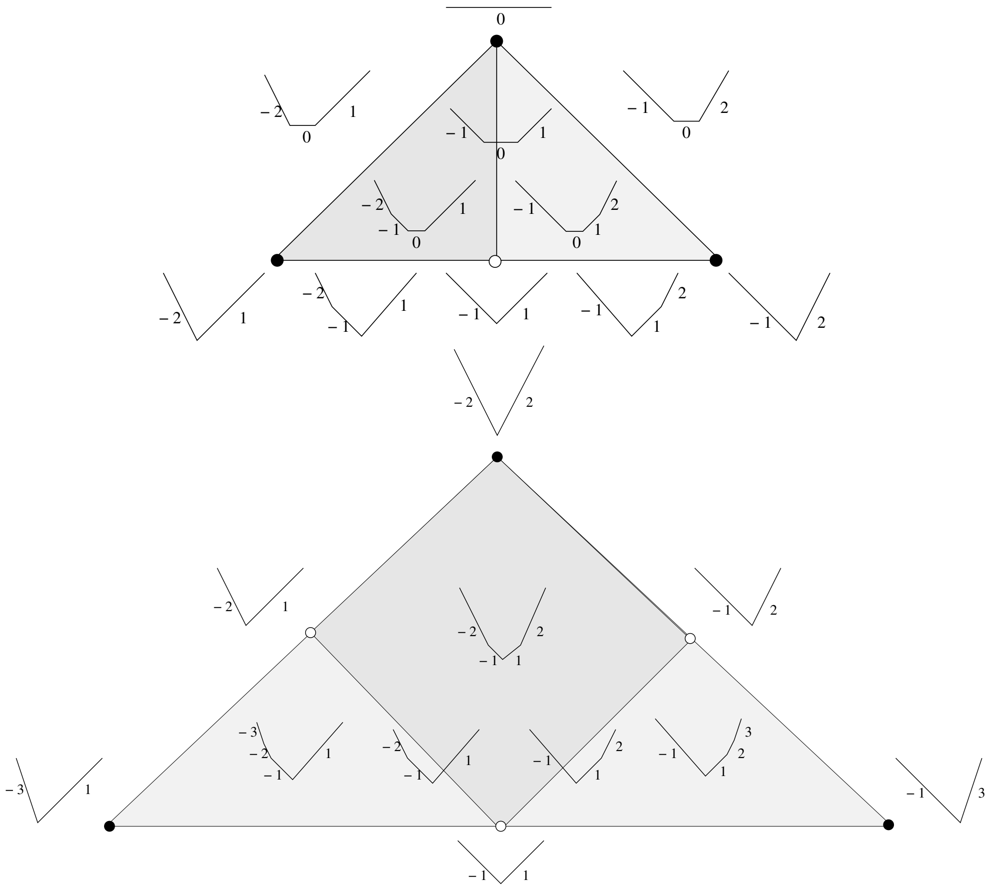

Fig. 2: (Top): The polyhedral cell complex $R(3v)/\mathbb{1}$ on $\Gamma = S^1$. The three black vertices are the extremals, and they correspond to the three divisors which are linearly equivalent to $3v$ and have the form $3w$. We have presented S^1 as the line segment $[0, 1]$ with points 0 and 1 identified.

(Bottom): The polyhedral cell complex $R(4v)/\mathbb{1}$ on $\Gamma = S^1$ is a subdivided tetrahedron, a cone over this subdivided triangle with the cone-point corresponding to the constant function. (The labels of most 1-cells are suppressed, but may be read off from the incident vertices or 2-cells.) The cone-point plus the three black vertices are the extremals.

positive so that the values of f at the two ends of the loop e agree. The cells not incident to the cone point yield the cell complex $P_d(circle)$, and are given by sequences $(a_1, \ldots, a_{i+2})$ such that all $a_i \neq 0$. To finish the computation of the f-vector for $P_d(circle)$, we pick an ordered pair $[j, k]$ with $j, k \geq 1$ and $j + k = i + 2$ to denote the number of negative and positive a_k's, respectively. After setting $a_1 = -\ell$, we note that the number of ways to pick the remaining negative a_k's is given by $\binom{\ell-1}{j-1}$, and the number of ways to pick a subset of positive a_k's such that $a_{i+2} - a_1 \leq d$ is given by $\binom{d-\ell}{k}$. Summing over possible ℓ, and using a standard identity involving binomial coefficients (for instance see [BQ03, Identity 136]), we obtain $\binom{d}{i+2}$ such tropical rational functions for each $[j, k]$. Since there are $i + 2$ such $[j, k]$'s, we get the above number of i-cells not incident to the cone point. For the case of $d = 4$, see Figure 2.

Example 19. (Circle. Cell structure of $|D|$ as a simplex) In Examples 17 and 18, we saw that having to choose a model, even one with only one vertex, gives $|D|$ a cell structure of a subdivided simplex. Moreover, different choices of models, even if they contain only one vertex each, may give combinatorially different cell complex structures for $|D|$. We wish to describe $|D|$ as a simplex.

First, let us look at the embedding of $|D|$ in the symmetric product of the tropical curve. Let Γ be the circle

$\mathbb{R}/\mathbb{Z}$, and $D = d \cdot [0]$ be a divisor of degree d. The embedding of $|D|$ in $\mathrm{Sym}^d \Gamma = \mathrm{Sym}^d(\mathbb{R}/\mathbb{Z})$ is given by

$$\{x \in (0,1]^d \;:\; 0 < x_1 \leq x_2 \leq \dots x_d \leq 1, \;\; x_1 + x_2 + \cdots + x_d \in \mathbb{Z}\}.$$

To see this, first consider a tropical rational function g on the line segment $[0,1]$ with $(g) = x_1 + x_2 + \cdots + x_d - d \cdot 0$ and $g(1) = 0$. Then $g(0) = x_1 + x_2 + \cdots + x_d$. If $g(0) \in \mathbb{Z}$, then adding g and a function l with constant slope $g(0)$ on $[0,1]$ gives a tropical rational function $f = g + l$ on the circle with $(f) + D = x_1 + x_2 + \cdots + x_d$. It is easy to check that any $f \in R(D)$ can be obtained this way. Although this description gives $|D|$ a uniform coordinate system, this does not give us a cell complex structure.

In fact, $|D|$ can be realized as a $(d-1)$-dimensional simplex, on d vertices. There is a unique set of d points $v_1, v_1, \dots, v_d$ in Γ such that $D \sim dv_i$ for all $i = 1, \dots, d$. These d points are equally spaced along Γ. The extremals of $R(D)$ are

$$\mathcal{E} = \{f \in R(D) : (f) + D = d \cdot v_i \text{ for some } i = 1, 2, \dots, d\}.$$

Consider the $(d-1)$-dimensional simplex on vertices $V = \{dv_1, dv_2, \dots, dv_d\}$, that is, the simplicial complex containing a $(k-1)$-dimensional cell for any k subset of V. We would like to stratify $|D|$ into these cells. For any divisor $D' \in |D|$, elements in the same cell as D' are obtained from D' by weighted chip firing moves that do not change the cyclically-ordered composition $d = a_1 + a_2 + \cdots + a_k$ associated to divisor $a_1x_1 + a_2x_2 + \cdots + a_kx_k$ where $x_1, x_2, \dots, x_k$ are distinct and cyclically ordered along the circle (with a fixed orientation). The complement of the support of $D' = a_1x_1 + a_2x_2 + \cdots + a_kx_k$ consists of k segments. For each of these segments, there is a unique extremal in $R(D')$ that is maximal and constant on it. These k extremals of $R(D')$, which are naturally identified with extremals of $R(D)$, are precisely the vertices of the cell of D' and their convex hull is the cell of D'.

Example 20. (K_4 continued) As in Example 10, consider the graph K_4 with equal edge lengths and the canonical divisor K. The canonical divisor is defined in general in Section 4.2. The coarsest cell structure of $|K|$ consists of 14 vertices and topologically is the cone over the Petersen graph shown in Figure 1. The cone point is the canonical divisor K. The "cones" over the 3 subdivided edges of the Petersen graph are quadrangles. The maximal cells of $|K|$ consist of 12 triangles and 3 quadrangles. In particular, $|K|$ is not simplicial. The quadrangle obtained from "coning" over the bottom edge of the Petersen graph is shown in Figure 3.

4.1 Local structure of a cell complex

If B is a cell complex and x is a point in B, then the $\mathrm{link}(x, B)$ denotes the cell complex obtained by intersecting B with a sufficiently small sphere centered at x. We will define a triangulation of $\mathrm{link}(D, |D|)$ which is finer than the cell structure. Note that $|D|$ and $|D'|$ are isomorphic as cell complexes, so $\mathrm{link}(D, |D|) \cong \mathrm{link}(D, |D'|)$ for any $D' \sim D$.

Let $D' \in \mathrm{link}(D, |D|)$ and f be a rational function such that $D' = D + (f)$. Let $h_0 > h_1 > \cdots > h_n$ be the values taken on by f on the set of points that are either vertices of Γ or where f is not smooth. Notice that h_0 and h_n are maximum and minimum values of f, respectively. Since $D + (f) \in \mathrm{link}(D, |D|)$, we may assume that $h_0 - h_n$ is sufficiently small. Let $G = (\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma)$ be a chain of subgraphs of Γ where $\Gamma_i = \{x \in \Gamma : f(x) \geq h_i\}$.

Let $G' = (\Gamma'_1 \subset \Gamma'_2 \subset \cdots \subset \Gamma'_n = \Gamma)$ be the chain of *compactified* graphs, where Γ'_i is the union of edges of Γ_i that are between two vertices of Γ. Each cell can be subdivided by specifying more combinatorial data: the chain G' obtained this way and the slopes at the non-smooth points. We call this the *fine subdivision*.

For an effective divisor D, we can naturally associate the *firing poset* $\mathcal{P}_D$ as follows. An element of $\mathcal{P}_D$ is a weighted chip firing move without the information about the length, i.e. it is a closed subgraph $\Gamma' \subset \Gamma$ together with an integer c_e for each out-going direction e of Γ' such that for each point $x \in \Gamma'$ we have $\sum c_e \leq D(x)$ where the sum on the left is taken over the all outgoing directions e from x and $D(x)$ denotes the coefficient of x in D. We say that $(\Gamma', c') \leq (\Gamma'', c'')$ if $\Gamma' \subset \Gamma''$ and $c'_e \geq c''_e$ for each common outgoing direction e of Γ' and Γ''.

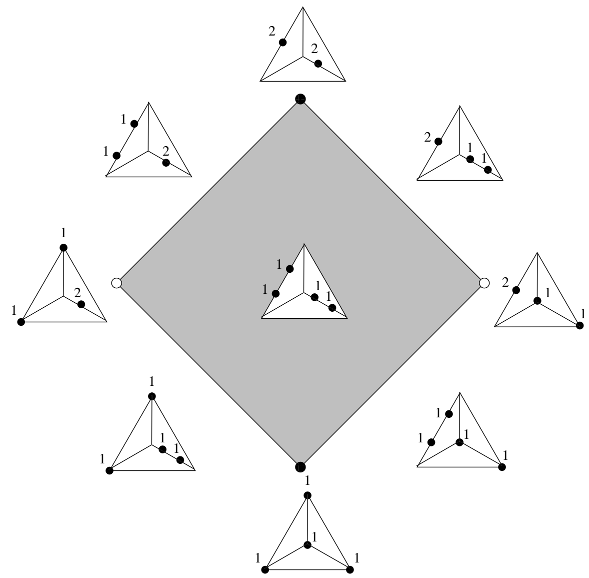

Fig. 3: A non-simplicial cell in the linear system $|K|$ for the complete graph on four vertices with edges of equal length.

Theorem 21. *The fine subdivision of the link of a divisor D in its linear system $|D|$ is a geometric realization of the order complex of the firing poset $\mathcal{P}_D$.*

Proof. By the discussion above, a cell in a fine subdivision $\operatorname{link}(D, |D|)$ corresponds to a unique chain in the firing poset. For any chain in the firing poset, we can construct an element in $\operatorname{link}(D, |D|)$ by performing the weighted chip firing moves in the order given by the chain, starting from the smallest element. The element constructed this way defines a cell in the fine subdivision. □

Note that the link of an element in $|D|$ does not depend on the precise location of the chips, but on the combinatorial data of the location. In other words, changing the edge lengths, without changing which edges the chips are on, does not affect the combinatorial structure of the link.

This Theorem, along with Proposition 13 allows us to explicitly describe the 1-cells incident to a 0-cell D' of $|D|$. For this, we need to define a specific subset of the weighted chip-firing moves. In particular, we call a weighted chip-firing move f (which is constant on Γ_1 and Γ_2) to be *doubly-connected* if Γ_1 and Γ_2 are both connected subgraphs.

Proposition 22. *Given $D' \in |D|$, and a model G such that $\operatorname{supp}(D') \subset V(G)$ (so that D' is a 0-cell in $|D|$), the 1-cells incident to D' correspond to the set of doubly-connected weighted chip-firing moves that are legal on chip configuration D' (up to combinatorial type).*

Proof. Let f be a weighted chip-firing move which is legal at D' that is constant on Γ_1 and Γ_2 such that $f(\Gamma_2) = f(\Gamma_1) - \epsilon$ for small $\epsilon > 0$. Then D'', defined as $D' + (f)$ has a chip on each of the line segments L_i connecting Γ_1 and Γ_2. Then the dimension of the corresponding cell of D'' is one if and only if Γ_1 and Γ_2 are both connected. □

4.2 Bergman subcomplex of $|K|$

Now we analyze the linear systems of an important family of divisors. The *canonical divisor* K on Γ is

$$K := \sum_{x \in \Gamma} (\operatorname{val}(x) - 2) \cdot x.$$

Vertices of valence two do not contribute to this sum so the divisor K is independent of the choice of model.

Let M be a matroid on a ground set E. The *Bergman fan* of M is the set of $w \in \mathbb{R}^E$ such that w attains its maximum at least twice on each circuit C of M. The only matroids considered here are *cographic matroids* of graphs. For a graph G with edge set E, the cographic matroid is the matroid on the ground set E whose dependent sets are cuts of G, i.e. the sets of edges whose complement is disconnected. The *Bergman complex* is the cell complex obtained by intersecting the Bergman fan with a sphere centered at the origin. The following result will be useful to us later.

Theorem 23. *[AK06]*

1. *The Bergman complex (with its fine subdivision) is a geometric realization of the order complex of the lattice of flats of M.*

2. *The Bergman fan is pure of codimension rank(M).*

Note that adding or removing parallel elements does not change the simplicial complex structure of the Bergman complex because the lattice of flats remains unchanged up to isomorphism. In particular, if G_1 and G_2 are two graphs, forming two models of the same tropical curve, then the corresponding cographic matroids have isomorphic Bergman complexes.

Lemma 24. *A subset of edges of a graph forms a flat of the cographic matroid if and only if its complement is a union of circuits of the graph.*

Suppose Γ has genus at least one but K_Γ is not effective. Let Γ' be the subgraph of Γ obtained by removing all the leaf edges recursively. Then the canonical divisor K' of Γ' is effective, and we can apply the following arguments for K' in Γ' or Γ.

Theorem 25. *The fine subdivision of* $\operatorname{link}(K, |K|)$ *contains the fine subdivision of the Bergman complex* $B(M^*(\Gamma))$ *as a subcomplex.*

Proof. The complement of a flat is a union of cocircuits, so the lattice of flats is isomorphic to the lattice of unions of cocircuits, ordered by reverse-inclusion. The cocircuits of the cographic matroid are the circuits of the graph. For the canonical divisor K, the proper union of circuits can always fire. Hence the proper part of the poset of union of circuits is a subposet of the firing poset, and so is the proper part of the lattice of flats. □

The Bergman complex may be a proper subcomplex of the link because there may be subgraphs that can fire on the canonical divisor but that are not union of circuits, e.g. two triangles connected by an edge in the graph of a triangular prism. Moreover, if Γ is not trivalent, there may be vertices that can fire more than one chip on each edge, so the firing poset may be strictly larger and so can the dimension of the order complex.

Example 26. (K_4 continued)

Let Γ be a tropical curve with the complete graph on four vertices as a model, with arbitrary edge lengths. Consider the canonical divisor K. In this case, the firing poset coincides with the lattice of unions of circuits, which is anti-isomorphic to the lattice of flats. Hence the link of the canonical divisor is isomorphic to the Bergman complex of the cographic matroid on the complete graph. Since the complete graph on four vertices is self-dual, its co-Bergman complex is the space of trees on five taxa, which is the Petersen graph [AK06].

See Figure 1. In the case when all edge lengths are equal, the quadrangles of $|K|$ described in Example 20 are subdivided in this fine subdivision of the $\operatorname{link}(K, |K|)$. Note that the link of the canonical divisor stays the same when we vary the edge lengths, while the generators and cell structure of $R(K)$ may change.

5 The induced map and projective embedding of a tropical curve

A finite set $\mathcal{F} = (f_1, \dots, f_r) \subset R(D)$ induces a map $\phi_{\mathcal{F}} \colon \Gamma \to \mathbb{TP}^{r-1}$, defined as $\phi_{\mathcal{F}}(x) = (f_1(x), \dots, f_r(x))$ for each $x \in \Gamma$. This is a map into $\mathbb{TP}^{r-1}$ rather than $\mathbb{R}^r$ as we take $\mathcal{F}$ to be defined up to translation by $\mathbb{1}$.

Theorem 27. *Let $\langle \mathcal{F} \rangle \subset R(D)$ be the tropical sub-semimodule of $R(D)$ generated by $\mathcal{F}$. Then $\langle \mathcal{F} \rangle / \mathbb{1}$ is homeomorphic to the tropical convex hull of the image of $\phi_{\mathcal{F}}$. In particular, if $\mathcal{F}$ generates $R(D)$, then $|D|$ is homeomorphic to the tropical convex hull of $\phi_{\mathcal{F}}(\Gamma)$.*

The *tropical convex hull* of a set is the tropical semi-module generated by the set.

Proof. The intuition behind this theorem is the result from [DS04] that the tropical convex hull of the rows of a matrix is isomorphic to the tropical convex hull of the columns. Here, the matrix $M_{\mathcal{F}}$ in question has entry $f_i(x)$ in row i and column x. As in [DS04], we define a convex set

$$P_{\mathcal{F}} = \{(y,z) \in (\mathbb{R}^r \times \mathbb{R}^\Gamma)/(\mathbb{1}, -\mathbb{1}) \,:\, y_i + z(x) \geq f_i(x)\}.$$

Let $B_{\mathcal{F}}$ be the union of bounded faces of $P_{\mathcal{F}}$, i.e. $B_{\mathcal{F}}$ contains points in the boundary of $P_{\mathcal{F}}$ that do not lie in the relative interior of an unbounded face of $P_{\mathcal{F}}$ in $(\mathbb{R}^r \times \mathbb{R}^\Gamma)/(\mathbb{1}, -\mathbb{1})$. We will show that $B_{\mathcal{F}}$ projects bijectively onto $\langle \mathcal{F} \rangle/\mathbb{1} \subset \mathbb{R}^\Gamma/\mathbb{1}$ on the one hand, and to $\operatorname{tconv} \phi_{\mathcal{F}}(\Gamma) \subset \mathbb{TP}^{r-1}$ on the other, establishing a homeomorphism. As in [DS04], we associate a *type* to $(y,z) \in P_{\mathcal{F}}$ as follows:

$$\operatorname{type}(y,z) := \{(i,x) \in [r] \times \Gamma : y_i + z(x) = f_i(x)\}.$$

In other words, a type is a collection of defining hyperplanes that contains (y,z), so elements in the relative interior of the same face have the same type. The recession cone of $P_{\mathcal{F}}$ is $\{(y,z) \in (\mathbb{R}^r \times \mathbb{R}^\Gamma)/(\mathbb{1}, -\mathbb{1}) \,:\, y_i + z(x) \geq 0\}$, which is the quotient of the positive orthant in $(\mathbb{R}^r \times \mathbb{R}^\Gamma)$ by $(\mathbb{1}, -\mathbb{1})$. Hence, a point $(y,z) \in P_{\mathcal{F}}$ lies in $B_{\mathcal{F}}$ if and only if we cannot add arbitrary positive multiples of any coordinate direction to it while staying in the same face of $P_{\mathcal{F}}$, which means keeping the same type. This holds if and only if

(1) The projection of $\operatorname{type}(y,z)$ onto $[r]$ is surjective, and

(2) The projection of $\operatorname{type}(y,z)$ onto Γ is surjective.

For $(y,z) \in P_{\mathcal{F}}$, these two conditions are equivalent respectively to

(1′) $y_i = \max\{f_i(x) - z(x) : x \in \Gamma\}$ for all $i \in [r]$, i.e. $y = M_{\mathcal{F}} \odot -z$, and

(2′) $z(x) = \max\{f_i(x) - y_i : i \in [r]\}$ for all $x \in \Gamma$, i.e. $z = -y \odot M_{\mathcal{F}}$.

where $M_{\mathcal{F}}$ is the $[r] \times \Gamma$ matrix with entry $f_i(x)$ in row i and column x, and $\odot$ is tropical matrix multiplication. These two conditions respectively imply that the projections of $B_{\mathcal{F}}$ onto $\mathbb{R}^\Gamma/\mathbb{1}$ and $\mathbb{R}^r/\mathbb{1}$ are one-to-one.

On the other hand, let $z \in \langle \mathcal{F} \rangle$, then $z = (u_1 \odot f_1) \oplus \cdots \oplus (u_r \odot f_r) = u \odot M_{\mathcal{F}}$ for some $u \in \mathbb{R}^r$ such that $z \geq u_i \odot f_i$ for each $i = 1, 2, \ldots, r$. Let $y \in \mathbb{R}^r$ such that $y_i = \min\{c : z \geq -c \odot f_i\}$ for $i = 1, 2, \ldots, r$; then $z = -y \odot M_{\mathcal{F}}$, so (y,z) satisfies (2′). Moreover, by construction, $-y_i \odot f_i(x) = z(x)$ for some x, so (y,z) satisfies (1). Thus $(y,z) \in B_{\mathcal{F}}$ and the set $B_{\mathcal{F}}$ projects surjectively onto $\langle \mathcal{F} \rangle/\mathbb{1} \subset \mathbb{R}^\Gamma/\mathbb{1}$. The image under the projection onto $\mathbb{R}^r/\mathbb{1}$ is the tropical convex hull of $\operatorname{image}(\phi_{\mathcal{F}})$, and the homeomorphism follows. □

Remark 28. All of the bounded faces of the convex set $P_{\mathcal{F}}$ are in fact vertices. If the union of bounded faces $B_{\mathcal{F}}$ contained a non-trivial line segment, then its projection $\langle \mathcal{F} \rangle/\mathbb{1}$ would as well, contradicting Lemma 11.

Example 29 (Circle, degree 3 divisor)**.** Let Γ be a circle of circumference 3, identified with $\mathbb{R}/3\mathbb{Z}$ and let D be the degree 3 divisor $[0] + [1] + [2]$. Let $f_0, f_1, f_2 \in R(D)$ be the extremals corresponding to divisors $3 \cdot [0], 3 \cdot [1]$, and $3 \cdot [2]$ respectively, and suppose $f_i([i]) = -1$ for each $i = 0, 1, 2$. Then the image of Γ under $\phi_{\mathcal{F}}$, for $\mathcal{F} = (f_0, f_1, f_2)$ is a union of three line segments between the points

$$\phi_{\mathcal{F}}([0]) = (-1,0,0), \quad \phi_{\mathcal{F}}([1]) = (0,-1,0), \quad \phi_{\mathcal{F}}([2]) = (0,0,-1) \quad \text{in } \mathbb{TP}^3.$$

In this case, the (max-) tropical convex hull of the image of $\phi_{\mathcal{F}}$ coincides with the usual convex hull and is a triangle. However, it is not the tropical convex hull of any proper subset of $\operatorname{image}(\phi_{\mathcal{F}})$. In particular, $|D|$ is not a tropical polytope, i.e. it is not the tropical convex hull of a finite set of points.

We know from [DS04] that tropically convex sets are contractible.

Corollary 30. *The sets $|D|$ and $R(D)$ are contractible.*

Tropical linear spaces are tropically convex [Spe08], so any tropical linear space containing the image $\phi_{\mathcal{F}}(\Gamma)$ must also contain its tropical convex hull.

Corollary 31. *Any tropical linear space in $\mathbb{TP}^{r-1}$ containing $\phi_{\mathcal{F}}(\Gamma)$ has dimension at least dim$(\langle \mathcal{F} \rangle)$.*

6 Conclusions and Open Questions

In this paper, we presented a number of properties of $|D|$ including verification that it is finitely generated as a tropical semi-module. We also provided some tools for explicitly understanding $|D|$ as a polyhedral cell complex such as a formula for the dimension of the face containing a given point, as well as applications such as using $|D|$ to embed an abstract tropical curve into tropical projective space.

There are many ways to continue this research for the future. It is quite tantalizing to investigate how the Baker-Norine rank of a divisor compares with the geometry and combinatorics of the associated linear system as a polyhedral cell complex. Also, is there any relation between $r(D)$ and the minimal number of generators of $R(D)$? How does the structure of $|D|$ change as we continuously move one point in the support of D or if we change the edge lengths of our metric graph while keeping the combinatorial type of the graph fixed?

In the case of finite graphs, i.e. divisors whose support lies within the set of vertices of the graph, can we combinatorially describe the associated linear systems? For example, is there a stabilization or an associated Ehrhart theory that one can use to count the sizes of such linear systems? Lastly, what other results from classical algebraic curve theory carry over to the theory of metric graphs (or tropical curves) and vice-versa?

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