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The Geometry of Lecture Hall Partitions and Quadratic Permutation Statistics

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Abstract. We take a geometric view of lecture hall partitions and anti-lecture hall compositions in order to settle some open questions about their enumeration. In the process, we discover an intrinsic connection between these families of partitions and certain quadratic permutation statistics. We define some unusual quadratic permutation statistics and derive results about their joint distributions with linear statistics. We show that certain specializations are equivalent to the lecture hall and anti-lecture hall theorems and another leads back to a special case of a Weyl group generating function that "ought to be better known."

Résumé. Nous regardons géométriquement les partitions amphithéâtre et les compositions planétarium afin de résoudre quelques questions énumératives ouvertes. Nous découvrons un lien intrinsèque entre ces familles des partitions et certaines statistiques quadratiques de permutation. Nous définissons quelques statistiques quadratiques peu communes des permutations et dérivons des résultats sur leurs distributions jointes avec des statistiques linéaires. Nous démontrons que certaines spécialisations sont équivalentes aux théorèmes amphithéâtre et planétarium. Une autre spécialisation mène à un cas spécial de la série génératrice d'un groupe de Weyl qui "devrait être mieux connue".

Keywords: lecture hall partitions, anti-lecture hall compositions, permutation statistics, lattice point enumeration, generating functions

1 Introduction

A *lecture hall partition* of length n is an integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ [BME97] satisfying

$$0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \ldots \leq \frac{\lambda_n}{n}.$$

An *anti-lecture hall composition* of length n is an integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ [CS03] satisfying

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \ldots \geq \frac{\lambda_n}{n} \geq 0.$$

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These intriguing combinatorial objects and their various generalizations have been the subject of several papers and they have been shown to be related to Bott's formula in the theory of affine Coxeter groups [BME97, BME99], Euler's partition theorem [BME97, Yee01, SY08], the Gaussian polynomials [CLS07, CS04], the q-Chu-Vandermonde Identities [CLS07, CS04], the q-Gauss summation [ACS09], and the little Göllnitz partition theorems [CSS09]. In this paper we regard them from the point of view of lattice point enumeration and uncover several new results and connections.

The set $\mathbb{Z}^n \subset \mathbb{R}^n$ is the n-dimensional integer lattice and its elements are called lattice points. $\mathbb{Z}^n_{\geq 0}$ denotes the set of lattice points with all coordinates nonnegative. So, lecture hall partitions and anti-lecture hall compositions of length n can be viewed as lattice points in $\mathbb{Z}^n_{\geq 0}$.

Let L_n be the set of lecture hall partitions of length n and A_n, the set of anti-lecture hall compositions of length n. Define the subsets $L_n^{(t)}$ and $A_n^{(t)}$ by the constraints:

$$L_n^{(t)}: \quad 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \ldots \leq \frac{\lambda_n}{n} \leq t$$

and

$$A_n^{(t)}: \quad t \geq \frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \ldots \geq \frac{\lambda_n}{n} \geq 0.$$

The following was shown in [CLS07]

Theorem 1.1 *For integer $t \geq 0$,*

$$\left| L_n^{(t)} \right| = (t+1)^n = \left| A_n^{(t)} \right|.$$

Let Q_t^n denote the lattice points in the n-dimensional cube of width t:

$$Q_t^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n_{\geq 0} \mid \ 0 \leq x_i \leq t, \ \ 1 \leq i \leq n\}.$$

Matthias Beck observed [Bec09] that since also $|Q_t^n| = (t+1)^n$, there should be some natural bijections with $L_n^{(t)}$ and $A_n^{(t)}$.

In Section 2, we prove two simple bijections

$$\Theta : \mathbb{Z}^n_{\geq 0} \to L_n$$

and

$$\Phi : \mathbb{Z}^n_{\geq 0} \to A_n$$

with the property that for every $t \geq 0$,

$$\Theta^{-1}(L_n^{(t)}) = Q_n^t = \Phi^{-1}(A_n^{(t)}).$$

Previously, a bijection between $L_n^{(t)}$ and $A_n^{(t)}$ was known [CLS07], but it depended on t, it did not extend to L_n and A_n, and it did not explain the relationship between their generating functions. In contrast, a new bijection $L_n \to A_n$ reveals the functional relationship between their generating functions and restricts to a bijection between $L_n^{(t)}$ and $A_n^{(t)}$. What emerges is a characterization of L_n and A_n in terms of (new) permutation statistics.

In Section 3, we use the bijections Θ and Φ to derive generating functions for L_n and A_n in terms of permutation statistics and show how to derive one from the other. Similar ideas underlie the computation of the refined generating function for L_n in [BME99] and A_n in [CS04], but the connection with permutation statistics, a key ingredient in the relationship between L_n and A_n, was missed.

In Section 4, we show how the generating functions derived in Section 3 imply new results about distributions of quadratic permutation statistics and connections with affine Coxeter groups.

2 The bijections

The bijections between points in the cube and the lecture hall partitions and anti-lecture hall compositions have simple descriptions in terms of permutations and their inversion sequences, so we first review some notation and results.

2.1 Permutation statistics and stable sorting

Let S_n be the set of permutations of $\{1, 2, \ldots, n\}$. For $\pi \in S_n$, an *inversion* of π is a pair (i, j) such that $i < j$, but $\pi_i > \pi_j$. The number of inversions of π is denoted $\mathrm{inv}(\pi)$. A *descent* of π is a position i such that $1 \leq i < n$ and $\pi_i > \pi_{i+1}$. The set of all descents of π is denoted $\mathrm{Des}(\pi)$ and its size is $\mathrm{des}(\pi) = |\mathrm{Des}(\pi)|$.

Define the *inversion sequence* of π as the sequence $\epsilon(\pi) = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, where ϵ_i is the number of elements of $\{1, \ldots, n\}$ to the right of i, in π, which are smaller than i. Then $\mathrm{inv}(\pi) = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$.

It is well-known ([Knu73], p. 12) that the mapping $\pi \to \epsilon(\pi)$ is a bijection between S_n and integer sequences I_n, where

$$I_n = \{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \mid 0 \leq \epsilon_i < i\}.$$

For $\pi \in S_n$, although in general $\epsilon(\pi) \neq \epsilon(\pi^{-1})$, it is known that ([Knu73], p. 14-15):

$$\mathrm{inv}(\pi) = \mathrm{inv}(\pi^{-1}). \tag{1}$$

A permutation $\pi \in S_n$ *stably sorts* a sequence $s = (s_1, \ldots, s_n)$ into weakly increasing order if

$$s_{\pi_1} \leq s_{\pi_2} \leq \ldots \leq s_{\pi_n}$$

and if $i \in \mathrm{Des}(\pi)$ then $s_{\pi_i} < s_{\pi_{i+1}}$, that is, equal elements of s retain their relative order. For every sequence s of length n there is a unique $\pi \in S_n$ such that π stably sorts s into weakly increasing order.

Let $(w_1 \leq w_2 \leq \ldots \leq w_n)$ denote a weakly increasing sequence and $(w_1 \geq w_2 \geq \ldots \geq w_n)$ a weakly decreasing sequence. For a sequence $s = (s_1, \ldots, s_n)$ and $\pi \in S_n$, define $\pi(s)$ by $\pi(s) = (s_{\pi_1}, s_{\pi_2}, \ldots, s_{\pi_n})$.

Define

$$S_n/(w_1 \leq w_2 \leq \ldots \leq w_n) = \{\pi \in S_n \mid \text{if } i \in \mathrm{Des}(\pi^{-1}) \text{ then } w_i < w_{i+1}\}$$

and

$$S_n/(w_1 \geq w_2 \geq \ldots \geq w_n) = \{\pi \in S_n \mid \text{if } i \in \mathrm{Des}(\pi^{-1}) \text{ then } w_i > w_{i+1}\}$$

Informally, $\pi \in S_n/w$ iff π^{-1} is the unique permutation in S_n that stably sorts $\pi(w)$ into w.

Define

$$I_n/w = \{\epsilon \in I_n \mid \text{if } w_i = w_{i+1} \text{ then } \epsilon_i \geq \epsilon_{i+1}\}.$$

It is straightforward to prove the following lemma, which characterizes the multiset permutations of $\{w_1, \ldots, w_n\}$ in terms of their inversion sequences.

Lemma 2.1 *Given $w = (w_1 \leq w_2 \leq \ldots \leq w_n)$ or $w = (w_1 \geq w_2 \geq \ldots \geq w_n)$, the mapping $\pi \to \epsilon(\pi)$ on S_n restricts to a bijection between S_n/w and I_n/w. The mapping $\pi \to \pi(w)$ is a bijection between S_n/w and (distinguishable) permutations of w. In particular, there is a bijection between permutations of w and inversion sequences in I_n/w.*

2.2 The bijection for lecture hall partitions

Bijection $\Theta : \mathbb{Z}^n_{\geq 0} \to L_n$:

For $p \in \mathbb{Z}^n_{\geq 0}$, define $\Theta(p)$ as follows:

1. Let π^{-1} be the unique permutation that stably sorts p into weakly increasing order $(w_1 \leq w_2 \leq ... \leq w_n) = \pi^{-1}(p)$
2. Let $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ be the inversion sequence of π

Then $\Theta(p) = \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_i = iw_i - \epsilon_i, \quad i = 1, 2, \ldots, n.$

Example 2.1 Let $p = (9, 0, 3, 3, 5, 4, 3, 8, 1, 8, 2, 9) \in \mathbb{Z}^{12}_{\geq 0}$. Then $w = (0, 1, 2, 3, 3, 3, 4, 5, 8, 8, 9, 9)$ and $\pi = (11, 1, 4, 5, 8, 7, 6, 9, 2, 10, 3, 12)$ and $\epsilon(\pi) = (0, 0, 0, 2, 2, 2, 3, 4, 2, 1, 10, 0)$. So

$$\begin{aligned}
\Theta(p) &= \lambda \\
&= (0-0, 2-0, 6-0, 12-2, 15-2, 18-2, 28-3, 40-4, 72-2, 80-1, 99-10, 108-0) \\
&= (0, 2, 6, 10, 13, 16, 25, 36, 70, 79, 89, 108).
\end{aligned}$$

To check that $\Theta(p) = \lambda \in L_n$, verify that

$$0 \leq \frac{0}{1} \leq \frac{2}{2} \leq \frac{6}{3} \leq \frac{10}{4} \leq \frac{13}{5} \leq \frac{16}{6} \leq \frac{25}{7} \leq \frac{36}{8} \leq \frac{70}{9} \leq \frac{79}{10} \leq \frac{89}{11} \leq \frac{108}{12}.$$

Also, note that $p \in Q^{12}_9$, since its largest coordinate is 9 and that $\Theta(p) = \lambda \in L^{(9)}_n$, since $\lambda_{12}/12 = (108)/(12) \leq 9$.

Theorem 2.2 *Θ is a bijection between lattice points in $\mathbb{Z}^n_{\geq 0}$ and lecture hall partitions of length n. In fact, $\Theta(Q^n_t) = L^{(t)}_n$.*

Proof: First, to prove $\Theta(Q^n_t) \subseteq L^{(t)}_n$, let $p \in Q^n_t$ and $\lambda = \Theta(p)$. From the definition of Θ, $\lambda_i = iw_i - \epsilon_i$, where $w = (w_1 \leq w_2 \leq \ldots \leq w_n)$ is the sorted sequence of coordinates of p and $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) = \epsilon(\pi)$ for the unique $\pi \in S_n/w$ with $\pi(w) = p$. By Lemma 2.1, $\epsilon(\pi) \in I_n/w$, so $0 \leq \epsilon_i < i$ and if $w_i = w_{i+1}$, then $\epsilon_i \geq \epsilon_{i+1}$.

To show that $\lambda \in L_n^{(t)}$, we must show that

$$0 \leq \frac{1w_1 - \epsilon_1}{1} \leq \ldots \leq \frac{iw_i - \epsilon_i}{i} \leq \frac{(i+1)w_{i+1} - \epsilon_{i+1}}{i+1} \leq \ldots \leq \frac{nw_n - \epsilon_n}{n} \leq t.$$

Clearly, the first inequality holds, since $w_1 \geq 0$ and $\epsilon_1 = 0$. Also, since $p \in Q_t^n$, $w_n \leq t$, so the last inequality holds.

To show $\frac{iw_i - \epsilon_i}{i} \leq \frac{(i+1)w_{i+1} - \epsilon_{i+1}}{i+1}$, consider the relationship between w_i and w_{i+1}. If $w_i = w_{i+1}$, then since $\epsilon_i \geq \epsilon_{i+1}$,

$$\frac{iw_i - \epsilon_i}{i} = w_{i+1} - \frac{\epsilon_i}{i} \leq w_{i+1} - \frac{\epsilon_{i+1}}{i} \leq w_{i+1} - \frac{\epsilon_{i+1}}{i+1} = \frac{(i+1)w_{i+1} - \epsilon_{i+1}}{i+1}.$$

Otherwise, $w_{i+1} \geq w_i + 1$, so since $0 \leq \epsilon_{i+1} < i + 1$,

$$\frac{(i+1)w_{i+1} - \epsilon_{i+1}}{i+1} \geq w_i + 1 - \frac{\epsilon_{i+1}}{i+1} \geq w_i + 1 - \frac{i}{i+1} > w_i \geq \frac{iw_i - \epsilon_i}{i}.$$

To complete the proof that $\Theta(Q_t^n) = L_n^{(t)}$, since by Theorem 1.1, $|L_n^{(t)}| = |Q_t^n|$, it suffices to show that Θ is one-to-one. Suppose $\Theta(p) = \lambda = \Theta(r)$ for $p, r \in Q_t^n$. Then $\lambda = (w_1 - \epsilon_1, \ldots, iw_i - \epsilon_i, \ldots, nw_n - \epsilon_n)$ for some $w = (w_1 \leq w_2 \leq \ldots \leq w_n)$ and ϵ satisfying $\epsilon \in I_n/w$, and in particular, $0 \leq \epsilon_i < i$. But this determines w uniquely as

$$w = (\lceil \lambda_1/1 \rceil, \lceil \lambda_2/2 \rceil, \ldots, \lceil \lambda_n/n \rceil) \tag{2}$$

and thus ϵ uniquely as

$$\epsilon_i = iw_i - \lambda_i.$$

There is a unique permutation $\pi \in S_n$ with inversion sequence ϵ and by Lemma 2.1, $\pi \in S_n/w$. Then, by the definition of Θ, $\pi(w) = p$ and $\pi(w) = r$. Thus $p = r$ and therefore Θ is a bijection. □

2.3 The bijection for anti-lecture hall compositions

Bijection $\Phi : \mathbb{Z}_{\geq 0}^n \to L_n$:

For $p \in \mathbb{Z}_{\geq 0}^n$, define $\Phi(p)$ as follows:

1. Let π^{-1} be the unique permutation that stably sorts p into weakly decreasing order $(w_1 \geq w_2 \geq ... \geq w_n) = \pi^{-1}(p)$
2. Let $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ be the inversion sequence of π

Then $\Phi(p) = \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_i = iw_i + \epsilon_i, \quad i = 1, 2, \ldots, n.$

Example 2.2 Let $p = (9, 0, 3, 3, 5, 4, 3, 8, 1, 8, 2, 9) \in \mathbb{Z}_{\geq 0}^{12}$. Then $w = (9, 9, 8, 8, 5, 4, 3, 3, 3, 2, 1, 0)$ and $\pi = (1, 12, 7, 8, 5, 6, 9, 3, 11, 4, 10, 2)$ and $\epsilon(\pi) = (0, 0, 1, 1, 3, 3, 5, 5, 3, 1, 3, 10)$. So

$$\Phi(p) = \lambda = (9, 18, 25, 33, 28, 27, 26, 29, 30, 21, 14, 10).$$

Theorem 2.3 *Φ is a bijection between lattice points in $\mathbb{Z}^n_{\geq 0}$ and anti-lecture hall compositions of length n. In fact, $\Phi(Q^n_t) = A^{(t)}_n$.*

Proof: In the same spirit as the proof of Theorem 2.2, first show that λ resulting from $\Phi(p)$ is, in fact, an anti-lecture hall composition. Then, cite Theorem 1.1 to show $\left|A^{(t)}_n\right| = |Q^n_t|$. To complete the bijective proof, show that Φ is one-to-one by observing that if $\Phi(p) = \lambda = \Phi(r)$, then because

$$w = (\lfloor \lambda_1/1 \rfloor, \lfloor \lambda_2/2 \rfloor, \ldots, \lfloor \lambda_n/n \rfloor), \tag{3}$$

we know, from Lemma 2.1 that $p = r$. □

3 Generating Functions

In this section we will derive generating functions for lecture hall partitions and anti-lecture hall compositions via the bijections Θ and Φ. We need the following additional observations about permutations.

Lemma 3.1 *If $\pi \in S_n$ stably sorts $(p_1, \ldots, p_n)$ into weakly increasing order and $\sigma \in S_n$ stably sorts $(p_n, \ldots, p_1)$ into weakly decreasing order then $\sigma_i = n + 1 - \pi_{n+1-i}$.*

Lemma 3.2 *Let $\sigma, \pi \in S_n$ be related by $\sigma_i = n + 1 - \pi_{n+1-i}$. Then their inverses are similarly related: $\sigma^{-1}_i = n + 1 - \pi^{-1}_{n+1-i}$.*

Lemma 3.3 *If $\sigma, \pi \in S_n$ are related by $\sigma_i = n+1-\pi_{n+1-i}$, then $\mathrm{des}(\sigma) = \mathrm{des}(\pi)$ and $\mathrm{inv}(\sigma) = \mathrm{inv}(\pi)$.*

For a point $p \in \mathbb{Z}^n_{\geq 0}$, the *weight* of p is $|p| = p_1 + \ldots + p_n$. For $\lambda \in A_n$, let

$$\lfloor \lambda \rfloor = (\lfloor \lambda_1/1 \rfloor, \lfloor \lambda_2/2 \rfloor, \ldots, \lfloor \lambda_n/n \rfloor).$$

Note from (3) that, for $\lambda \in A_n$,

$$|\lfloor \lambda \rfloor| = |\Phi^{-1}(\lambda)|.$$

Similarly, for $\lambda \in L_n$, let

$$\lceil \lambda \rceil = (\lceil \lambda_1/1 \rceil, \lceil \lambda_2/2 \rceil, \ldots, \lceil \lambda_n/n \rceil).$$

Then from (2) for $\lambda \in L_n$,

$$|\lceil \lambda \rceil| = |\Theta^{-1}(\lambda)|.$$

Define

$$L_n(u, q) = \sum_{\lambda \in L_n} u^{|\lceil \lambda \rceil|} q^{|\lambda|} \quad \text{and} \quad A_n(u, q) = \sum_{\lambda \in A_n} u^{|\lfloor \lambda \rfloor|} q^{|\lambda|}.$$

It was shown in [BME99] that

$$L_n(u, q) = \prod_{i=1}^{n} \frac{1 + uq^i}{1 - u^2 q^{n+i}} \tag{4}$$

and in [CS04] that

$$A_n(u,q) = \prod_{i=1}^{n} \frac{1+uq^i}{1-u^2q^{1+i}}. \tag{5}$$

Although a relationship between $A_n(u,q)$ and $L_n(u,q)$ can be deduced from (4) and (5), each generating function was derived independently and until now the relationship could not be explained combinatorially.

From Theorem 2.2, the mapping $p \to \Theta(p)$ is a bijection $\mathbb{Z}^n_{\geq 0} \to L_n$ and if $\lambda = \Theta(p)$ then $|p| = |\lceil \lambda \rceil|$. Thus

$$L_n(u,q) = \sum_{p \in \mathbb{Z}^n_{\geq 0}} u^{|p|} q^{|\Theta(p)|}.$$

Similarly, from Theorem 2.3, the mapping $p \to \Phi(p)$ is a bijection $\mathbb{Z}^n_{\geq 0} \to A_n$ and if $\lambda = \Phi(p)$ then $|p| = |\lfloor \lambda \rfloor|$. So,

$$A_n(u,q) = \sum_{p \in \mathbb{Z}^n_{\geq 0}} u^{|p|} q^{|\Phi(p)|}.$$

For the first time we are able to show that $L_n(u,q)$ can be derived from $A_n(u,q)$. Define the reverse of a sequence $s = (s_1, s_2, \ldots, s_n)$ by $s^{\text{rev}} = (s_n, s_{n-1}, \ldots, s_1)$.

Theorem 3.4

$$L_n(u,q) = A_n(uq^{n+1}, q^{-1}).$$

Proof: For $p \in \mathbb{Z}^n_{\geq 0}$, we compare the contribution of $\Phi(p)$ to $A_n(u,q)$ with the contribution of $\Theta(p^{\text{rev}})$ to $L_n(u,q)$. Let π^{-1} be the permutation that stably sorts p into weakly decreasing order $w = (w_1 \geq \ldots \geq w_n)$ and let σ^{-1} be the permutation that stably sorts p^{rev} into weakly increasing order w^{rev}. Then

$$\begin{aligned}
\Phi(p) &= (w_1 + \epsilon_1(\pi), 2w_2 + \epsilon_2(\pi), \ldots, nw_n + \epsilon_n(\pi)) \\
\Theta(p^{\text{rev}}) &= (w_n - \epsilon_1(\sigma), 2w_{n-1} - \epsilon_2(\sigma), \ldots, nw_1 - \epsilon_n(\sigma)).
\end{aligned}$$

So

$$u^{|p|} q^{|\Phi(p)|} = u^{|w|} q^{\sum_i^n i w_i} q^{\text{inv}(\pi)}$$

and

$$\begin{aligned}
u^{|p^{\text{rev}}|} q^{|\Theta(p^{\text{rev}})|} &= u^{|w^{\text{rev}}|} q^{\sum_i^n (n+1-i) w_i} q^{-\text{inv}(\sigma)} \\
&= (uq^{n+1})^{|w|} q^{-\sum_i^n i w_i} q^{-\text{inv}(\sigma)} = (uq^{n+1})^{|p|} q^{-|\Phi(p)|},
\end{aligned}$$

where we have used $\text{inv}(\sigma) = \text{inv}(\pi)$ from Lemma 3.3. Note finally that summing over all $p \in \mathbb{Z}^n_{\geq 0}$ is equivalent to summing over all $p^{\text{rev}} \in \mathbb{Z}^n_{\geq 0}$, so

$$\begin{aligned}
L_n(u,q) = \sum_{p \in \mathbb{Z}^n_{\geq 0}} u^{|p|} q^{|\Theta(p)|} &= \sum_{p^{\text{rev}} \in \mathbb{Z}^n_{\geq 0}} u^{|p^{rev}|} q^{|\Theta(p^{rev})|} \\
&= \sum_{p \in \mathbb{Z}^n_{\geq 0}} (uq^{n+1})^{|p|} q^{-|\Phi(p)|} = A_n(uq^{n+1}, q^{-1}).
\end{aligned}$$

□

Now we derive the generating function for A_n in terms of permutation statistics.

Theorem 3.5

$$A_n(u,q) = \sum_{\pi \in S_n} \frac{q^{\mathrm{inv}(\pi)} \prod_{i \in \mathrm{Des}(\pi)} u^i q^{i(i+1)/2}}{(1-uq)(1-u^2q^{1+2}) \cdots (1-u^n q^{1+2+\ldots+n})}$$

Proof: For $T \subseteq \mathbb{Z}^n_{\geq 0}$, define F_T by

$$F_T(u, z_1, \ldots, z_n) = \sum_{\lambda \in T} u^{|\lambda|} z_1^{\lambda_1} z_2^{\lambda_2} \ldots z_n^{\lambda_n}.$$

Given $D \subseteq \{1, 2, \ldots, n-1\}$, define

$$S_D = \{(w_1 \geq w_2 \geq \ldots \geq w_n) \in \mathbb{Z}^n_{\geq 0} \mid w_i > w_{i+1} \text{ if } i \in D\}.$$

Then

$$F_{S_D}(u, z_1, \ldots, z_n) = \frac{\prod_{i \in D} u^i z_1 z_2 \cdots z_i}{(1-uz_1)(1-u^2 z_1 z_2) \cdots (1-u^n z_1 z_2 \cdots z_n)}.$$

We count A_n from $\mathbb{Z}^n_{\geq 0}$ via Φ. Use the permutations $\pi \in S_n$ to partition the points $p \in \mathbb{Z}^n_{\geq 0}$ into sets T_π defined by

$$T_\pi = \{p \mid p = \pi(w_1 \geq w_2 \geq \ldots \geq w_n) \text{ such that } i \in \mathrm{Des}(\pi^{-1}) \rightarrow w_i > w_{i+1}\}.$$

So, we are partitioning the points according to the permutation π such that π^{-1} stably sorts p into weakly decreasing order. The bijection $\Phi : \mathbb{Z}^n_{\geq 0} \rightarrow A_n$ does the following to the points in T_π: They are first mapped onto the points in $S_{\mathrm{Des}(\pi^{-1})}$. Then for each i, the ith coordinate is multiplied by i and added to $\epsilon_i(\pi)$. So in the generating function

$$z_1^{\epsilon_1(\pi)} \cdots z_n^{\epsilon_n(\pi)} F_{S_{\mathrm{Des}(\pi^{-1})}}(u, z_1, z_2^2, \ldots, z_n^n),$$

u keeps track of the weight of $p \in T_\pi$ and the variables z_i track the weight of $\Phi(p)$. Putting this together,

$$\begin{aligned} A_n(u, z_1, \ldots, z_n) &= \sum_{\lambda \in A_n} u^{|\lfloor \lambda \rfloor|} z_1^{\lambda_1} \cdots z_n^{\lambda_n} = \sum_{\pi \in S_n} \sum_{p \in T_\pi} u^{|p|} z_1^{\Theta(p)_1} \cdots z_n^{\Theta(p)_n} \\ &= \sum_{\pi \in S_n} \sum_{p \in T_\pi} z_1^{\epsilon_1(\pi)} \cdots z_n^{\epsilon_n(\pi)} F_{S_{\mathrm{Des}(\pi^{-1})}}(u, z_1, z_2^2, \ldots, z_n^n) \\ &= \sum_{\pi \in S_n} \frac{z_1^{\epsilon_1(\pi)} \cdots z_n^{\epsilon_n(\pi)} \prod_{i \in \mathrm{Des}(\pi^{-1})} u^i z_1 z_2^2 \cdots z_i^i}{(1-uz_1)(1-u^2 z_1 z_2^2) \cdots (1-u^n z_1 z_2^2 \cdots z_n^n)}. \end{aligned}$$

Setting all $z_i = q$, and using (1), which states that $\mathrm{inv}(\pi) = \mathrm{inv}(\pi^{-1})$,

$$A_n(u,q) = \sum_{\pi \in S_n} \frac{q^{\mathrm{inv}(\pi)} \prod_{i \in \mathrm{Des}(\pi)} u^i q^{i(i+1)/2}}{(1-uq)(1-u^2q^{1+2}) \cdots (1-u^n q^{1+2+\ldots+n})}.$$

□

Theorem 3.6

$$L_n(u,q) = \sum_{\pi \in S_n} \frac{q^{-\mathrm{inv}(\pi)} \prod_{i \in \mathrm{Des}(\pi)} (uq^{(n+1)})^i q^{-i(i+1)/2}}{\prod_{i=1}^{n} (1 - u^i q^{i(n+1) - i(i+1)/2})}.$$

Proof: From Theorem 3.4, $L_n(u,q) = A_n(uq^{n+1}, q^{-1})$, so apply Theorem 3.5. □

Combining Theorems 3.5 and 3.6 with equations (4) and (5) will have implications about the distribution of certain permutation statistics, discussed in the next section.

4 Quadratic Permutation Statistics

Define the q-integer $[\ n\ ]_q$ by $[\ n\ ]_1 = 1$ and for $q \neq 1$, by $[\ n\ ]_q = (1-q^n)/(1-q)$. In Section 2.1 we defined the permutation statistics inv and des. The *major index* of $\pi \in S_n$ is the sum of the descent positions: $\mathrm{maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i$. It is known that

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = \prod_{i=1}^{n} [\ i\ ]_q, \tag{6}$$

and that inv and maj are equally distributed over all permutations [Mac60].

Motivated by Theorems 3.5 and 3.6, we introduce quadratic permutation statistics sq and bin:

$$\mathrm{sq}(\pi) \ = \sum_{i \in \mathrm{Des}(\pi)} i^2 \qquad \text{and} \qquad \mathrm{bin}(\pi) \ = \sum_{i \in \mathrm{Des}(\pi)} \binom{i+1}{2}.$$

Because of the way "inv" is involved with the distribution of these quadratic statistics, we also define

$$\begin{aligned} \mathrm{sqin}(\pi) &= \mathrm{sq}(\pi) + \mathrm{inv}(\pi) \\ \mathrm{binv}(\pi) &= \mathrm{bin}(\pi) + \mathrm{inv}(\pi) \end{aligned}$$

and prove two distribution theorems that refine (6). The first comes from the enumeration of anti-lecture hall compositions.

Theorem 4.1

$$\sum_{\pi \in S_n} u^{\mathrm{maj}(\pi)} q^{\mathrm{binv}(\pi)} = \prod_{i=1}^{n} (1 - u^i q^{\binom{i+1}{2}}) \frac{1 + uq^i}{1 - u^2 q^{1+i}}.$$

Proof: Restate the generating function for $A_n(u,q)$ in Theorem 3.5 in terms of the new permutation statistics and apply equation (5). □

Setting $q = 1$ in Theorem 4.1 gives (6). Setting $u = 1$ in Theorem 4.1 gives the following interesting generating function for the symmetric group.

Corollary 4.2

$$\sum_{\pi \in S_n} q^{\mathrm{binv}(\pi)} = \prod_{i=1}^{n} [\,2\,]_{q^i} \frac{[\,i(i+1)/2\,]_q}{[\,i+1\,]_q}$$

In Theorem 4.1, setting $q = q^{-1}$ and then $u = q^{n+1}$ gives an unusual variation of (6).

Corollary 4.3

$$\sum_{\pi \in S_n} q^{(n+1)\mathrm{maj}(\pi) - \mathrm{binv}(\pi)} = \prod_{i=1}^{n} [\,i\,]_{q^{2(n-i)+1}}.$$

Proof: By Theorem 3.4, $A_n(u, q^{n+1}, q^{-1}) = L_n(u, q)$. Equate $L_n(u, q)$ in Theorem 3.6 and equation (4), setting $u = 1$, and simplify:

$$\sum_{\pi \in S_n} q^{(n+1)\mathrm{maj}(\pi) - \mathrm{binv}(\pi)} \quad = \quad \prod_{i=1}^{n} (1 - q^{i(2n-i+1)/2}) \prod_{i=1}^{n} \frac{1}{1 - q^{2i-1}}.$$

The result follows by observing that

$$1 - q^{i(2n-i+1)/2} \;=\; \begin{cases} 1 - q^{k(2(n-k)+1)} \;=\; [\,k\,]_{q^{2(n-k)+1}} (1 - q^{2(n-k)+1}) & \text{if } i = 2k \\ 1 - q^{(2k+1)(n-k)} \;=\; [\,n-k\,]_{q^{2k+1}} (1 - q^{2k+1}) & \text{if } i = 2k+1. \end{cases}$$

□

The second distribution theorem has the following form.

Theorem 4.4

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{sqin}(\pi)} = \prod_{i=1}^{n} [\,i\,]_{qt^i}$$

Before proving Theorem 4.4, we observe that it has the following specializations. Setting $t = 1$ in Theorem 4.4 gives (6). Setting $q = 1$ in Theorem 4.4 gives the following, which appears to be a new observation:

Corollary 4.5

$$\sum_{\pi \in S_n} t^{\mathrm{sqin}(\pi)} = \prod_{i=1}^{n} [\,i\,]_{t^i}.$$

Setting $q = q^n$ and $t = 1/q$ in Theorem 4.4 gives:

Corollary 4.6

$$\sum_{\pi \in S_n} q^{nmaj(\pi) - \mathrm{sqin}(\pi)} = \prod_{i=1}^{n} [\,i\,]_{q^{n-i}}.$$

In [SW98], Stembridge and Waugh derive a Weyl group generating function which, especially in the case of the symmetric group, they felt "ought to be better known". In [Zab03], Zabrocki gave a simple combinatorial proof of that special case, which was exactly Corollary 4.6 above. It appears that we have come full circle, since lecture hall partitions originally arose in Eriksson's work on affine Coxeter groups [BME97].

Setting $q = q^{2n+1}$ and $t = 1/q^2$ in Theorem 4.4 gives:

Corollary 4.7

$$\sum_{\pi \in S_n} q^{(2n+1)maj(\pi) - 2\mathrm{sqin}(\pi)} = \prod_{i=1}^{n} [\, i \,]_{q^{2(n-i)+1}},$$

We finish this section with a proof of Theorem 4.4, which follows the same strategy as Zabrocki's proof of Corollary 4.6 above. In contrast to the proof of Theorem 4.3, this is a direct and elementary proof which does not rely on the theory of lecture hall partitions or affine Coxeter groups. We have not as yet found a similar approach to Theorem 4.3.

Proof: (of Theorem 4.4) Expand the product in Theorem 4.4 as

$$\prod_{i=1}^{n} [\, i \,]_{qt^i} = (1)(1+qt^2)(1+qt^3+(qt^3)^2)\ldots \tag{7}$$

$$= \sum_{(r_1,\ldots,r_n)} q^{r_1+\ldots+r_n} t^{1r_1+2r_2+\ldots+nr_n} \tag{8}$$

where the sum is over the $n!$ sequences $(r_1, \ldots, r_n)$ satisfying $0 \le r < i$. So, we will establish a bijection from S_n to these sequences with the property that if π maps to $(r_1, \ldots, r_n)$, then $\mathrm{maj}(\pi) = r_1 + \ldots + r_n$ and $\mathrm{sqin}(\pi) = 1r_1 + 2r_2 + \ldots + nr_n$.

Given π, let $\epsilon = \epsilon(\pi^{-1})$ be the inversion sequence of π^{-1}. Define r by $r_i = \epsilon_i - \epsilon_{i+1} + i$ if $i \in \mathrm{Des}(\pi)$ and $r_i = \epsilon_i - \epsilon_{i+1}$, otherwise. Observe that $\epsilon_i < \epsilon_{i+1}$ if and only if $i \in \mathrm{Des}(\pi)$. By definition, $0 \le \epsilon_i < i$ for every i Thus $0 \le r_i < i$ for every i. Clearly $r_1 + \ldots + r_n = \mathrm{maj}(\pi)$ and

$$\sum_{i=1}^{n} ir_i = \sum_{i \in \mathrm{Des}(\pi)} i^2 + \sum_{i=1}^{n} i(\epsilon_i - \epsilon_{i+1}) = \mathrm{sq}(\pi) + \mathrm{inv}(\pi^{-1}) = \mathrm{sq}(\pi) + \mathrm{inv}(\pi).$$

Finally, observe that ϵ, and therefore π^{-1} and π, can be recovered from r: $\epsilon_n = r_n$ and for $i < n$, given r_i and ϵ_{i+1}, it must be that $\epsilon_i = r_i + \epsilon_{i+1}$ if $r_i + \epsilon_{i+1} < i$ and otherwise, $\epsilon_i = r_i + \epsilon_{i+1} - i$. □

5 Further directions

We mention a few questions suggested by this work. Are there other areas where quadratic permutation statistics arise naturally? Other joint distribution results? Can we give a direct and elementary proof of Theorem 4.4 on the joint distribution of maj and binv that is independent of the theory of lecture hall partitions and Weyl groups? The lecture hall theorem came from the theory of affine Coxeter groups and Bott's formula; do anti-lecture hall compositions have any place in this theory?

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References

[ACS09] George E. Andrews, Sylvie Corteel, and Carla D. Savage. On q-series identities arising from lecture hall partitions. *Int. J. Number Theory*, 5(2):327–337, 2009.

[Bec09] Matthias Beck. Personal communication, March, 2009.

[BME97] Mireille Bousquet-Mélou and Kimmo Eriksson. Lecture hall partitions. *Ramanujan J.*, 1(1):101–111, 1997.

[BME99] Mireille Bousquet-Mélou and Kimmo Eriksson. A refinement of the lecture hall theorem. *J. Combin. Theory Ser. A*, 86(1):63–84, 1999.

[CLS07] Sylvie Corteel, Sunyoung Lee, and Carla D. Savage. Enumeration of sequences constrained by the ratio of consecutive parts. *Sém. Lothar. Combin.*, 54A:Art. B54Aa, 12 pp. (electronic), 2005/07.

[CS03] Sylvie Corteel and Carla D. Savage. Anti-lecture hall compositions. *Discrete Math.*, 263(1-3):275–280, 2003.

[CS04] Sylvie Corteel and Carla D. Savage. Lecture hall theorems, q-series and truncated objects. *J. Combin. Theory Ser. A*, 108(2):217–245, 2004.

[CSS09] Sylvie Corteel, Carla D. Savage, and Andrew V. Sills. Lecture hall sequences, q-series, and asymmetric partition identities. *Developments in Mathematics*, 2009. to appear.

[Knu73] Donald E. Knuth. *The art of computer programming. Volume 3*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1973. Sorting and searching, Addison-Wesley Series in Computer Science and Information Processing.

[Mac60] Percy A. MacMahon. *Combinatory analysis*. Two volumes (bound as one). Chelsea Publishing Co., New York, 1960.

[SW98] John R. Stembridge and Debra J. Waugh. A Weyl group generating function that ought to be better known. *Indag. Math. (N.S.)*, 9(3):451–457, 1998.

[SY08] Carla D. Savage and Ae Ja Yee. Euler's partition theorem and the combinatorics of ℓ-sequences. *J. Combin. Theory Ser. A*, 115(6):967–996, 2008.

[Yee01] Ae Ja Yee. On the combinatorics of lecture hall partitions. *Ramanujan J.*, 5(3):247–262, 2001.

[Zab03] Mike Zabrocki. A bijective proof of an unusual symmetric group generating function. 2003. preprint, `arXiv:math/0910301v1`.