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Tropical secant graphs of monomial curves

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Abstract. We construct and study an embedded weighted balanced graph in $\mathbb{R}^{n+1}$ parameterized by a strictly increasing sequence of n coprime numbers $\{i_1, \ldots, i_n\}$, called the *tropical secant surface graph*. We identify it with the tropicalization of a surface in $\mathbb{C}^{n+1}$ parameterized by binomials. Using this graph, we construct the tropicalization of the first secant variety of a monomial projective curve with exponent vector $(0, i_1, \ldots, i_n)$, which can be described by a balanced graph called the *tropical secant graph*. The combinatorics involved in computing the degree of this classical secant variety is non-trivial. One earlier approach to this is due to K. Ranestad. Using techniques from tropical geometry, we give algorithms to effectively compute this degree (as well as its multidegree) and the Newton polytope of the first secant variety of any given monomial curve in $\mathbb{P}^4$.

Résumé. On construit et on étude un graphe plongé dans $\mathbb{R}^{n+1}$ paramétrisé par une suite strictement croissante de n nombres entiers $\{i_1, \ldots, i_n\}$, premiers entre eux. Ce graphe s'appelle *graphe tropical surface sécante*. On montre que ce graphe est la tropicalisation d'une surface dans $\mathbb{C}^{n+1}$ paramétrisé par des binômes. On utilise ce graphe pour construire la tropicalisation de la première sécante d'une courbe monomiale ayant comme vecteur d'exponents $(0, i_1, \ldots, i_n)$. On répresent ce variété tropical pour un graphe balancé (le *graphe tropical sécante*). La combinatoire qu'on utilise pour le calcul du degré de ces variétés sécantes classiques n'est pas triviale, et a été developé par K. Ranestad. En utilisant des techniques de la géométrie tropicale, on donne des algorithmes qui calculent le degré (même le multidegré) et le polytope de Newton de la première sécante d'une courbe monomiale de $\mathbb{P}^4$.

Keywords: monomial curves, secant varieties, resolution graphs, tropical geometry, Newton polytope

1 Introduction

In this paper, we define and study an abstract graph (the *abstract tropical secant surface graph*) which we embed in $\mathbb{R}^{n+1}$, assigning integer coordinates to each node. This graph is parameterized by a sequence of n coprime positive integers $i_1 < \ldots < i_n$. The abstract graph is constructed by gluing two caterpillar trees and several star trees, according to the combinatorics of the given integer sequence. Our embedding has a key feature: we can endow this graph with weights on all edges in such a way that it satisfies the balancing condition (Theorem 3). We call this weighted graph the *tropical secant surface graph* or *master graph* (Section 2). As the name suggests, this balanced graph is closely related to a tropical surface and it will be the cornerstone of our paper. More precisely, it is the building block for constructing the tropicalization of a threefold: the first secant variety of a monomial projective curve whose set of

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exponents is $\{0, i_1, \ldots, i_n\}$. By definition, this secant variety is the closure of the union of lines that meet the curve in two distinct points. These varieties have been studied extensively in the literature (Cox and Sidman, 2007; Ranestad, 2006). We describe this tropical connection in Section 6.

The tropicalization of the first secant variety of a monomial projective curve *strictly contains*, as a *subfan*, the set of all tropical lines between any two points in the tropicalization of the monomial curve itself, i.e. points that are obtained as coordinatewise minima of two points in the classical plane spanned by the lattice $\Lambda = \langle \mathbf{1}(0, i_1, \ldots, i_n)\rangle$. The latter is the first tropical secant variety of the corresponding classical line in the n-dimensional tropical projective torus $\mathbb{TP}^n = \mathbb{R}^{n+1}/(1, \ldots, 1)$. The union of these tropical lines is precisely the cone from the classical line $\mathbb{R}\langle(0, i_1, \ldots, i_n)\rangle$ over the pure 1-dimensional subfan of the secondary fan of the point configuration $\{0, i_1, \ldots, i_n\} \subset \mathbb{R}$ consisting of all regular subdivisions with the property that two of its facets contains all $n+1$ points. By (Theorem 3.1, Develin, 2006), we know that this subfan is precisely the cone from the plane $\mathbb{R} \otimes \Lambda$ over the complex of lower faces of the cyclic polytope $C(2, n-1)$ (i.e. $n-1$ points in dimension 2). This complex is the subgraph of the tropical secant graph consisting of the chain graph with $n-1$ nodes $E_{i_1}, \ldots, E_{i_{n-1}}$, depicted in Figure 1.

In recent years, tropical geometry has provided a new approach to attack implicitization problems (Dickenstein et al., 2007; Sturmfels et al., 2007; Cueto et al., 2010). In particular, tropicalization interplays nicely with several classical constructions, such as Hadamard products of subvarieties of tori. Using such techniques, we can effectively compute the Chow polytope of these secant varieties, as we discuss in Section 7. In the case of the secants of monomial curves in $\mathbb{P}^4$, the Chow polytopes coincide with the Newton polytopes of these hypersurfaces. Interpolation techniques can then be used to obtain their defining equations.

As one may suspect, computing the tropicalization of an algebraic variety without information on its defining ideal is not an easy task. Such methods rely on a parametric representation of the variety and the characterization of tropical varieties in terms of valuations (Bieri and Groves, 1984), and they are known as *geometric tropicalization* (Theorem 7). As we explain in Section 4, the main difficulty lies in finding a suitable compactification of the variety such that its boundary has simple normal crossings, or combinatorial normal crossings in the case of surfaces. However, this geometric construction does not provide information about the tropical variety as a weighted set: the multiplicities are missing in the construction of Hacking et al. (2009) and they are essential for tropical implicitization methods. We give a formula to compute these numbers in Theorem 8. The combinatorics involved in the construction of such compactifications is non-trivial, since they are the combinatorial counterpart of the algebro-geometric process of *resolution of singularities*.

In the case of surfaces, the resolution can be achieved in theory by blowing up plane curves at finitely many points, as described in Section 5. We then use the rational parameterization of the original surface to obtain a resolution of this surface from the resolution of the arrangement of plane curves in $\mathbb{T}^2$. In practice, knowing which points to blow up and how the intersection multiplicities of proper transforms and exceptional divisors are carried along the various blow-ups can be a combinatorial challenge. However, the surfaces studied in this paper (binomial surfaces obtained from a dehomogeneization of the first secant of monomial projective curves) have very rich combinatorial structures, and we can make full use of this feature to compute their tropicalizations via resolutions. Indeed, our methods allow us to read off the intersection numbers of the boundary divisors directly from the master graphs, which encode the resolution diagrams of these surfaces (Figure 1). This is carried out in Section 3, in particular in Theorem 3.

Finally, we use this tropical surface to effectively compute the first secant variety of any monomial curve as a collection of 4-dimensional cones *with multiplicities* (Theorem 16). From this construction we recover the multidegree of this secant variety with respect to the rank-two lattice generated by the all-one's vector and the exponent vector parameterizing the curve. The degree of this variety was previously worked out in (Ranestad, 2006), and our work gives similar combinatorial formulas for this degree in terms of the exponent vector. But tropical methods enable us to obtain more information, namely the Chow polytope of the secant variety. We illustrate all our results in Example 18 which was inspired by (Ranestad, 2006).

2 The master graph

In this section, we describe the main object of this paper: the master graph. We start by defining an abstract graph, called the *abstract tropical secant surface graph*, parameterized by a list $(i_1, \ldots, i_n)$ of n distinct, coprime, nonnegative integers. Throughout the paper, we set $n \geq 4$ and we call $i_0 = 0$ to simplify notation. We construct this abstract graph by gluing three different families of graphs along the common labeled nodes D_{i_j}, as depicted in Figure 1. The first two graphs $G_{E,D}$ and $G_{h,D}$ are caterpillar trees with $2n-1$ and $2n$ nodes, grouped in two levels, with labels $D_0, D_{i_1}, \ldots, D_{i_n}$, $E_{i_1}, \ldots, E_{i_{n-1}}$ and $h_{i_1}, \ldots, h_{i_{n-1}}$ respectively. The third family of graphs is parameterized by subsets of the index set $\{0, i_1, \ldots, i_n\}$ of size at least two, which are obtained by intersecting an arithmetic progression of integers with the index set. Note that several arithmetic progressions can give the same subset of $\{0, i_1, \ldots, i_n\}$ and all of them will give the same node $F_{\underline{a}}$ in the graph. If $\underline{a} = \{i_{j_1}, \ldots, i_{j_k}\}$ then the graph $G_{F_{\underline{a}},D}$ has $k+1$ nodes and k edges: a central node $F_{\underline{a}}$ and k nodes labeled $D_{i_{j_1}}, \ldots, D_{i_{j_k}}$. The central node is connected to the other k nodes in the graph.

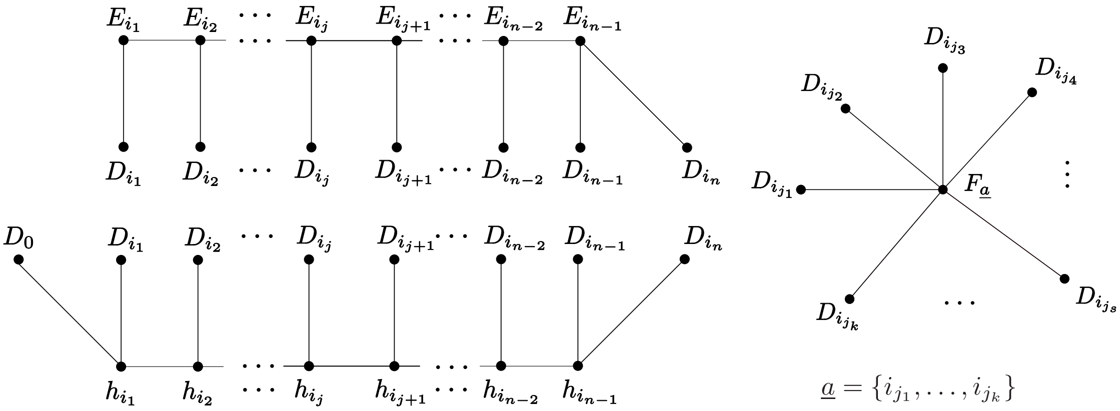

Fig. 1: The graphs $G_{E,D}$, $G_{h,D}$ and $G_{F_{\underline{a}},D}$ glue together to form the *abstract tropical secant surface graph*.

Next, we embed this graph in $\mathbb{R}^{n+1}$, mapping each node to an integer vector, as in Definition 1. Our chosen embedding has addition data: a weight on each edge that makes the graph *balanced*. We call this weighted graph the *tropical secant surface graph* or *master graph*. For a numerical example, see Figure 2.

Definition 1 *The* master graph *is a weighted graph in* $\mathbb{R}^{n+1}$ *parameterized by* $\{i_1, \ldots, i_n\}$ *with nodes:*

(i) $D_{i_j} = e_j := (0, \ldots, 0, 1, 0, \ldots, 0)$ $\quad (0 \leq j \leq n)$,

(ii) $E_{i_j} = (0, i_1, \ldots, i_{j-1}, i_j, \ldots, i_j)\,, h_{i_j} = (-i_j, -i_j, \ldots, -i_j, -i_{j+1}, \ldots, -i_n)$ $\quad (1 \leq j \leq n-1)$,

(iii) $F_{\underline{a}} = \sum_{i_j \in \underline{a}} e_j$ *where* $\underline{a} \subseteq \{0, i_1, \ldots, i_n\}$ *has size at least two and is obtained by intersecting an arithmetic progression of integers with the index set* $\{0, i_1, \ldots, i_n\}$.

Its edges agree with the edges of the abstract tropical secant surface graph, and have weights:

(i) $m_{D_{i_0},h_{i_1}} = 1$, $m_{D_{i_n},E_{i_{n-1}}} = \gcd(i_1, \ldots, i_{n-1})$, $m_{D_{i_n},h_{i_{n-1}}} = i_n$,

(ii) $m_{D_{i_j},E_{i_j}} = \gcd(i_1, \ldots, i_j)$, $m_{D_{i_j},h_{i_j}} = \gcd(i_j, \ldots, i_n)$ $(1 \leq j \leq n-1)$,

(iii) $m_{E_{i_j},E_{i_{j+1}}} = \gcd(i_1, \ldots, i_j)$, $m_{h_{i_j},h_{i_{j+1}}} = \gcd(i_{j+1}, \ldots, i_n)$ $(1 \leq j \leq n-2)$,

(iv) $m_{F_{\underline{a}},D_{i_j}} = \sum_r \varphi(r)$, *where we sum over the common differences* r *of all arithmetic progressions containing* i_j *and giving the same subset* $\underline{a}$. *Here,* φ *denotes Euler's phi function.*

Definition 2 *Let* $(G, m) \subset \mathbb{R}^N$ *be a weighted graph where each node has integer coordinates. Let* w *be a node in* G *and let* $\{w_1, \ldots, w_r\}$ *be the set of nodes adjacent to* w. *Consider the primitive lattices* $\Lambda_w = \mathbb{R}\langle w\rangle \cap \mathbb{Z}^N$ *and* $\Lambda_{w,w_i} = \mathbb{R}\langle w, w_i\rangle \cap \mathbb{Z}^N$. *Then* $\Lambda_{w,w_i}/\Lambda_w$ *is a rank one lattice, and it admits a unique generator* u_i *lifting to the cone* $\mathbb{R}_{\geq 0}\langle w, w_i\rangle/\mathbb{R}\langle w\rangle$. *We say that the* node w *is* balanced *if* $\sum_{i=1}^r m(w_i, w)u_i = 0 \in \mathbb{R}^N/\mathbb{R}\langle w\rangle$. *If all nodes are balanced, then* G *satisfies the* balancing condition.

Theorem 3 *The master graph satisfies the* balancing condition.

Remark 4 *If the arithmetic progression* $\underline{a}$ *has two elements, then* $F_{\underline{a}}$ *is a bivalent node and we can safely eliminate it from the graph if desired, replacing its two adjacent edges by a single edge. Both edges have the same multiplicity, which we assign to the new edge. To simplify notation, we keep these bivalent nodes.*

3 The master graph is a tropical surface

In this section, we explain the suggestive name "tropical secant surface graph." More concretely, we show that the master graph is the tropicalization of a surface in $\mathbb{C}^{n+1}$ parameterized by the binomial map $(\lambda, w) \mapsto (1-\lambda, w^{i_1} - \lambda, \ldots, w^{i_n} - \lambda)$. Before that, we review the basics of tropical geometry.

Definition 5 *Given a variety* $X \subset \mathbb{C}^N$ *with defining ideal* $I = I_X$, *we define the* tropicalization *of* X *as*

$$\mathcal{T}X = \mathcal{T}I = \{w \in \mathbb{R}^N : in_w(I) \text{ does not contain a monomial}\}.$$

Here, $in_w(I) = \langle in_w(f) : f \in I\rangle$, *and if* $f = \sum_\alpha c_\alpha \underline{x}^\alpha$ *where all* $c_\alpha \neq 0$, *then* $in_w(f) = \sum_{\alpha \cdot w = W} c_\alpha \underline{x}^\alpha$ *where* $W = \min\{\alpha \cdot w : c_\alpha \neq 0\}$. *In the case of an embedded projective variety* $X \subset \mathbb{P}^N$, *the tropicalization of* X *is defined as* $\mathcal{T}(X') \subset \mathbb{R}^{N+1}$ *where* X' *is the affine cone over* X *in* $\mathbb{C}^{N+1}$.

Although it may not be clear from Definition 5, tropicalizations are toric in nature. More precisely, let $\mathbb{T}^N = (\mathbb{C}^*)^N$ be the algebraic torus. Let Y be a subvariety of $\mathbb{T}^N$ with defining ideal $I_Y \subseteq \mathbb{C}[\mathbb{T}^N] = \mathbb{C}[y_1^{\pm}, \ldots, y_N^{\pm}]$. We define the tropicalization of $Y \subset \mathbb{T}^N$ as

$$\mathcal{T}Y = \{v \in \mathbb{R}^N : 1 \notin \text{in}_v(I_Y)\}.$$

Here, the initial ideal with respect to a vector v is the same as that in Definition 5. Consider the Zariski closure $\overline{Y}$ of Y in $\mathbb{C}^N$. It is easy to see that $\mathcal{T}Y$ equals $\mathcal{T}\overline{Y}$. Indeed, this follows from the fact that I_Y is the saturation ideal $\left(I_{\overline{Y}}\mathbb{C}[\mathbb{T}^N] : (y_1 \cdots y_N)^\infty\right)$ and $I_{\overline{Y}} = I_Y \cap \mathbb{C}[y_1, \ldots, y_N]$. Therefore, if we start with an irreducible variety $X \subset \mathbb{C}^N$ not contained in a coordinate hyperplane, then we can consider the very affine variety $Y = X \cap \mathbb{T}^N$, which has the same dimension as X. The tropical variety $\mathcal{T}Y$ is a pure polyhedral subfan of the Gröbner fan of I and it preserves an important invariant of Y: both objects have the same dimension (Bieri and Groves, 1984). We can choose to study $\mathcal{T}Y$ or $\mathcal{T}X$, and both sets will give us equivalent information about X. This approach will be useful in subsequent sections.

Tropical implicitization is a recently developed technique to approach classical implicitization problems (Sturmfels and Tevelev, 2008). For instance, when Y is a codimension-one hypersurface, $I_Y = \langle g \rangle$ is principal and $\mathcal{T}Y$ is the union of non-maximal cones in the normal fan of the Newton polytope of g, so knowing $\mathcal{T}Y$ can help us in finding g. But to achieve this, we need to compute $\mathcal{T}Y$ without explicitly knowing I_Y. We show how to do this in Section 4.

A point $w \in \mathcal{T}X$ is called *regular* if $\mathcal{T}X$ is a linear space locally near w. We can assign a positive integer number to regular points of the tropical variety, to have good properties. More precisely, we define the *multiplicity* m_w of a regular point w as the sum of multiplicities of all minimal associated primes of the initial ideal $\text{in}_w(I)$. For a given maximal cone σ in $\mathcal{T}X$, we define its multiplicity as the multiplicity at a regular point w in σ, that is, the multiplicity of any point in the relative interior. One can show that this assignment does not depend on the choice of w and that with these multiplicities, the tropical variety satisfies the *balancing condition* (Corollary 3.4, Sturmfels and Tevelev, 2008).

In the case of projective varieties, or in general, when we have a torus action, the tropical variety $\mathcal{T}X$ has a *lineality space*, that is, the maximal linear space contained in all cones of the fan $\mathcal{T}X$. For example, the lineality space of a tropical hypersurface $\mathcal{T}(g)$ will equal the orthogonal complement of the affine span of the Newton polytope of g, after appropriate translation to the origin. The extreme cases correspond to toric varieties globally parameterized by a monomial map with associated matrix A. Their tropicalizations $\mathcal{T}X$ will be classical linear spaces: the row span of A. In particular, $\mathcal{T}X$ coincides with its lineality space as sets with constant multiplicity *one* (Dickenstein et al., 2007).

We now realize the master graph as a tropical surface in $\mathbb{R}^{n+1}$:

Theorem 6 *Fix a strictly increasing sequence* $(0, i_1, \ldots, i_n)$ *of coprime integers. Let* Z *be the surface in* $\mathbb{C}^{n+1}$ *parameterized by* $(\lambda, \omega) \mapsto (1-\lambda, \omega^{i_1} - \lambda, \ldots, \omega^{i_n} - \lambda)$*. Then, the tropical surface* $\mathcal{T}Z \subset \mathbb{R}^{n+1}$ *coincides with the cone over the master graph as weighted polyhedral fans, with the convention that we assign the weight* $m_{D_{i_1}, E_{i_1}} + m_{F_{\underline{e}}, D_{i_1}}$ *to the cone over the edge* $D_{i_1}E_{i_1}$ *and we disregard the cone over the edge* $D_{i_1}F_{\underline{e}}$*, if the ending sequence* $\underline{e} = \{i_1, \ldots, i_n\}$ *gives a node* $F_{\underline{e}}$ *in the master graph.*

The proof of this statement involves techniques from geometric tropicalization and resolution of singularities of plane curves. Beautiful combinatorics emerge from them, as we will see in the next sections.

4 Geometric Tropicalization

In this section, we present the basics of geometric tropicalization. The spirit of this approach relies on computing the tropicalization of subvarieties of tori by analyzing the combinatorics of their boundary in a suitable compactification of the torus and of the subvariety therein. In what follows, we describe the method and its applications to implicitizations of subvarieties of tori.

Let $f_1, \ldots, f_N$ be Laurent polynomials in $\mathbb{C}[t_1^{\pm}, \ldots, t_r^{\pm}]$ and consider the rational map $\mathbf{f} \colon \mathbb{T}^r \dashrightarrow \mathbb{T}^N$, $\mathbf{f} = (f_1, \ldots, f_N)$. For simplicity, we will assume that the fiber of $\mathbf{f}$ over a generic point of $Y \subset \mathbb{T}^N$ is finite. Our goal is to compute the tropicalization $\mathcal{T}Y$ of the closure of the image of the map $\mathbf{f}$ inside the torus without knowledge of its defining ideal. When the coefficients of $f_1, \ldots, f_N$ are generic with respect to their Newton polytopes, a method for constructing $\mathcal{T}Y$ was given in (Thm 2.1, Sturmfels et al., 2007) and proved in (Thm 5.1, Sturmfels and Tevelev, 2008). We describe an algorithm proposed in (§5, Sturmfels and Tevelev, 2008) which may be applied to maps $\mathbf{f}$ which are non-generic. For simplicity, we state it for the case of parametric *surfaces*, although the method generalizes to higher dimensions as well.

Theorem 7 (Geometric Tropicalization (Hacking et al., 2009, §2)) *Let $\mathbb{T}^N$ be the N-dimensional torus over $\mathbb{C}$ with coordinate functions $t_1, \ldots, t_N$, and let Y be a closed* surface *in $\mathbb{T}^N$. Suppose Y is smooth and $\overline{Y} \supset Y$ is any compactification whose boundary $D = \overline{Y} \setminus Y$ is a smooth divisor with simple normal crossings. Let $D_1, \ldots, D_m$ be the irreducible components of D, and write $\Delta_{Y,D}$ for the intersection complex of the boundary divisor D, i.e. the graph on $\{1, \ldots, m\}$ defined by*

$$\{k_i, k_j\} \in \Delta_{Y,D} \iff D_{k_i} \cap D_{k_j} \neq \emptyset.$$

Define the integer vectors $[D_k] := (val_{D_k}(t_1), \ldots, val_{D_k}(t_N)) \in \mathbb{Z}^N$ ($k = 1, \ldots, m$), where $val_{D_k}(t_j)$ is the order of zero-poles of t_j along D_k. For any $\sigma \in \Delta_{Y,D}$, let $[\sigma]$ be the cone in $\mathbb{Z}^N$ spanned by $\{[D_k] : k \in \sigma\}$ and let $\mathbb{R}_{\geq 0}[\sigma]$ be the cone in $\mathbb{R}^N$ spanned by the same integer vectors. Then,

$$\mathcal{T}Y = \bigcup_{\sigma \in \Delta_{Y,D}} \mathbb{R}_{\geq 0}[\sigma].$$

We complement the previous result by a formula giving the multiplicities of regular points in tropical surfaces. A similar formula will hold in higher dimensions:

Theorem 8 *(Cueto, 2011) In the notation of Theorem 7, the multiplicity of a regular point w in the tropical surface equals:*

$$m_w = \sum_{\substack{\sigma \in \Delta_{Y,D} \\ w \in \mathbb{R}_{\geq 0}[\sigma]}} (D_{k_1} \cdot D_{k_2})\, \mathit{index}\big((\mathbb{R} \otimes_{\mathbb{Z}} [\sigma]) \cap \mathbb{Z}^N : \mathbb{Z}[\sigma]\big),$$

where $D_{k_1} \cdot D_{k_2}$ denotes the intersection number of these divisors and we sum over all two-dimensional cones σ whose associated rational cone $\mathbb{R}_{\geq 0}[\sigma]$ contains the point w.

To compute $\mathcal{T}Y$ using the previous theorems, we require a compactification $\overline{Y} \supset Y$ whose boundary has simple normal crossings (SNC). In words, all components of the divisor D should be smooth and they show intersect "as transversally as possible." One method for producing such a *tropical compactification* is taking the closure $\overline{Y}$ of Y in $\mathbb{P}^N \supset \mathbb{T}^N$ and finding a resolution of singularities for the boundary $\overline{Y} \setminus Y$. This latter step can be difficult. However, in the case of surfaces, it is enough to require the boundary to have *combinatorial normal crossings* (CNC), that is, "no three divisors intersect at a point" (Sturmfels and Tevelev, 2008). We describe the resolution process for our binomial surface Z in the next section.

5 Combinatorics of Monomial Curves

In this section, we compute the tropical variety of the surface Z described in Theorem 6. Let $f_{i_j} := \omega^{i_j} - \lambda$ ($0 \leq j \leq n$) and consider the parameterization $\mathbf{f} \colon \mathbb{C}^2 \to Z$ given by these $n+1$ polynomials. Since geometric tropicalization involves subvarieties of tori, we restrict our domain to $X = \mathbb{T}^2 \smallsetminus \bigcup_{j=1}^{n} (f_{i_j} = 0)$.

We give a compactification of X which, in turn, gives a *tropical compactification* of $Z \cap \mathbb{T}^{n+1}$ with CNC boundary via the map $\mathbf{f}$.

First, we naively compactify X inside $\mathbb{P}^2$. The components of the boundary divisor are $D_{i_j} = (f^h_{i_j}(\omega, \lambda, u) = 0)$ and $D_\infty = (u = 0)$, where $f^h_{i_j}$ is the homogenization of f_{i_j} with respect to the new variable u. We encounter three types of singularities: the origin, the point $(0 : 1 : 0)$ at infinity, and isolated singularities in $\mathbb{T}^2$. We resolve them by blowing up these points and contracting divisors with negative self-intersection (encoded by superfluous bivalent nodes), in a way that preserves the CNC condition. The resolutions diagrams will *precisely* be the graphs in Figure 1, where h_1 corresponds to the divisor D_∞. The nodes E_{i_j} $(1 \leq j \leq n-1)$ and h_{i_j} $(2 \leq j \leq n-1)$ will correspond to exceptional divisors. All intersection multiplicities will equal one, so to compute the multiplicities of the edges in $\mathcal{T}Z$ involving nodes h_{i_j} or E_{i_j}, we only need to calculate indices of suitable lattices associated to these edges.

We now describe the resolution process at each one of our three types of singular points. At the origin, all curves D_{i_j} (except for D_0) intersect and they are tangential to each other. For any j, the strict transform of a given D_{i_j}, after a single blow-up, equals D_{i_j-1}, so we can resolve this singularity after i_{n-1}-blow-ups. The exceptional divisors are labeled E_k $(1 \leq k \leq i_{n-1})$ and all of them give bivalent nodes in the resolution diagram, except for the $n-1$ nodes E_{i_j}. We eliminate the bivalent nodes by contraction. By induction, we see that the valuation of each exceptional divisor is the integer vectors E_{i_j} from Theorem 3.

At infinity, the resolution process is more delicate. Here, the singular point $p = (0 : 1 : 0)$ corresponds to the intersection of D_∞ and all divisors D_{i_j} with $i_j \geq 2$. However, we know that p is a singular point of all prime divisors D_{i_j}. Therefore, we first need to perform a blow-up to smooth them out. More precisely, if π denotes this blow-up and H is the exceptional divisor, we obtain $\pi^*(D_{i_j}) = D_{i_j} + (i_j - 1)H$, $\pi^*(D_\infty) = D'_\infty + H$, where $H = (t = 0)$, and $D'_{i_j} = (\omega - t^{i_j - 1} = 0)$, $D'_\infty = (w = 0)$ are the strict transforms. Therefore, the new setting is very similar to the one we described before for the singularity of the boundary D at the origin. The main difference with the resolution at the origin is that along the series of blow-ups, the strict transform of H will continue to be tangential to the divisors intersecting at a "fat point", whereas H was not present in the resolution at the origin. All exceptional divisors will be denoted by h_k $(k = 2, \ldots, i_n)$ and again we only keep the non-bivalent nodes h_{i_j} $(2 \leq j \leq n)$ after appropriate contractions. For simplicity, we denote the strict transform of D_∞ by h_1. At the end of the resolution process H gets contracted, explaining why we do not see it in the resolution diagram (Figure 1). As expected, we recover the integer vectors h_{i_j} from Theorem 3.

Finally, we come to multiple intersections among the divisors D_{i_j} in $\mathbb{T}^2$. If (λ, ω) satisfies $f_{i_j} = \lambda - \omega^{i_j} = 0$ and $f_{i_k} = \lambda - \omega^{i_k} = 0$, then $\omega^{i_j} = \lambda = \omega^{i_k}$, so ω is a primitive r-th root of unity for some $r \mid (i_k - i_j)$. Alternatively, $i_j \equiv i_k \equiv s \pmod r$, $\omega = e^{2\pi i p/r}$ and $\lambda = \omega^s$ for p coprime to r. All other curves $(f_{i_l} = 0)$ with $i_l \equiv s \pmod r$ will also meet at (λ, ω). We represent this crossing point (λ, ω) by $x_{p,r,s}$ and the index set of curves meeting at $x_{p,r,s}$ by $\underline{a}_{r,s}$, or $\underline{a}$ for short. That is,

$$x_{p,r,s} = (e^{2\pi i p s/r}, e^{2\pi i p/r}), \quad \underline{a} = \underline{a}_{r,s} := \{i_j | i_j \equiv s \ (\mathrm{mod}\, r)\}.$$

Furthermore, the curves $D_{i_j} = (f_{i_j} = 0)$ meeting at $x_{p,r,s}$ intersect transversally.

If three or more curves meet at a point, we blow up this point to separate the curves. To simplify notations, we also blow up crossings with $|\underline{a}| = 2$. After a single blow-up at each crossing point $x_{p,r,s}$ we obtain a new divisor $F_{\underline{a}, x_{p,r,s}}$ (the exceptional divisor associated to the point $x_{p,r,s}$) which intersects the proper transform of all D_{i_j} normally, for $j \in \underline{a}$. After studying the coefficient of $F_{\underline{a}, x_{p,r,s}}$ in the pull-back of each character of the torus $\mathbb{T}^{n+1}$ under the map $\mathbf{f}$, we get the node $F_{\underline{a}} = [F_{\underline{a}, x_{p,r,s}}] = \sum_{i_j \in \underline{a}} e_j$, as desired. The resolution diagram will correspond to the graph in the right-hand side of Figure 1.

Finally, we use Theorem 8 to compute the multiplicity of the edge $F_{\underline{a}}D_{i_j}$ in $\mathcal{T}Z$. All summands equal one and so the multiplicity is just the number of such summands, that is, the number of points $x_{p,r,s}$ such that $F_{\underline{a}} = [F_{\underline{a},x_{p,r,s}}]$. This number equals the sum $\sum_l \varphi(l)$ over all common differences l giving $\underline{a}$.

6 The tropical secant graph is a Hadamard product

In this section, we use the master graph to effectively compute the tropicalization of the first secant variety of a monomial projective curve C. Without loss of generality, we may assume that the curve is parameterized as $(1 : t^{i_1} : \ldots : t^{i_n})$, where $0 < i_1 < \ldots < i_n$ are coprime integers. By definition,

$$Sec^1(C) = \overline{\{a \cdot p + b \cdot q : a, b \in \mathbb{C}, p, q \in C\}} \subset \mathbb{P}^n.$$

As discussed in Section 3, tropicalizations are toric in nature. Thus, for the rest of this section, instead of looking at the projective varieties C and $Sec^1(C)$, we study the corresponding very affine varieties which are intersections of their affine cones in $\mathbb{R}^{n+1}$ with the algebraic torus $\mathbb{T}^{n+1}$. To simplify notation, we will also denote them by C and $Sec^1(C)$ in a way that is clear from the context. Tropicalizations of projective varieties and their corresponding very affine varieties are the same.

We parameterize this secant variety by the *secant map* $\phi\colon \mathbb{T}^4 \to \mathbb{T}^{n+1}$, $\phi(a, b, s, t) = (as^{i_k} + bt^{i_k})_{0 \leq k \leq n}$. After a monomial change of coordinates $b = -\lambda a$ and $t = \omega s$, this map can be written as

$$\phi(a, s, \omega, \lambda) = \left(as^{i_k}\left(\omega^{i_k} - \lambda\right)\right)_{0 \leq k \leq n}.$$

From this observation, it is natural to consider the Hadamard product of subvarieties of tori:

Definition 9 *Let $X, Y \subset \mathbb{T}^N$ be two subvarieties of tori. The* Hadamard product *of X and Y equals*

$$X \centerdot Y = \overline{\{(x_1y_1, \ldots, x_Ny_N) \mid x \in X, y \in Y\}} \subset \mathbb{T}^N.$$

From the construction, we get the following characterization of our secant variety:

Proposition 10 *The first secant variety $Sec^1(C) \subset \mathbb{R}^{n+1}$ of the monomial curve C parameterized by $t \mapsto (1 : t^{i_1} : \ldots : t^{i_n}) \in \mathbb{P}^n$ equals $C \centerdot Z \subset \mathbb{T}^{n+1}$ where Z is the surface parameterized by $(\lambda, \omega) \mapsto (1 - \lambda, \omega^{i_1} - \lambda, \ldots, \omega^{i_n} - \lambda)$.*

We now explain the relationship between Hadamard products and their tropicalization:

Proposition 11 *(Corollary 13, Cueto et al., 2010) Given C, Z as in Proposition 10, then as* sets

$$\mathcal{T}Sec^1(C) = \mathcal{T}C + \mathcal{T}Z, \tag{1}$$

where the sum on the (RHS) denotes the Minkowski sum in $\mathbb{R}^{n+1}$.

As we mentioned earlier, $\mathcal{T}C = \mathbb{R}\langle \mathbf{1}, (0, i_1, \ldots, i_n)\rangle$ with constant weight one. By construction, the lineality space of $\mathcal{T}Z \subset \mathbb{R}^{n+1}$ is the origin, and the lineality space of $\mathcal{T}Sec^1(C) \subset \mathbb{R}^{n+1}$ equals $\mathcal{T}C$.

As occurs in general with Hadamard products and their tropicalizations, the right-hand side of (1) has no canonical fan structure. Some maximal cones can be subdivided, whereas others can be merged into bigger cones. Hence, we present this set as a collection of four-dimensional weighted cones in $\mathbb{R}^{n+1}$ obtained as a Minkowski sum of maximal cones in $\mathcal{T}C$ and $\mathcal{T}Z$. The multiplicity at a regular point would simply be the sum of multiplicities of all cones in the collection containing it. Moreover, we will be able to express this number in terms of the multiplicities in $\mathcal{T}Z$, using the following result from (Sturmfels and Tevelev, 2008) that shows the interplay between maps on tori and their tropicalization. Let $\alpha\colon \mathbb{T}^r \to \mathbb{T}^N$ be a homomorphism of tori, that is, a monomial map whose exponents are encoded in a matrix $A \in \mathbb{Z}^{N \times r}$.

Theorem 12 *(Sturmfels and Tevelev, 2008) Let $V \subset \mathbb{T}^r$ be a subvariety. Then $\mathcal{T}(\alpha(V)) = A(\mathcal{T}V)$.*

Moreover, if α induces a generically finite morphism of degree δ on V, then the multiplicity of $\mathcal{T}(\alpha(V))$ at a regular point w is

$$m_w = \frac{1}{\delta} \cdot \sum_v m_v \cdot \textit{ index } (\mathbb{L}_w \cap \mathbb{Z}^N : A(\mathbb{L}_v \cap \mathbb{Z}^r)), \tag{2}$$

where the sum is over all points $v \in \mathcal{T}V$ with $Av = w$. We also assume that the number of such v is finite, and that all of them are regular in $\mathcal{T}V$. In this setting, $\mathbb{L}_v, \mathbb{L}_w$ denote the linear spans of neighborhoods of $v \in \mathcal{T}V$ and $w \in A(\mathcal{T}V)$ respectively.

The key fact in the computation of multiplicities for $\mathcal{T}Sec^1(C)$ is that we can express the Hadamard product in terms of the monomial map $\alpha\colon \mathbb{T}^{2n+2} \to \mathbb{T}^{n+1}$ given by the matrix $A = (I_{n+1} \mid I_{n+1}) \in \mathbb{Z}^{(n+1)\times 2(n+1)}$. The subvariety $V \subset \mathbb{T}^{2n+2}$ is the Cartesian product $C \times Z$, where we consider each surface inside the torus. From (Cueto et al., 2010), we have $\mathcal{T}V = \mathcal{T}(C \times Z) = \mathcal{T}C \times \mathcal{T}Z$ and the multiplicity m_v at a regular point $v = (c, z)$ of V equals m_z. By dimension arguments, we see that α is generically finite when restricted to V, so we can use formula (2) to compute multiplicities in $\mathcal{T}Sec^1(C)$.

Lemma 13 *For $V = C \times Z$ and α as above, the generic fiber of $\alpha_{|V}$ has size 2, hence $\delta = 2$.*

Next, we compute the fiber of a regular point w in $\mathcal{T}(\alpha(V))$ under the linear map A. The strategy will be to pick all possible pairs of maximal cones σ, σ' in $\mathcal{T}Z$ and to compute the dimension of $(\mathbb{R}\sigma + \mathcal{T}C) \bigcap (\mathbb{R}\sigma' + \mathcal{T}C)$. If this dimension is strictly less than four, then we know that generic points in $\mathcal{T}C \times \sigma$ and $\mathcal{T}C \times \sigma'$ belong to different fibers of A. If it equals four, we compute the fiber of A at any point in the intersection. In particular, we conclude:

Lemma 14 *(i) The cones $\langle D_0, h_{i_1}\rangle + \mathcal{T}C$, $\langle F_{\{0,i_1,\ldots,i_n\}}, D_{i_j}\rangle + \mathcal{T}C$ ($0 \le j \le n$), $\langle D_{i_n}, E_{i_{n-1}}\rangle + \mathcal{T}C$ and $\langle D_{i_n}, h_{i_{n-1}}\rangle + \mathcal{T}C$ are not maximal, so we disregard them together with the node $F_{\{0,i_1,\ldots,i_n\}}$.*

(ii) For all $1 \le j \le n-2$, we have equalities $\langle E_{i_j}, D_{i_j}\rangle + \mathcal{T}C = \langle h_{i_j}, D_{i_j}\rangle + \mathcal{T}C$ and $\langle E_{i_j}, E_{i_{j+1}}\rangle + \mathcal{T}C = \langle h_{i_j}, h_{i_{j+1}}\rangle + \mathcal{T}C$ because $E_{i_j} \equiv h_{i_j}$ modulo $\mathcal{T}C$. Hence, we disregard all nodes h_{i_j}.

(iii) $i_1 \cdot F_{\underline{e}} = E_{i_1}$ and $(i_n - i_{n-1}) \cdot F_{\underline{b}} \equiv E_{i_{n-1}}$ modulo $\mathcal{T}C$, where $\underline{e} = \{i_1, \ldots, i_n\}$ and $\underline{b} = \{0, i_1, \ldots, i_{n-1}\}$. Thus, the maximal cones $\mathbb{R}\langle F_{\underline{e}}, D_{i_1}\rangle + \mathcal{T}C$ and $\mathbb{R}\langle E_{i_1}, D_{i_1}\rangle + \mathcal{T}C$ coincide, as well as $\mathbb{R}\langle F_{\underline{b}}, D_{i_{n-1}}\rangle + \mathcal{T}C$ and $\mathbb{R}\langle E_{i_{n-1}}, D_{i_{n-1}}\rangle + \mathcal{T}C$.

(iv) All other fibers have size one.

As a consequence of this lemma, in numerical examples we will identify the nodes E_{i_1} and $F_{\underline{e}}$, as well as $E_{i_{n-1}}$ and $F_{\underline{b}}$. In this identification, the nodes $F_{\underline{e}}$ and $F_{\underline{b}}$ are removed, and the edges adjacent to the nodes $F_{\underline{e}}$ and $F_{\underline{b}}$ are added to those of E_{i_1} and $E_{i_{n-1}}$. We also merge the corresponding edges $E_{i_1}D_{i_1}$ and $F_{\underline{e}}D_{i_1}$ (resp. $E_{i_{n-1}}D_{i_{n-1}}$ and $F_{\underline{b}}D_{i_{n-1}}$) in the tropical secant graph, assigning the sum of their weights to the new edge.

The indices involved in (2) are calculated as follows. Let $l_1 = \mathbf{1}$ and $l_2 = (0, i_1, \ldots, i_n)$ be the generators of $\mathcal{T}C$. For each edge of $\mathcal{T}Z$, we pick its two end points x, y. The index in (2) associated to a point $v \in \mathcal{T}C + \mathbb{R}_{\ge 0}\langle x, y\rangle \subset \mathcal{T}C + \mathcal{T}Z$ is the quotient of the gcd of the 4-minors of the matrix $(x \mid y \mid l_1 \mid l_2)$ by the gcd of the 2-minors of the matrix $(x \mid y)$. These gcd's are computed as the product of the nonzero diagonal elements of the Smith normal form of each matrix. Here is our main result:

Definition 15 *The* tropical secant graph *is a weighted subgraph of the master graph in* $\mathbb{R}^{n+1}$*, with nodes:*

(i) $D_{i_j} = e_j := (0, \dots, 0, 1, 0, \dots, 0)$ *(*$0 \leq j \leq n$*),*

(ii) $E_{i_j} = (0, i_1, \dots, i_{j-1}, i_j, \dots, i_j) = \sum_{k<j} i_k \cdot e_k + i_j \cdot (\sum_{k \geq j} e_k)$ *(*$1 \leq j \leq n-1$*),*

(iii) $F_{\underline{a}} = \sum_{i_j \in \underline{a}} e_j$ *where* $\underline{a} \subsetneq \{0, i_1, \dots, i_n\}$ *varies among all proper subsets containing at least two elements that are obtained from an arithmetic progression.*

The edges are a subset of the edges of the master graph. Their positive weights are assigned as follows:

(i) $m_{E_{i_j}, E_{i_{j+1}}} = \gcd(i_1, \dots, i_j) \gcd_{j<t<n} (i_n - i_t)$ *(*$1 \leq j \leq n-2$*),*

(ii) $m_{D_{i_j}, E_{i_j}} = \gcd\big(\gcd(i_1, \dots, i_{j-1}) \gcd_{j<s\leq n}(i_s - i_j)\,;\, \gcd_{0\leq k<j}(i_j - i_k) \gcd(i_{j+1}, \dots, i_n)\big)$ *(*$1 \leq j \leq n-1$*),*

(iii) $m_{F_{\underline{a}}, D_{i_j}} = \frac{1}{2} \sum_r \varphi(r) \cdot \gcd\big(\gcd_{i_l, i_k \notin \underline{a}} (\mid i_l - i_k \mid)\,;\, \gcd_{\substack{i_l, i_k \in \underline{a} \\ l,k \neq j}} (\mid i_l - i_k \mid)\big)$ *(*$i_j \in \underline{a}$*, where the sum runs over all common differences* r *of arithmetic progressions giving the subset* $\underline{a}$*).*

(By convention, a gcd over an empty set of indices is taken to be 0.)

Theorem 16 *Given a monomial curve* C *with primitive exponent vector* $(0, i_1, \dots, i_n)$*,* $0 = i_0 < i_1 < \dots < i_n$*, the tropicalization of the first secant variety of* C *can be characterized set-theoretically as a collection of 4-dimensional weighted cones (with no fan structure). Each cone has a 2-dimensional lineality space with basis given by the* intrinsic lattice $\Lambda = \langle (1, \dots, 1), (0, i_1, \dots, i_n) \rangle$*. The collection is obtained as the cone from the subspace* $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda$ *over the tropical secant graph, preserving all weights.*

7 The Newton polytope of the secant graph for $\mathbb{P}^4$

In this section, we focus our attention on the *inverse problem*. That is, given the tropical variety of an irreducible hypersurface, we wish to recover its defining equation. A first step towards a satisfactory answer would consist of computing the *Newton polytope* of the defining equation $f = \sum_a c_a \underline{x}^a$, i.e. the convex hull of integer vectors a such that $\underline{x}^a$ appears with a nonzero coefficient in f. This will let us find the defining equation via interpolation.

We now explain the connection between $\mathcal{T}(f)$ and NP(f) for an irreducible polynomial f in $n+1$ variables defined over $\mathbb{C}$. For a vector $w \in \mathbb{R}^{n+1}$, the initial form $\text{in}_w(f)$ is a monomial if and only if w is in the interior of a maximal cone (chamber) of the normal fan of NP(f). The tropical variety of the hypersurface $(f = 0)$ is the union of codimension one cones of the normal fan of NP(f). The multiplicity of a maximal cone in $\mathcal{T}(f)$ is the lattice length of the edge of NP(f) normal to that cone.

A construction for the Newton polytope NP(f) from its normal fan $\mathcal{T}(f)$ equipped with multiplicities was developed in Dickenstein et al. (2007). We describe this *ray-shooting algorithm* in Theorem 17:

Theorem 17 *Suppose* $w \in \mathbb{R}^{n+1}$ *is a generic vector so that the ray* $(w + \mathbb{R}_{>0}\, e_i)$ *intersects* $\mathcal{T}(f)$ *only at regular points of* $\mathcal{T}(f)$*, for all* i*. Let* $\mathcal{P}^w$ *be the vertex of the polytope* $\mathcal{P} = NP(f)$ *that attains the maximum of* $\{w \cdot x : x \in \mathcal{P}\}$*. Then the* i^{th} *coordinate of* $\mathcal{P}^w$ *equals* $\sum_v m_v \cdot |l_i^v|$*, where the sum is taken over all points* $v \in \mathcal{T}(f) \cap (w + \mathbb{R}_{>0} e_i)$*,* m_v *is the multiplicity of* v *in* $\mathcal{T}(f)$*, and* l_i^v *is the* i^{th} *coordinate of the primitive integral normal vector* l^v *to the maximal cone in* $\mathcal{T}(f)$ *containing* v*.*

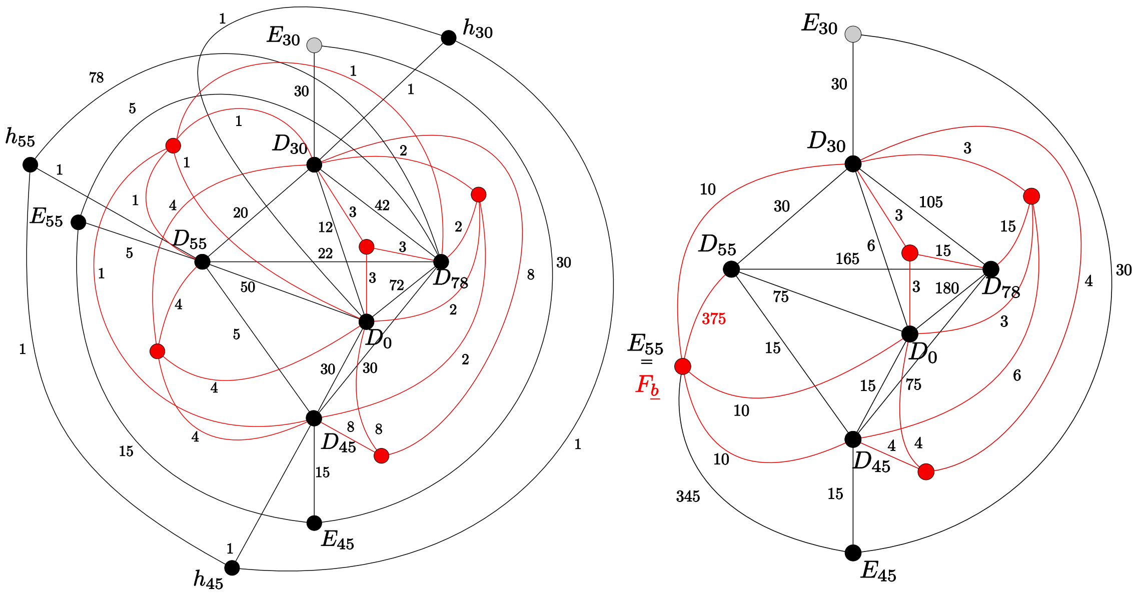

Fig. 2: The master graph and the tropical secant graph of the monomial curve $(1 : t^{30} : t^{45} : t^{55} : t^{78})$.

Note that we do not need a fan structure on $\mathcal{T}(f)$ to use Theorem 17. A description of $\mathcal{T}(f)$ as a set, together with a way to compute the multiplicities at regular points, gives us enough information to compute vertices of $\mathrm{NP}(f)$ in any generic directions. Computing a single vertex using Theorem 17 will give us the multidegree of f with respect to the grading given by the intrinsic lattice Λ from Theorem 16.

The entire polytope $\mathrm{NP}(f)$ can be computed by iterating the ray-shooting algorithm with different objective vectors (one per chamber). A method to choose these vectors appropriately was developed in (Algorithm 2, Cueto et al., 2010): the *walking algorithm*. The core of the method is to keep track of the cones that we meet while ray-shooting from a given objective vector, to use the list of such cones to walk from chamber to chamber in the normal fan of $\mathrm{NP}(f)$, picking objective vectors along the way, and to repeat the shooting algorithm with these new vectors. We illustrate these methods with an example.

Example 18 *The first secant variety of the monomial curve* $t \mapsto (1 : t^{30} : t^{45} : t^{55} : t^{78})$ *in* $\mathbb{P}^4$ *is known to be a hypersurface of degree 1820 (Example 3.3, Ranestad, 2006). We use geometric tropicalization to compute the tropicalization of this variety. By Theorems 6 and 16, we construct the two graphs in Figure 2: the leftmost picture corresponds to the master graph, whereas the rightmost picture is the tropical secant graph. The ten nodes in the tropical secant graph have coordinates* $D_0 = e_0$, $D_{30} = e_1$, $D_{45} = e_2$, $D_{55} = e_3$, $D_{78} = e_4$, $E_{30} = (0,30,30,30,30)$, $E_{45} = (0,30,45,45,45)$, $F_{\{0,30,45,55\}} \equiv E_{55} = (0,30,45,55,55)$, $F_{\{0,30,78\}} = (1,1,0,0,1)$, $F_{\{0,30,45,78\}} = (1,1,1,0,1)$, *and* $F_{\{0,30,45\}} = (1,1,1,0,0)$. *The master graph has the five extra nodes* $h_{30} = (-30,-30,-45,-55,-78)$, $h_{45} = (-45,-45,-45,-55,-78)$, $h_{55} = (-55,-55,-55,-55,-78)$, $F_{\{0,30,45,55,78\}} = (1,1,1,1,1)$, *and* $F_{\{0,30,45,55\}} = (1,1,1,1,0)$. *The unlabeled nodes in Figure 2 indicate nodes of type* $F_{\underline{a}}$, *where the subset* $\underline{a}$ *consists of the indices of all nodes* D_{i_j} *adjacent to the unlabeled node. Notice that the nodes* E_{55} *and* $F_{\underline{b}}$ *coincide in the tropical secant graph, as predicted by Lemma 14.*

Finally, we apply the ray-shooting and walking algorithms to recover the Newton polytope of this hypersurface. Its multidegree with respect to the lattice $\Lambda = \mathbb{Z}\langle \mathbf{1}, (0, 30, 45, 55, 78)\rangle$ *is* $(1\,820, 76\,950)$. *The polytope has 24 vertices and* f*-vector* $(24, 38, 16)$. *Using* `LattE` *we see that it contains* $7\,566\,849$ *lattice points, which gives an upper bound for the number of monomials in the defining equation.*

The implicitization methods discussed in this section can be generalized to monomial curves in higher dimensional projective spaces, where the first secant has no longer codimension one. In this case, one can recover the *Chow polytope* of the secant variety by a natural generalization of the ray-shooting method: the *orthant-shooting algorithm* (Theorem 2.2, Dickenstein et al., 2007). Instead of shooting rays, we shoot orthants (i.e. cones spanned by vectors in the canonical basis of $\mathbb{R}^{n+1}$) of dimension equal to the codimension of our variety. A formula similar to the one described in Theorem 17 will give us the vertex of the Chow polytope associated to the input objective vector. However, an analog to the *walking algorithm* still needs to be developed, since there is, a priori, no canonical way of ordering the intersection points for walking along the complement of the tropical variety. We hope to pursue this direction in the near future.

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