Stokes polyhedra for X-shaped polyminos

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Definitions. Consider a pair of *interlacing regular convex polygons*, each with $2(n+2)$ vertices, which we will be referring to as *red* and *black* ones. One can place these vertices on the unit circle $|z| = 1$ in the complex plane; the vertices of the red polygon at $\epsilon^{2k}, k = 0, \ldots, 2n-1$, of the black polygon at $\epsilon^{2k+1}, k = 0, \ldots, 2n-1$; here $\epsilon = \exp(i\pi/(2n+2))$.

We assign to the vertices of each polygon alternating (within each polygon) signs. Note that all the pairwise intersections of red and black sides are oriented consistently. We declare the corresponding orientation positive.

Definition 1 *A collection (perhaps, empty) of non-intersecting chords connecting vertices of opposing sign in one of the polygons will be called a* quadrangulation *of that polygon. We will color these chords according to which polygon they partition, and orient from $-$ to $+$. A maximal quadrangulation partition the polygon into* 4*-sided polygons which we will be calling the* squares.

A pair of quadrangulations of red and black polygons is compatible *if any pair of intersecting oriented chords (from different polygons) form a positive frame, just as the intersecting sides of the interlacing polygons do.*

All pairs of compatible quadrangulations for $n = 1$ are shown below.

g Fix (complete) quadrangulation q_ν for the *black* polygon. Complete *red* quadrangulations compatible with q_ν form a finite set $P^0_{q_\nu}$; incomplete red quadrangulations compatible with q_ν form a sublattice of the power set of $P^0_{q_\nu}$. As it turned out,

Proposition 1 *There exists a convex polyhedron P_{q_ν} with the set of vertices $P^0_{q_\nu}$ such that the lattice of incomplete quadrangulations compatible with q_ν coincides with the lattice of faces of P_{q_ν}.*

We refer to the polyhedron P_{q_ν} as *Stokes* polyhedron. The family of Stokes polyhedra was introduced in [2], which also noticed that it interpolates between the (combinatorial) d-cubes and Stasheff polyhedra it was noticed that the polyhedra P_{q_ν}

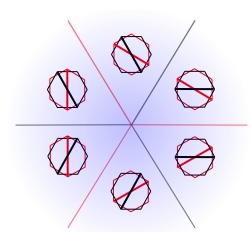

Figure 1: Complete pair of compatible quadrangulations for $n = 1$ are shown. The Stokes polyhedron here is the segment, with two endpoints corresponding to two red quadrangulations compatible with the black quadrangulations in the right third of the display.

Motivation. Our interest to the pairs of compatible quadrangulations stems from the fact that they describe the topology of the stratification of the space of polynomials $\{z^{n+2} + \sum_{l=1}^{n} a_l z^{n+1-l}\} \cong \mathbb{C}^n$ by the union of Stokes and antiStokes sets (see [2, 1]).

Specifically, for any collection of chords defining a pair of quadrangulations, $q = (q_h, q_v)$, let $\mathbb{K}_q$ be the cone $\mathbb{R}_+^q$ of *nonnegative weights* on chords in q. Inclusions $i : q' \to q$ induce embeddings of the corresponding cones $\mathbb{K}_{q'} \to \mathbb{K}_q$ (by assigning zero weights to the missing chords), and the inductive limit of the corresponding direct system is a polyhedron Λ, stratified by the relatively open cones $\mathbb{R}_{>0}^q$.

The resulting stratified space is homeomorphic to the pair (Stokes, antiStokes) in $\mathbb{C}^n$.

Polyminos. It is easier to visualize a quadrangulation q_v as a *polymino*, which is understood here as a collection of $(n+1)$ unit squares with some sides identified isometrically; the polymino is then the polygon obtained by gluing these sides.

The sides of the squares of the polymino that correspond to the exterior sides of the black polygon will be called *exterior sides* as well; the sides corresponding to the chords will be called *interior*. We will call a square of a polymino a *leaf* if it has three exterior sides, a *turning square* if it has exactly two adjacent exterior sides and a *separating square* if it has exactly two opposite exterior sides.

In a *snake* polymino all non-leaf squares are turning; in a *band* polymino all non-leaf squares are separating. These two classes of polymino represent two extremes in terms of complexity of the Stokes polyhedra. Indeed, P_{q_v} for the band polymino is combinatorially a cube (in $\mathbb{R}^n$). As for the *snake polyminos*, there are 2^{n-1} types (if we fix one end, the the snake polymino is completely characterized whether the turns at the turning squares are left or right).

Perhaps unexpectedly, snake polymino all have combinatorially equivalent fans, regardless of the type: The Stokes polyhedra for snake polyminos are combinatorially equivalent to the associahedra, or Stasheff polyhedra, see [3].

We remark that over the past 20 years, the associahedra became a staple of algebraic, analytic and combinatorics and combinatorial geometry; a good overview can be found in [3]

Enumeration for X-polyminos Our main result is the enumeration of the vertices in the 4-parametric family of X-polyminos.

Definition 2 *An polymino is called an X-polymino, if all but one of its squares are either turning squares or leaf squares. The exceptional square (which is not required to be a turning or leaf one, but can happen to be one) is called* central. *If, upon removal of the central square the polymino splits into 4 polyminos (which are necessarily snake polyminos) of sizes n_1, n_2, n_3 and n_4 (in cyclic order around the internal square), we will denote such polymino as $q_{v_\mathbf{n}}$, where $\mathbf{n} = (n_1, n_2, n_3, n_4)$.*

Theorem 1 *The generating function for the number of vertices of Stokes polyhedra corresponding to X-polyminos of size* $\mathbf{n}$ *is*

$$T(\mathbf{z}) = \frac{(C(z_1) - C(z_2)(C(z_2) - C(z_3))(C(z_3) - C(z_4))(C(z_4) - C(z_1))}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_4)(z_4 - z_1)}, \tag{1}$$

where

$$C(z) = \sqrt{1 - 4z}.$$

Examples. Using generating function 1 one can immediately derive the generating functions for various 1-parametric families of X-polymino. For example, the T-series $\mathbf{n} = (1, 0, 1, n)$ has the generating function (for the number of vertices of the corresponding Stokes polyhedron) given by

$$\frac{\partial^2 T(z_1, z_2, z_3, z_4)}{\partial z_1 \partial z_3}\Big|_{\mathbf{z}=(0,0,0,z)} = \frac{2 - 2\sqrt{1-4z} - z\left(3 + \sqrt{1-4z}\right) - 6z^2}{2z^3}$$

which translates at once into $T_{(1,0,1,n)} = 2C_{n+2} + C_{n+1}$. The generating function begins with $5 + 12z + 33z^2 + 98z^3 + 306z^4 + 990z^5 + 3289z^6 + 11154z^7 + \ldots$.

References

[1] Digital library of mathematical functions.

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[3] G. M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.