A generalization of the alcove model and its applications

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Abstract. The alcove model of the first author and Postnikov describes highest weight crystals of semisimple Lie algebras. We present a generalization, called the quantum alcove model, and conjecture that it uniformly describes tensor products of column shape Kirillov-Reshetikhin crystals, for all untwisted affine types. We prove the conjecture in types A and C. We also present evidence for the fact that a related statistic computes the energy function.

Résumé. Le modèle des alcôves du premier auteur et Postnikov décrit les cristaux de plus haut poids des algèbres de Lie semi-simples. Nous présentons une généralisation, appelée le modèle des alcôves quantique, et nous conjecturons qu'il décrit dans une manière uniforme les produits tensoriels des cristaux de Kirillov-Reshetikhin de type colonne, pour toutes les types affines symétriques. Nous prouvons la conjecture dans les types A et C. Nous fournissons aussi des preuves qu'une statistique associée donne la fonction d'énergie.

Keywords: Kirillov-Reshetikhin crystals, energy function, alcove model, quantum Bruhat graph, Kashiwara-Nakashima columns

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1 Introduction

Kashiwara's crystals [[Kashiwara(1991)]] encode the structure of certain bases (called crystal bases) for highest weight representations of quantum groups $U_q(\mathfrak{g})$ as q goes to zero. The first author and Postnikov [[Lenart and Postnikov(2007), Lenart and Postnikov(2008)]] defined the so-called alcove model for highest weight crystals associated to a semisimple Lie algebra $\mathfrak{g}$ (in fact, the model was defined more generally, for symmetrizable Kac-Moody algebras $\mathfrak{g}$). A related model is the one of Gaussent-Littelmann, based on LS-galleries [[Gaussent and Littelmann(2005)]]. Both models are discrete counterparts of the celebrated Littelmann path model.

In this paper we define a generalization of the alcove model, which we call the quantum alcove model, as it is based on enumerating paths in the so-called quantum Bruhat graph of the corresponding finite Weyl group. This graph first appeared in connection with the quantum cohomology of flag varieties [[Fulton and Woodward(2004)]]. The path enumeration is determined by the choice of a certain sequence of alcoves (an alcove path), like in the classical alcove model. If we restrict to paths in the usual Bruhat graph, we recover the classical alcove model. The mentioned paths in the quantum Bruhat graph first

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appeared in [[Lenart(2012)]], where they index the terms in the specialization $t = 0$ of the Ram-Yip formula [[Ram and Yip(2011)]] for Macdonald polynomials $P_\lambda(X; q, t)$.

We define crystal operators in the quantum alcove model, both classical ones f_i, $i > 0$, and the affine one f_0. The main conjecture is that the new model uniformly describes tensor products of column shape Kirillov-Reshetikhin (KR) crystals [[Kirillov and Reshetikhin(1990)]], for all untwisted affine types. We prove the conjecture in types A and C, by showing that the bijections constructed in [[Lenart(2012)]], from the objects of the quantum alcove model to tensor products of Kashiwara-Nakashima (KN) columns [[Kashiwara and Nakashima(1994)]], are affine crystal isomorphisms (indeed, a column shape KR crystal is realized by a KN column in these cases). The first author is working with S. Naito, A. Schilling, and M. Shimozono on a uniform proof of the conjecture for all untwisted affine types.

If the conjecture is true, then the quantum alcove model has the following two applications. The first one is to the *energy function* on a tensor product of KR crystals, which endows it with an affine grading. We present evidence for the fact that the so-called level statistic in the Ram-Yip formula mentioned above expresses the energy function. On another hand, the authors plan to realize the *combinatorial R-matrix* (i.e., the affine crystal isomorphism commuting two factors in a tensor product) by extending to the quantum alcove model the alcove model version of Schützenberger's *jeu de taquin* on Young tableaux in [[Lenart(2007)]]; the latter is based on so-called *Yang-Baxter moves*.

2 Background

2.1 Root systems

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra, whose rank is r. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible *root system*, $\mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}$ the real span of the roots, and $\Phi^+ \subset \Phi$ the set of positive roots. The sign of the root α, denoted $\mathrm{sgn}(\alpha)$, is defined to be 1 if $\alpha \in \Phi^+$, and -1 otherwise. Let $\rho := \frac{1}{2}(\sum_{\alpha\in\Phi^+}\alpha)$. Let $\alpha_1, \ldots, \alpha_r \in \Phi^+$ be the corresponding *simple roots*. We denote $\langle\cdot,\cdot\rangle$ the nondegenerate scalar product on $\mathfrak{h}^*_{\mathbb{R}}$ induced by the Killing form. Given a root α, we consider the corresponding *coroot* $\alpha^\vee := 2\alpha/\langle\alpha,\alpha\rangle$ and reflection s_α. If $\alpha = \sum_i c_i\alpha_i$, then the *height* of α, denoted by $\mathrm{ht}(\alpha)$, is given by $\mathrm{ht}(\alpha) = \sum_i c_i$. We will denote by $\widetilde{\alpha}$ the highest root in Φ^+, and we let $\theta = -\widetilde{\alpha}$.

Let W be the corresponding *Weyl group*, whose Coxeter generators are denoted, as usual, by $s_i := s_{\alpha_i}$. The length function on W is denoted by $\ell(\cdot)$. The *Bruhat order* on W is defined by its covers $w \lessdot ws_\alpha$, for $\alpha \in \Phi^+$, if $\ell(ws_\alpha) = \ell(w) + 1$. The mentioned covers correspond to the labeled directed edges of the *Bruhat graph* on W:

$$w \xrightarrow{\alpha} ws_\alpha \quad \text{for } w \lessdot ws_\alpha. \tag{1}$$

The *weight lattice* Λ is given by

$$\Lambda = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \,:\, \langle\lambda,\alpha^\vee\rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi\}\,. \tag{2}$$

The weight lattice Λ is generated by the *fundamental weights* $\omega_1, \ldots \omega_r$, which form the dual basis to the basis of simple coroots, i.e., $\langle\omega_i, \alpha_j^\vee\rangle = \delta_{ij}$. The set Λ^+ of *dominant weights* is given by

$$\Lambda^+ := \left\{\lambda \in \Lambda \,:\, \langle\lambda,\alpha^\vee\rangle \geq 0 \text{ for any } \alpha \in \Phi^+\right\}. \tag{3}$$

Let $\mathbb{Z}[\Lambda]$ be the group algebra of the weight lattice Λ, which has the $\mathbb{Z}$-basis of formal exponents x^λ, for $\lambda \in \Lambda$, with multiplication $x^\lambda \cdot x^\mu = x^{\lambda+\mu}$.

Given $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we denote by $s_{\alpha,k}$ the reflection in the affine hyperplane

$$H_{\alpha,k} := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \, : \, \langle \lambda, \alpha^\vee \rangle = k\} \, . \tag{4}$$

These reflections generate the *affine Weyl group* W_{aff} for the *dual root system* $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$. The hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}_{\mathbb{R}}^*$ into open regions, called *alcoves.* The *fundamental alcove* $A_\circ$ is given by

$$A_\circ := \left\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \,|\, 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Phi^+ \right\} . \tag{5}$$

Define $w \lhd ws_\alpha$, for $\alpha \in \Phi^+$, if $\ell(ws_\alpha) = \ell(w) - 2\langle \rho, \alpha^\vee \rangle + 1$. The *quantum Bruhat graph* [[Fulton and Woodward(2004)]] is defined by adding to the Bruhat graph (1) the following edges labeled by positive roots α:

$$w \stackrel{\alpha}{\longrightarrow} ws_\alpha \quad \text{ for } w \lhd ws_\alpha . \tag{6}$$

We will need the following properties of the quantum Bruhat graph [[Lenart, Naito, Schilling and Shimozono (2012)]].

Lemma 2.1. *Let $w \in W$. We have $w^{-1}(\theta) > 0$ if and only if $w \lhd s_\theta w$. We also have $w^{-1}(\theta) < 0$ if and only if $s_\theta w \lhd w$.*

Proposition 2.2. *Let $w \in W$, let α be a simple reflection, $\beta \in \Phi^+$, and assume $s_\alpha w \neq ws_\beta$. Then $w \lessdot s_\alpha w$ and $w \longrightarrow ws_\beta$ if and only if $ws_\beta \lessdot s_\alpha ws_\beta$ and $s_\alpha w \longrightarrow s_\alpha ws_\beta$. Furthermore, in this context we have $w \lessdot ws_\beta$ if and only if $s_\alpha w \lessdot s_\alpha ws_\beta$.*

Proposition 2.3. *Let $w \in W$, $\beta \in \Phi^+$, and assume $s_\theta w \neq ws_\beta$. Then $w \lhd s_\theta w$ and $w \longrightarrow ws_\beta$ if and only if $ws_\beta \lhd s_\theta ws_\beta$ and $s_\theta w \longrightarrow s_\theta ws_\beta$.*

2.2 Kirillov-Reshetikhin (KR) crystals

A $\mathfrak{g}$-*crystal* is a nonempty set B together with maps $e_i, f_i : B \to B \cup \{\mathbf{0}\}$ for $i \in I$ (I indexes the simple roots, as usual, and $\mathbf{0} \notin B$), and $\text{wt} : B \to \Lambda$. We require $b' = f_i(b)$ if and only if $b = e_i(b')$. The maps e_i and f_i are called crystal operators and are represented as arrows $b \to b' = f_i(b)$; thus they endow B with the structure of a colored directed graph. For $b \in B$, we set $\varepsilon_i(b) = \max\{k \mid e_i^k(b) \neq \mathbf{0}\}$, $\varphi_i(b) = \max\{k \mid f_i^k(b) \neq \mathbf{0}\}$. Given two $\mathfrak{g}$-crystals B_1 and B_2, we define their tensor product $B_1 \otimes B_2$ as follows. As a set, $B_1 \otimes B_2$ is the Cartesian product of the two sets. For $b = b_1 \otimes b_2 \in B_1 \otimes B_2$, the weight function is simply $\text{wt}(b) = \text{wt}(b_1) + \text{wt}(b_2)$. The crystal operator f_i is given by

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \varphi_i(b_2), \\ b_1 \otimes f_i(b_2) & \text{otherwise,} \end{cases}$$

while $e_i(b)$ is defined similarly. The *highest weight crystal* $B(\lambda)$ of highest weight $\lambda \in \Lambda^+$ is a certain crystal with a unique element u_λ such that $e_i(u_\lambda) = \mathbf{0}$ for all $i \in I$ and $\text{wt}(u_\lambda) = \lambda$. It encodes the structure of the crystal basis of the $U_q(\mathfrak{g})$-irreducible representation with highest weight λ as q goes to 0.

A *Kirillov-Reshetikhin (KR) crystal* [[Kirillov and Reshetikhin(1990)]] is a finite crystal $B^{r,s}$ for an affine algebra, associated to a rectangle of height r and length s. We now describe the KR crystals $B^{r,1}$ for type $A_{n-1}^{(1)}$ and $C_n^{(1)}$, where $r \in \{1, 2, \ldots, n-1\}$, and $r \in \{1, 2, \ldots, n\}$, respectively. As a classical type $A_{n-1}^{(1)}$ (resp. $C_n^{(1)}$) crystal, the KR crystal $B^{r,1}$ is isomorphic to the corresponding $B(\omega_r)$.

In type $A_{n-1}^{(1)}$, $b \in B(\omega_r)$ is represented by a strictly increasing filling with entries in $[n] := \{1, \dots, n\}$ of a height r column. To apply f_i on a tensor product of such elements, we need the following signature rule. For $1 \leq i \leq n-1$, the i-signature of a column filling is a word in $\{i, i+1\}$. The word is empty if the column contains both or neither letters, otherwise it is made up of one of the two letters from $\{i, i+1\}$ contained in the column. The 0-signature is computed the same way on the letters $\{n, 1\}$. To apply a crystal operator f_i, first compute the i-signature of the filling by concatenating the i-signatures of each column, and then reduce the i-signature by deleting $(i+1)i$ terms successively ($1n$ terms if $i = 0$). Eventually we obtain a reduced i-signature of the form $i \dots i(i+1) \dots (i+1)$ (the reduced 0-signature is of the form $n \dots n1 \dots 1$). Then the action of f_i for $i > 0$ (resp. e_i) is defined by changing the rightmost i to an $i+1$ (resp. leftmost $i+1$ to an i) in the corresponding column. Similarly f_0 (resp. e_0) changes the rightmost n to a 1 (resp. leftmost 1 to an n) in the corresponding column, and sorts the column ascendingly.

Example 2.4. Let $n = 3$. The 0-signature of the filling $\begin{array}{|c|c|c|}\hline 2&1&1\\\hline 3&2\\\cline{1-2}\end{array}$ is 311, which is already reduced. So we have $f_0\left(\begin{array}{|c|c|c|}\hline 2&1&1\\\hline 3&2\\\cline{1-2}\end{array}\right) = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2}\end{array}$.

In type $C_n^{(1)}$, $B(\omega_r)$ is represented by *Kashiwara-Nakashima (KN) columns* [[Kashiwara and Nakashima(1994)]] of height r, with entries in the set $\{1 < \cdots < n < \overline{n} < \cdots < \overline{1}\}$. The crystal operators are defined by a related signature rule.

Certain Demazure crystals for affine Lie algebras are isomorphic as classical crystals to tensor products of KR crystals [[Fourier, Schilling and Shimozono(2007)]]. Under this isomorphism only the KR arrows in Definition 2.5 below correspond to arrows in the related affine Demazure crystal.

Definition 2.5. An arrow $b \to f_i(b)$ is called a *Demazure arrow* if $i \neq 0$, or $i = 0$ and $\varphi_i(b) \geq 2$.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ be a partition, which is interpreted as a dominant weight in classical types; λ' is the conjugate partition. Let $B^{\otimes\lambda} = \bigotimes_{i=1}^{\lambda_1} B^{\lambda'_i,1}$. The *energy function* D is a statistic on $B^{\otimes\lambda}$. It is defined by summing the local energies of column pairs. We will only need the following property of the energy function, which defines it as an affine grading on the crystal $B^{\otimes\lambda}$.

Theorem 2.6 ([Schilling and Tingley(2011)]). *The energy is preserved by the classical crystal operators f_i. If $b \to f_0(b)$ is a Demazure arrow, then $D(f_0(b)) = D(b) - 1$.*

It follows that the energy is determined up to a constant on the connected components of the subgraph of the affine crystal $B^{\otimes\lambda}$ containing only the Demazure arrows. Note that there is only one connected component if all the tensor factors are *perfect* crystals. For instance, $B^{k,1}$ is perfect in type $A_{n-1}^{(1)}$, but not in type $C_n^{(1)}$.

In types A and C, and conjecturally in types B and D, there is another statistic on $B^{\otimes\lambda}$, called the *charge*, which is obtained by translating a certain statistic in the Ram-Yip formula for Macdonald polynomials (i.e., the level statistic in (8)) to the model based on KN columns [[Lenart(2012)]]; this is done by using certain bijections, see Section 4. The charge statistic is related to the energy function by the following theorem.

Theorem 2.7 ([Lenart and Schilling(2011)]). *Let $B^{\otimes\lambda}$ be a tensor product of KR crystals in type $A_{n-1}^{(1)}$ or type $C_n^{(1)}$. For all $b \in B^{\otimes\lambda}$, we have $D(b) = -\mathrm{charge}(b)$.*

The previous theorem is conjectured to also hold in types $B_n^{(1)}$ and $D_n^{(1)}$ [[Lenart and Schilling(2011)]]. The charge gives a much easier method to compute the energy than the recursive one based on Theorem 2.6.

3 The quantum alcove model

3.1 λ-chains and admissible subsets

We say that two alcoves are *adjacent* if they are distinct and have a common wall. Given a pair of adjacent alcoves A and B, we write $A \xrightarrow{\beta} B$ if the common wall is of the form $H_{\beta,k}$ and the root $\beta \in \Phi$ points in the direction from A to B.

Definition 3.1 ([Lenart and Postnikov(2007)])**.** An *alcove path* is a sequence of alcoves $(A_0, A_1, \dots, A_m)$ such that A_{j-1} and A_j are adjacent, for $j = 1, \dots m$. We say that an alcove path is *reduced* if it has minimal length among all alcove paths from A_0 to A_m.

Let $A_\lambda = A_\circ + \lambda$ be the translation of the fundamental alcove $A_\circ$ by the weight λ.

Definition 3.2 ([Lenart and Postnikov(2007)])**.** The sequence of roots $(\beta_1, \beta_2, \dots, \beta_m)$ is called a λ-*chain* if

$$A_0 = A_\circ \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_m} A_m = A_{-\lambda}$$

is a reduced alcove path.

We now fix a dominant weight λ and an alcove path $\Pi = (A_0, \dots, A_m)$ from $A_0 = A_\circ$ to $A_m = A_{-\lambda}$. Note that Π is determined by the corresponding λ-chain of positive roots $\Gamma := (\beta_1, \dots, \beta_m)$. We let $r_i := s_{\beta_i}$, and let $\widehat{r}_i$ be the affine reflection in the hyperplane containing the common face of A_{i-1} and A_i, for $i = 1, \dots, m$; in other words, $\widehat{r}_i := s_{\beta_i, -l_i}$, where $l_i := |\{j < i\,;\, \beta_j = \beta_i\}|$. We define $\widetilde{l}_i := \langle \lambda, \beta_i^\vee \rangle - l_i = |\{j \geq i\,;\, \beta_j = \beta_i\}|$.

Example 3.3. Consider the dominant weight $\lambda = 3\varepsilon_1 + 2\varepsilon_2$ in the root system A_2 (cf. Section 4.1 and the notation therein). The corresponding λ-chain is $(\alpha_{23}, \alpha_{13}, \alpha_{23}, \alpha_{13}, \alpha_{12}, \alpha_{13})$. The corresponding levels l_i are $(0, 0, 1, 1, 0, 2)$ and $\widetilde{l}_i$ are $\{2, 3, 1, 2, 1, 1\}$. The alcove path is shown in Figure 1(a); here A_0 is shaded, and $A_0 - \lambda$ is the alcove at the end of the path.

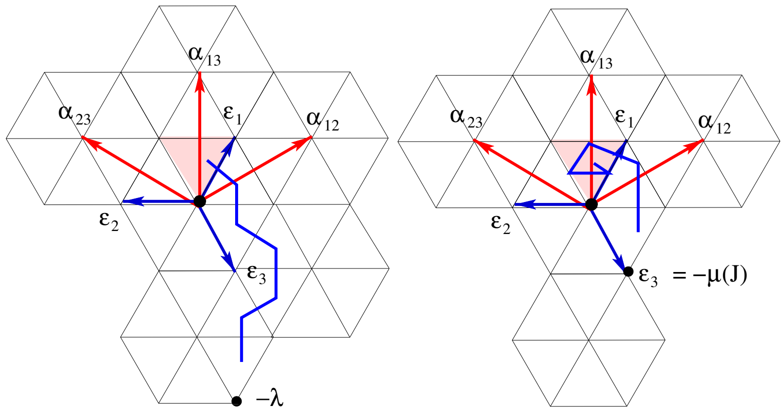
Let $J = \{j_1 < j_2 < \cdots < j_s\} \subseteq [m]$ be a subset of $[m]$. The elements of J are called *folding positions*. We fold Π in the hyperplanes corresponding to these positions and obtain a folded path, see Example 3.5 and Figure 1(b). Like Π, the folded path can be recorded by a sequence of roots, namely $\Delta = \Gamma(J) = (\gamma_1, \gamma_2, \dots, \gamma_m)$; here $\gamma_k = r_{j_1} r_{j_2} \dots r_{j_p}(\beta_k)$, with j_p the largest folding position less than k. We define $\gamma_\infty := r_{j_1} r_{j_2} \dots r_{j_s}(\rho)$. Upon folding, the hyperplane separating the alcoves A_{k-1} and A_k in Π is mapped to

$$H_{|\gamma_k|, -l_k^\Delta} = \widehat{r}_{j_1} \widehat{r}_{j_2} \dots \widehat{r}_{j_p}(H_{\beta_k, -l_k})\,, \tag{7}$$

for some l_k^Δ, which is defined by this relation.

Given $i \in J$, we say that i is a *positive folding position* if $\gamma_i > 0$, and a *negative folding position* if $\gamma_i < 0$. We denote the positive folding positions by J^+, and the negative ones by J^-. We call $\mu = \mu(J) := -\widehat{r}_{j_1} \widehat{r}_{j_2} \dots \widehat{r}_{j_s}(-\lambda)$ the *weight* of J. We define

$$\mathrm{level}(J) := \sum_{j \in J^-} \widetilde{l}_j. \tag{8}$$



(a) Γ for $\lambda = 3\varepsilon_1 + 2\varepsilon_2$ (b) $\Gamma(J)$ for $J = \{1, 2, 3, 5\}$

Fig. 1: Unfolded and folded λ-chain

Definition 3.4. A subset $J = \{j_1 < j_2 < \cdots < j_s\} \subseteq [m]$ (possibly empty) is an *admissible subset* if we have the following path in the quantum Bruhat graph on W:

$$1 \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} r_{j_1} r_{j_2} \xrightarrow{\beta_{j_3}} \cdots \xrightarrow{\beta_{j_s}} r_{j_1} r_{j_2} \cdots r_{j_s} \,. \tag{9}$$

We call $\Delta = \Gamma(J)$ an *admissible folding*. We let $\mathcal{A} = \mathcal{A}(\lambda)$ be the collection of admissible subsets.

Example 3.5. We continue Example 3.3. Let $J = \{1, 2, 3, 5\}$, then $\Delta = \Gamma(J) = \{\alpha_{23}, \alpha_{12}, \alpha_{31}, \alpha_{23}, \alpha_{21}, \alpha_{13}\}$. The folded path is shown in Figure 1(b). We have $J^+ = \{1, 2\}$, $J^- = \{3, 5\}$, $\mu(J) = -\varepsilon_3$, and $\operatorname{level}(J) = 2$. We check that J is admissible in (23).

Given $J \subseteq [m]$ and $\alpha \in \Phi$, we will use the following notation:

$$I_\alpha = I_\alpha(\Delta) := \{i \in [m] \,|\, \gamma_i = \pm\alpha\}\,, \quad L_\alpha = L_\alpha(\Delta) := \left\{ l_i^\Delta \,|\, i \in I_\alpha \right\},$$
$$\widehat{I}_\alpha = \widehat{I}_\alpha(\Delta) := I_\alpha \cup \{\infty\}, \quad \widehat{L}_\alpha = \widehat{L}_\alpha(\Delta) := L_\alpha \cup \{l_\alpha^\infty\},$$

where $l_\alpha^\infty := \langle \mu(J), \operatorname{sgn}(\alpha)\alpha^\vee \rangle$. We will use $\widehat{L}_\alpha$ to define the crystal operators on admissible subsets in the following sections. The following graphical representation of $\widehat{L}_\alpha$ is useful for such purposes. Let

$$\widehat{I}_\alpha = \{i_1 < i_2 < \cdots < i_n \le m < i_{n+1} = \infty\} \text{ and } \varepsilon_i := \begin{cases} 1 & \text{if } i \notin J \\ -1 & \text{if } i \in J \end{cases}.$$

If $\alpha > 0$, we define the continuous piecewise-linear function $g_\alpha : [0, n + \frac{1}{2}] \to \mathbb{R}$ by

$$g_\alpha(0) = -\frac{1}{2}, \quad g'_\alpha(x) = \begin{cases} \operatorname{sgn}(\gamma_{i_k}) & \text{if } x \in (k-1, k-\frac{1}{2}),\ k = 1, \ldots, n \\ \varepsilon_{i_k} \operatorname{sgn}(\gamma_{i_k}) & \text{if } x \in (k-\frac{1}{2}, k),\ k = 1, \ldots, n \\ \operatorname{sgn}(\langle \gamma_\infty, \alpha^\vee \rangle) & \text{if } x \in (n, n+\frac{1}{2}). \end{cases} \tag{10}$$

If $\alpha < 0$, we define g_α to be the graph obtained by reflecting $g_{-\alpha}$ in the x-axis. By [[Lenart and Postnikov(2008)]], we have

$$\mathrm{sgn}(\alpha) l_{i_k}^{\Delta} = g_\alpha\left(k - \frac{1}{2}\right), k = 1, \ldots, n, \text{ and } \mathrm{sgn}(\alpha) l_\alpha^\infty := \langle \mu(J), \alpha^\vee \rangle = g_\alpha\left(n + \frac{1}{2}\right). \quad (11)$$

Example 3.6. We continue Example 3.5. The graphs of g_{α_2} and g_θ are given in Figure 2.

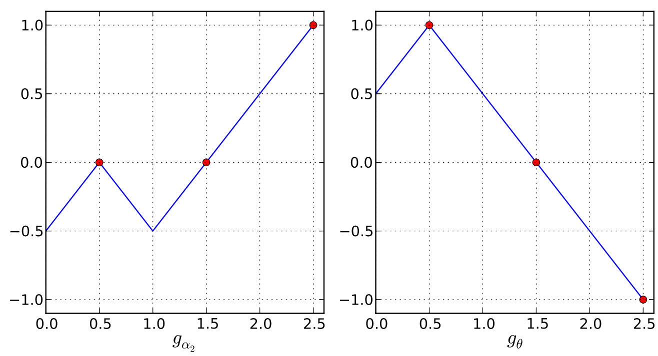


Fig. 2:

3.2 Crystal operators

We will now define *crystal operators* on the collection $\mathcal{A} = \mathcal{A}(\lambda)$ of admissible subsets corresponding to our fixed λ-chain. Let J be such an admissible subset, let Δ be the associated admissible folding, and $L(\Delta) = (l_i^\Delta)_{i \in [m]}$ its level sequence. Also recall from Section 3.1 the definitions of the finite sequences $I_\alpha(\Delta)$, $\widehat{I}_\alpha(\Delta)$, $L_\alpha(\Delta)$, and $\widehat{L}_\alpha(\Delta)$, where α is a root, as well as the related notation. Let $\delta_{i,j}$ be the Kronecker delta function.

Throughout we fix α_p, which is a simple root if $p > 0$, or θ if $p = 0$. We define f_p on admissible subsets. We will use the convention $J \backslash \{\infty\} = J \cup \{\infty\} = J$. Let M be the maximum of g_{α_p}, and suppose that $M \geq \delta_{p,0}$. Let m be the minimum index i in $\widehat{I}_{\alpha_p}(\Delta)$ for which we have $\mathrm{sgn}(\alpha_p) l_i^\Delta = M$.

Proposition 3.7. *Given the above setup, the following hold.*

(1) *If $m \neq \infty$, then $\gamma_m = \alpha_p$ and $m \in J$.*

(2) *If $M > \delta_{p,0}$ then m has a predecessor k in $\widehat{I}_{\alpha_p}(\Delta)$ such that*

$$\gamma_k = \alpha_p,\ k \notin J, \textit{ and } \mathrm{sgn}(\alpha_p) l_k^\Delta = M - 1.$$

Based on the previous proposition, we define

$$f_p(J) := \begin{cases} (J \backslash \{m\}) \cup \{k\} & \text{if } M > \delta_{p,0} \\ \mathbf{0} & \text{otherwise .} \end{cases} \quad (12)$$

To define the crystal operator e_p, we assume that $M > \langle \mu(J), \alpha^\vee \rangle$. Let k be the maximum index i in $I_{\alpha_p}(\Delta)$ for which we have $\mathrm{sgn}(\alpha_p) l_i^\Delta = M$, and let m be the successor of k in $\widehat{I}_{\alpha_p}(\Delta)$.

Proposition 3.8. *Given the above setup, the following hold.*
(1) *We have* $\gamma_k = \alpha_p$ *and* $k \in J$.
(2) *If* $m \neq \infty$ *then*

$$\gamma_m = -\alpha_p,\ m \notin J,\ \text{ and } \operatorname{sgn}(\alpha_p) l_m^{\Delta} = M - 1.$$

We define

$$e_p(J) := \begin{cases} (J \setminus \{k\}) \cup \{m\} & \text{if } M > \langle \mu(J), \alpha^{\vee} \rangle \\ \mathbf{0} & \text{otherwise .} \end{cases} \tag{13}$$

Note that $f_p(J) = J'$ if and only if $e_p(J') = J$.

Example 3.9. We continue Example 3.6. We find $f_2(J)$ by noting that $\widehat{I}_{\alpha_2} = \{1, 4, \infty\}$. From g_{α_2} in Figure 2 we can see that $\widehat{L}_{\alpha_2} = \{0, 0, 1\}$, so $k = 4, m = \infty$, and $f_2(J) = J \cup \{4\} = \{1, 2, 3, 4, 5\}$. We can see from Figure 2 that the maximum of $g_\theta = 1$, hence $f_0(J) = \mathbf{0}$.

Theorem 3.10. *If* $f_p(J)$ *is defined, then it is also an admissible subset. Similarly for* $e_p(J)$.

Proof: Suppose $p \neq 0$. We consider f_p first. The cases corresponding to $m \neq \infty$ and $m = \infty$ can be proved in similar ways, so we only consider the first case. Let $J = \{j_1 < j_2 < \ldots < j_s\}$, and let $w_i = r_{j_1} r_{j_2} \ldots r_{j_i}$. Based on Proposition 3.7, let $a < b$ be such that

$$j_a < k < j_{a+1} < \cdots < j_b = m < j_{b+1};$$

if $a = 0$ or $b + 1 > s$, then the corresponding indices j_a, respectively j_{b+1}, are missing. To show that $(J \setminus \{m\}) \cup \{k\}$ is an admissible subset, it is enough to prove

$$w_a \longrightarrow w_a r_k \longrightarrow w_a r_k r_{j_{a+1}} \longrightarrow \ldots \longrightarrow w_a r_k r_{j_{a+1}} \ldots r_{j_{b-1}} = w_b. \tag{14}$$

By our choice of k, we have

$$w_a(\beta_k) = \alpha_p \iff w_a^{-1}(\alpha_p) = \beta_k > 0 \iff w_a \lessdot s_p w_a = w_a r_k. \tag{15}$$

So we can rewrite (14) as

$$w_a \longrightarrow s_p w_a \longrightarrow s_p w_{a+1} \longrightarrow \ldots \longrightarrow s_p w_{b-1} = w_b. \tag{16}$$

We will now prove that (16) is a path in the quantum Bruhat graph. Observe

$$s_p w_{i-1} = w_i \iff w_{i-1}(\beta_{j_i}) = \pm \alpha_p \iff j_i \in I_\alpha.$$

Our choice of k and b implies that we have

$$s_p w_{i-1} \neq w_i \ \text{ for } a < i < b \tag{17}$$

(otherwise $j_i \in I_\alpha$ for $k < j_i < j_b$), and $s_p w_{b-1} = w_b$ since $j_b \in I_\alpha$. Since J is admissible, we have

$$w_{i-1} \longrightarrow w_i. \tag{18}$$

With (15) as the base case, assume by induction that $w_{i-1} \lessdot s_p w_{i-1}$. We can apply Proposition 2.2 to conclude that $w_i \lessdot s_p w_i$ and

$$s_p w_{i-1} \longrightarrow s_p w_i \ \text{ for } a < i < b. \tag{19}$$

The proof for $e_p(J)$ is similar. The above proof follows through for $p = 0$ with $\lessdot$ replaced by $\triangleleft$ with the help of Lemma 2.1 and Proposition 2.3. □

3.3 Main conjecture

The following is our main conjecture about the quantum alcove model. The setup is that of untwisted affine root systems.

Conjecture 3.11. (1) $\mathcal{A}(\lambda)$ *is isomorphic to the subgraph of* $B^{\otimes\lambda}$ *containing only the Demazure arrows.*
(2) *If* b *corresponds to* J *under the isomorphism in* (1)*, then the energy is given by* $D(b) = -\text{level}(J)$.

The reason for which we restrict to the Demazure arrows is that the other 0-arrows in $B^{\otimes\lambda}$ are realized in the quantum alcove model by removing/adding more than one element of an admissible subset; this rule is not yet understood, but we hope to elucidate it in the future. Part (1) of the conjecture is addressed in Section 4 for types A and C, whereas part (2) in these cases is essentially Theorem 2.7. There is even more evidence for the conjecture, namely the sets $\mathcal{A}(\lambda)$ and $B^{\otimes\lambda}$ have the same cardinality in type D and E (assuming that only certain fundamental weights appear in λ in type E). This follows by combining the specialization $t = 0$ of the Ram-Yip formula for Macdonald polynomials [[Ram and Yip(2011)]], with the relation between Macdonald polynomials and affine Demazure characters [[Ion(2003)]], as well as the relation between Demazure and KR crystals [[Fourier and Littelmann(2006)]]. In type D, it is also known that the generating functions of $-D(b)$ and $\text{level}(J)$ agree [[Schilling and Tingley(2011), Corollary 9.5]].

Assuming that part (1) of Conjecture 3.11 holds, part (2) can be approached by showing that the level statistic satisfies the recursive definition of energy in Theorem 2.6. The part of the recursion involving the crystal operators f_i, $i \neq 0$, is proved below.

Proposition 3.12. *Let* λ *be a dominant weight and* $J \in \mathcal{A}(\lambda)$*. If* $i \neq 0$ *then* $\text{level}(f_i(J)) = \text{level}(J)$.

Proof: By definition, $f_i(J)$ is obtained from J by adding a folding position and possibly removing one, and both of these are positive. We will show that the other folding positions in J do not change sign. This follows from the proof of Theorem 3.10. We continue using notation from that proof. For $w \in W$ and β a positive root, $w(\beta) > 0$ if and only if $\ell(w) < \ell(ws_\beta)$. Hence $\gamma_{j_i} = w_{i-1}(\beta_{j_i}) > 0$ if and only if $\ell(w_{i-1}) < \ell(w_i)$. By Proposition 2.2, the edge in (18) corresponding to J is a cover if and only if the edge in (19) corresponding to $f_i(J)$ is a cover, so we don't introduce any new negative folding positions. □

4 The quantum alcove model in types A and C

4.1 Type A

We start with the basic facts about the root system of type A_{n-1}. We can identify the space $\mathfrak{h}_{\mathbb{R}}^*$ with the quotient $V := \mathbb{R}^n/\mathbb{R}(1,\ldots,1)$, where $\mathbb{R}(1,\ldots,1)$ denotes the subspace in $\mathbb{R}^n$ spanned by the vector $(1,\ldots,1)$. Let $\varepsilon_1,\ldots,\varepsilon_n \in V$ be the images of the coordinate vectors in $\mathbb{R}^n$. The root system is $\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j \ : \ i \neq j,\ 1 \leq i,j \leq n\}$. The simple roots are $\alpha_i = \alpha_{i,i+1}$, for $i = 1,\ldots,n-1$. The highest root $\widetilde{\alpha} = \alpha_{1n}$, so $\theta = \alpha_{n1}$. The weight lattice is $\Lambda = \mathbb{Z}^n/\mathbb{Z}(1,\ldots,1)$. The fundamental weights are $\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$, for $i = 1,\ldots,n-1$. A dominant weight $\lambda = \lambda_1\varepsilon_1 + \ldots + \lambda_{n-1}\varepsilon_{n-1}$ is identified with the partition $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0)$ having at most $n-1$ parts. Note that $\rho = (n-1, n-2, \ldots, 0)$. Considering the Young diagram of the dominant weight λ as a concatenation of columns, whose heights are $\lambda'_1, \lambda'_2, \ldots$, corresponds to expressing λ as $\omega_{\lambda'_1} + \omega_{\lambda'_2} + \ldots$ (as usual, λ' is the conjugate partition to λ).

The Weyl group W is the symmetric group S_n, which acts on V by permuting the coordinates $\varepsilon_1,\ldots,\varepsilon_n$. Permutations $w \in S_n$ are written in one-line notation $w = w(1)\ldots w(n)$. For simplicity, we use the same notation (i,j) with $1 \le i < j \le n$ for the root α_{ij} and the reflection $s_{\alpha_{ij}}$, which is the transposition t_{ij} of i and j.

We now consider the specialization of the alcove model to type A. For any $k = 1, \ldots, n-1$, we have the following ω_k-chain, from $A_\circ$ to $A_{-\omega_k}$, denoted by $\Gamma(k)$ [[Lenart and Postnikov(2007)]]:

$$\begin{array}{llll}((k,k+1), & (k,k+2) & ,\ldots, & (k,n), \\ \ (k-1,k+1), & (k-1,k+2) & ,\ldots, & (k-1,n), \\ \ \vdots & \vdots & & \vdots \\ \ (1,k+1), & (1,k+2) & ,\ldots, & (1,n))\,. \end{array} \tag{20}$$

Example 4.1. For $n = 4, \Gamma(2)$ can be visualized as obtained from the following broken column, by pairing row numbers in the top and bottom parts in the prescribed order.

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \\ \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array} \quad , \quad \Gamma(2) = \{(2,3),(2,4),(1,3),(1,4)\}\,.$$

Note the top part of the above broken column corresponds to ω_2.

We construct a λ-chain $\Gamma = (\beta_1, \beta_2, \ldots, \beta_m)$ as the concatenation $\Gamma := \Gamma^1 \ldots \Gamma^{\lambda_1}$, where $\Gamma^j = \Gamma(\lambda'_j)$. Let $J = \{j_1 < \cdots < j_s\}$ be a set of folding positions in Γ, not necessarily admissible, and let T be the corresponding list of roots of Γ. The factorization of Γ induces a factorization on T as $T = T^1 T^2 \ldots T^{\lambda_1}$, and on $\Delta = \Gamma(J)$ as $\Delta = \Delta^1 \ldots \Delta^{\lambda_1}$. We denote by $T^1 \ldots T^j$ the permutation obtained via multiplication by the transpositions in $T^1, \ldots, T^j$ considered from left to right. For $w \in W$, written $w = w_1 w_2 \ldots w_n$, let $w[i,j] = w_i \ldots w_j$. To each J we can associate a filling of a Young diagram λ.

Definition 4.2. Let $\pi_j = \pi_j(T) = T^1 \ldots T^j$. We define the *filling map*, which produces a filling of the Young diagram λ, by

$$\mathrm{fill}(J) = \mathrm{fill}(T) = C_1 \ldots C_{\lambda_1}; \text{ here } C_i = \pi_i[1, \lambda'_i]. \tag{21}$$

We need the circular order $\prec_i$ on $[n]$ starting at i, namely $i \prec_i i+1 \prec_i \ldots \prec_i n \prec_i 1 \prec_i \ldots \prec_i i-1$. It is convenient to think of this order in terms of the numbers $1, \ldots, n$ arranged on a circle clockwise. We make the convention that, whenever we write $a \prec b \prec c \prec \ldots$, we refer to the circular order $\prec = \prec_a$. We have the following description of the edges of the quantum Bruhat graph in type A.

Proposition 4.3 ([Lenart(2012)])**.** *For $1 \le i < j \le n$ we have an edge $w \xrightarrow{(i,j)} w(i,j)$ if and only if there is no k such that $i < k < j$ and $w(i) \prec w(k) \prec w(j)$.*

Example 4.4. We restate Examples 3.3 and 3.5 in more detail. Let $n = 3$ and $\lambda = (3,2,0)$, which is identified with $3\varepsilon_1 + 2\varepsilon_2 = 2\omega_2 + \omega_1$, and corresponds to the Young diagram $\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2}\end{array}$. We have

$$\Gamma = \Gamma^1\Gamma^2\Gamma^3 = \Gamma(2)\Gamma(2)\Gamma(1) = \{\underline{(2,3)}, \underline{(1,3)} \,|\, \underline{(2,3)}, (1,3) \,|\, \underline{(1,2)}, (1,3)\},$$

where we underlined the roots in positions $J = \{1, 2, 3, 5\}$. Then

$$T = \{(2,3), (1,3) \mid (2,3) \mid (1,2)\}, \text{ and}$$

$$\Gamma(J) = \Delta = \Delta^1\Delta^2\Delta^3 = \{\underline{(2,3)}, \underline{(1,2)} \mid \underline{(3,1)}, (2,3) \mid \underline{(2,1)}, (1,3)\}, \tag{22}$$

where we again underlined the folding positions. We write permutations in (9) as broken columns. Based on Proposition 4.3, note that J is admissible since

$$\begin{array}{c}\boxed{1}\\\boxed{\mathbf{2}}\\ \\\boxed{\mathbf{3}}\end{array} \lessdot \begin{array}{c}\boxed{\mathbf{1}}\\\boxed{3}\\ \\\boxed{\mathbf{2}}\end{array} \lessdot \begin{array}{c}\boxed{2}\\\boxed{3}\\ \\\boxed{1}\end{array} \Big| \begin{array}{c}\boxed{2}\\\boxed{\mathbf{3}}\\ \\\boxed{\mathbf{1}}\end{array} \lhd \begin{array}{c}\boxed{2}\\\boxed{1}\\ \\\boxed{3}\end{array} \Big| \begin{array}{c}\boxed{\mathbf{2}}\\ \\\boxed{\mathbf{1}}\\\boxed{3}\end{array} \lhd \begin{array}{c}\boxed{1}\\ \\\boxed{2}\\\boxed{3}\end{array} \Big|. \tag{23}$$

By considering the top part of the last column in each segment and by concatenating these columns left to right, we obtain $\mathrm{fill}(J)$, i.e., $\mathrm{fill}(J) = \begin{array}{|c|c|c|}\hline 2 & 2 & 1\\\hline 3 & 1 & \\\cline{1-2}\end{array}$.

Definition 4.5. We define the *sorted filling map* $\mathrm{sfill}(J)$ by sorting ascendingly the columns of $\mathrm{fill}(J)$.

Theorem 4.6 ([Lenart(2012)])**.** *The map* sfill *is a bijection between* $\mathcal{A}(\lambda)$ *and* $B^{\otimes\lambda}$.

Theorem 4.7. *The map* sfill *preserves the affine crystal structures, with respect to Demazure arrows. In other words, given* $\mathrm{sfill}(J) = b$*, there is a Demazure arrow* $b \to f_i(b)$ *if and only if* $f_i(J) \neq \mathbf{0}$*, and we have* $f_i(b) = \mathrm{sfill}(f_i(J))$.

The main idea of the proof of Theorem 4.7 is the following. The signature of a filling, used to define the crystal operator f_i, can be interpreted as a graph similar to the graph of g_{α_i}, which is used to define the crystal operator on the corresponding admissible subsequence. The link between the two graphs is given by Lemma 4.8 below, called the level counting lemma, which we now explain.

Let $N_c(\sigma)$ denote the number of entries c in a filling σ. Let $\mathrm{ct}(\sigma) = (N_1(\sigma), \ldots, N_n(\sigma))$ be the content of σ. Let $\sigma[q]$ be the filling consisting of the columns $1, 2, \ldots, q$ of σ. Recall the factorization of Δ illustrated in (22) and the levels l_k^Δ defined in (7).

Lemma 4.8 ([Lenart(2011)], Proposition 4.1)**.** *Let* $J \subseteq [m]$*, and* $\sigma = \mathrm{fill}(J)$*. For a fixed* k*, let* $\gamma_k = (c, d)$ *be a root in* Δ^{q+1}*. We have*

$$\mathrm{sgn}(\gamma_k)\, l_k^\Delta = \langle \mathrm{ct}(\sigma[q]), \gamma_k^\vee \rangle = N_c(\sigma[q]) - N_d(\sigma[q]).$$

4.2 Type C

The results in type A are paralleled in type C. The affine crystal $B^{\otimes\lambda}$ is realized as a tensor product of KN columns [[Kashiwara and Nakashima(1994)]], with a related signature rule. In [[Lenart(2012), Theorem 6.1]], the first author provides a bijection between $\mathcal{A}(\lambda)$ and $B^{\otimes\lambda}$. This bijection can be shown to be an affine crystal isomorphism in the sense of Theorem 4.7.

References

[Fourier, Schilling and Shimozono(2007)] G. Fourier, A. Schilling, M. Shimozono. Demazure structure inside Kirillov-Reshetikhin crystals. *J. Algebra*, 309:386–404, 2007.

[Fourier and Littelmann(2006)] G. Fourier and P. Littelmann. Tensor product structure of affine Demazure modules and limit constructions. *Nagoya Math. J.*, 182:171–198, 2006.

[Fulton and Woodward(2004)] W. Fulton and C. Woodward. On the quantum product of Schubert classes. *J. Algebraic Geom.*, 13:641–661, 2004.

[Gaussent and Littelmann(2005)] S. Gaussent and P. Littelmann. LS-galleries, the path model and MV-cycles. *Duke Math. J.*, 127:35–88, 2005.

[Humphreys(1990)] J. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, 1990.

[Ion(2003)] B. Ion. Nonsymmetric Macdonald polynomials and Demazure characters. *Duke Math. J.*, 116:299–318, 2003.

[Kashiwara(1991)] M. Kashiwara. On crystal bases of the q-analogue of universal enveloping algebras. *Duke Math. J.*, 63:465–516, 1991.

[Kashiwara and Nakashima(1994)] M. Kashiwara and T. Nakashima. Crystal graphs for representations of the q-analogue of classical Lie algebras. *J. Algebra*, 165:295–345, 1994.

[Kirillov and Reshetikhin(1990)] A. N. Kirillov and N. Reshetikhin. Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras. *J. Soviet Math.*, 52:3156–3164, 1990.

[Lenart(2007)] C. Lenart. On the combinatorics of crystal graphs, I. Lusztig's involution. *Adv. Math.*, 211:204–243, 2007.

[Lenart(2012)] C. Lenart. From Macdonald polynomials to a charge statistic beyond type A. *J. Combin. Theory Ser. A*, 119:683–712, 2012.

[Lenart(2011)] C. Lenart. Hall-Littlewood polynomials, alcove walks and fillings of Young diagrams. *Discrete Math.*, 311:258–275, 2011

[Lenart, Naito, Schilling and Shimozono (2012)] C. Lenart, S. Naito, A. Schilling, and M. Shimozono. A uniform combinatorial model for Kirillov-Reshetikhin crystals and applications. Work in progress.

[Lenart and Postnikov(2008)] C. Lenart and A. Postnikov. A combinatorial model for crystals of Kac-Moody algebras. *Trans. Amer. Math. Soc.*, 360:4349–4381, 2008.

[Lenart and Postnikov(2007)] C. Lenart and A. Postnikov. Affine Weyl groups in K-theory and representation theory. *Int. Math. Res. Not.*, Art. ID rnm038, 1–65, 2007.

[Lenart and Schilling(2011)] C. Lenart and A. Schilling. Crystal energy functions via the charge in types A and C. *Math. Z.*, DOI 10.1007/s00209-012-1011-2.

[Ram and Yip(2011)] A. Ram and M. Yip. A combinatorial formula for Macdonald polynomials. *Adv. Math.*, 226: 309–331, 2011.

[Schilling and Tingley(2011)] A. Schilling and P. Tingley. Demazure crystals, Kirillov-Reshetikhin crystals, and the energy function. *Electronic J. Combin.*, 19(2) #P4, 2012.