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Flow polytopes and the Kostant partition function for signed graphs (extended abstract)

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Abstract. We establish the relationship between volumes of flow polytopes associated to signed graphs and the Kostant partition function. A special case of this relationship, namely, when the graphs are signless, has been studied in detail by Baldoni and Vergne using techniques of residues. In contrast with their approach, we provide combinatorial proofs inspired by the work of Postnikov and Stanley on flow polytopes. As an application of our results we study a distinguished family of flow polytopes: the Chan-Robbins-Yuen polytopes. Inspired by their beautiful volume formula $\prod_{k=0}^{n-2} Cat(k)$ for the type A_n case, where $Cat(k)$ is the k^{th} Catalan number, we introduce type C_{n+1} and D_{n+1} Chan-Robbins-Yuen polytopes along with intriguing conjectures about their volumes.

Résumé. Nous établissons la relation entre les volumes de polytopes de flux associés aux graphes signés et la fonction de partition de Kostant. Le cas particulier de cette relation où les graphes ne sont pas signés a été étudié en détail par Baldoni et Vergne en utilisant des techniques de résidus. Contrairement á leur approche, nous apportons des preuves combinatoires inspirées par l'analyse de Postnikov et Stanley sur les polytopes de flux. Comme mise en pratique des résultats, nous étudions une famille distinguée de polytopes de flux: les polytopes Chan-Robbins-Yuen. Inspirés par leur belle formule du volume $\prod_{k=0}^{n-2} Cat(k)$ pour le cas de type A_n (où $Cat(k)$ est le k-ème nombres de Catalan), nous présentons les polytopes Chan-Robbins-Yuen des types C_{n+1} et D_{n+1} accompagnés de conjectures intéressantes sur leurs volumes.

Keywords: Kostant partition function, flow polytopes, Chan-Robbins-Yuen polytope, Morris identity

1 Introduction

In this extended abstract we use combinatorial techniques to establish the relationship between volumes of flow polytopes associated to signed graphs and the Kostant partition function. Traditionally, flow polytopes are associated to loopless (and signless) graphs in the following way. Let G be a graph on the vertex set $[n+1]$, and let $\mathrm{in}(e)$ denote the smallest (initial) vertex of edge e and $\mathrm{fin}(e)$ the biggest (final) vertex of edge e. A **flow** is a function $\mathbf{f}$ from $E(G)$ to $\mathbb{R}_{\geq 0}$, where $\mathbf{f}(e)$ is the amount of fluid flowing on e from the smaller to the bigger vertices, so that the total fluid volume entering vertex 1 is one and leaving vertex $n+1$ is one, and there is conservation of fluid at the intermediate vertices. We can think of $\mathbf{f}$ as a vector in $\mathbb{R}_{\geq 0}^{\#E(G)}$. The **flow polytope** $\mathcal{F}_G$ of G is the set of all such flows $\mathbf{f}$ in $\mathbb{R}_{\geq 0}^{\#E(G)}$.

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Recall that the **Kostant partition function** K_G evaluated at the vector $\mathbf{a} \in \mathbb{Z}^{n+1}$ is defined as

$$K_G(\mathbf{a}) = \#\{(b_i)_{i\in[N]} \mid \sum_{i\in[N]} b_i\alpha_i = \mathbf{a} \text{ and } b_i \in \mathbb{Z}_{\geq 0}\}, \tag{1}$$

where $\{\{\alpha_1, \ldots, \alpha_N\}\}$ is the multiset of vectors corresponding to the multiset of edges of G under the correspondence which associates an edge (i, j), $i < j$, of G with a positive type A_n root $e_i - e_j$, where e_i is the i^{th} standard basis vector in $\mathbb{R}^{n+1}$.

The following connection between the volume of $\mathcal{F}_G$ for signless graphs G and the Kostant partition function was established by combinatorial methods by Postnikov and Stanley [13, 14] and by Baldoni and Vergne [1, 2] using residue techniques:

Theorem 1 ([13, 14]) *Given a loopless (signless) connected graph G on the vertex set $[n + 1]$, let $d_i = indeg_G(i)-1$ for $i \in \{2, \ldots, n\}$ where $indeg_G(i)$ is the indegree of vertex i. Then, the normalized volume* $\mathrm{vol}(\mathcal{F}_G)$ *of the flow polytope associated to the graph G is* $\mathrm{vol}(\mathcal{F}_G) = K_G(0, d_2, \ldots, d_n, -\sum_{i=2}^{n} d_i)$.

Kostant partition functions were introduced in and are a vital part of representation theory: weight multiplicities and tensor product multiplicities (e.g. Littlewood-Richardson coefficients) can be expressed in terms of them (see [6] and Steinberg's formula in [8, Sec. 24.4]). A salient feature of $K_G(\mathbf{a})$ is that it is a *piecewise quasipolynomial function* in $\mathbf{a}$ if G is fixed [7, 15]. Also, note that $K_G(\mathbf{a})$ can be obtained as a coefficient of the following generating series:

$$K_G(\mathbf{a}) = [\mathbf{x}^{\mathbf{a}}] \prod_{(i,j)\in E(G)} (1 - x_i x_j^{-1})^{-1}, \tag{2}$$

where $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} x_{n+1}^{a_{n+1}}$. In this abstract we give a generalization of Theorem 1 which establishes the connection between the volume of $\mathcal{F}_G$ for *signed* graphs G and a **dynamic** Kostant partition function with the following generating series:

$$K_G^{\text{dyn}}(\mathbf{a}) = [\mathbf{x}^{\mathbf{a}}] \prod_{(i,j,-)\in E(G)} (1 - x_i x_j^{-1})^{-1} \prod_{(i,j,+)\in E(G)} (1 - x_i - x_j)^{-1}, \tag{3}$$

where $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} x_{n+1}^{a_{n+1}}$. In simplified terms, our main result is the following:

Theorem 2 ([11]) *Given a loopless signed connected graph G on the vertex set $[n+1]$, let $d_i = indeg_G(i)-1$ for $i \in \{2, \ldots, n\}$, where $indeg_G(i)$ is the indegree of vertex i (the number of edges $(\cdot, i, -)$). The normalized volume* $\mathrm{vol}(\mathcal{F}_G)$ *of the flow polytope associated to graph G is*

$$\mathrm{vol}(\mathcal{F}_G) = K_G^{dyn}(0, d_2, \ldots, d_n, d_{n+1}),$$

where K_G^{dyn} has the generating series given in Equation (3).

An interesting example of flow polytopes for signless graphs is the case that $G = K_{n+1}$, the complete graph on $n + 1$ vertices. This is the Chan-Robbins-Yuen polytope $CRYA_n$ [4, 5] and its volume is $\prod_{k=0}^{n-2} Cat(k)$ where $Cat(k) = \frac{1}{k+1}\binom{2k}{k}$ is the k^{th} Catalan number. The volume formula for $CRYA_n$ was proved by Zeilberger [16], but no combinatorial proof is known. Here we introduce a flow polytope of signed graphs: $CRYD_n$ which is a type D_n analogue of $CRYA_n$. We give the following conjecture for its volume:

Conjecture 3 ([11]) *If $CRYD_n$ is the type D_n Chan-Robbins-Yuen polytopes (defined in Examples 7 (iv)), then* $\mathrm{vol}(CRYD_n) = 2^{(n-2)^2} \cdot \mathrm{vol}(CRYA_n)$.

The outline of this abstract is as follows: in Section 2 we give the necessary background on signed graphs, Kostant partition functions and flow polytopes. In Section 3 we talk about the vertices and Ehrhart functions of these polytopes. In Section 4 we show that certain operations on graphs, called reduction rules, are a way of encoding subdivisions of flow polytopes. Using these rules in Section 5 we state the Subdivision Lemma, which is a key ingredient to prove Theorem 1 and Theorem 2 in Section 6. In Section 7 we look at the volume of $CRYD_n$.

For a complete version of this extended abstract see [11].

2 Signed graphs, Kostant partition functions, and flows

In this section we define the concepts of graphs, Kostant partition functions and flows, all in the **signed** universe. One can think of these as the generalization of these concepts' signless counterparts from the type A_n (signless) root system to other classical types, such as C_{n+1} and D_{n+1}. We also define general flow polytopes, which are a main object of this paper.

Throughout this section, the graphs G on the vertex set $[n+1]$ that we consider are signed, that is there is a sign $\epsilon \in \{+,-\}$ assigned to each of its edges. We allow multiple edges and loops unless otherwise noted. The sign of a loop is always $+$, and a loop at vertex i is denoted by $(i,i,+)$. Denote by $(i,j,-)$ and $(i,j,+)$, $i<j$, a negative and a positive edge between vertices i and j, respectively. A positive edge, that is an edge labeled by $+$, is **positively incident** to both of its endpoints. A negative edge is positively incident to its smaller vertex and **negatively incident** to its greater endpoint. See Figure 2 for an example of the incidences. Denote by m_{ij}^{ϵ} the multiplicity of edge (i,j,ϵ) in G, $i \le j$, $\epsilon \in \{+,-\}$. To each edge (i,j,ϵ), $i \le j$, of G, associate the positive type C_{n+1} root $\mathrm{v}(i,j,\epsilon)$, where $\mathrm{v}(i,j,-) = e_i - e_j$ and $\mathrm{v}(i,j,+) = e_i + e_j$. Let $\{\{\alpha_1, \ldots, \alpha_N\}\}$ be the multiset of vectors corresponding to the multiset of edges of G. Note that $N = \sum_{1 \le i \le j \le n+1}(m_{ij}^- + m_{ij}^+)$.

The **Kostant partition function** K_G evaluated at the vector $\mathbf{a} \in \mathbb{Z}^{n+1}$ is defined as

$$K_G(\mathbf{a}) = \#\{(b_i)_{i\in[N]} \mid \sum_{i\in[N]} b_i\alpha_i = \mathbf{a} \text{ and } b_i \in \mathbb{Z}_{\geq 0}\}.$$

That is, $K_G(\mathbf{a})$ is the number of ways to write the vector $\mathbf{a}$ as an $\mathbb{N}$-linear combination of the positive type C_{n+1} roots corresponding to the edges of G, without regard to order.

Example 4 *For the signed graph G in Figure 1,* $K_G(1,3,0) = 2$, *since* $(1,3,0) = (e_1+e_2) + (2e_2) = (e_1 - e_2) + 2(2e_2)$.

Just like in the type A_n case, we would like to think of the vector $(b_i)_{i\in[N]}$ as a **flow**. For this we here give a precise definition of flows in the type C_{n+1} case, of which type A_n is of course a special case.

Let G be a signed graph on the vertex set $[n+1]$. Let $\{\{e_1, \ldots, e_N\}\}$ be the multiset of edges of G, and $\{\{\alpha_1, \ldots, \alpha_N\}\}$ the multiset of positive type C_{n+1} roots corresponding to the multiset of edges of G. Also, let M_G be the $(n+1) \times N$ matrix whose columns are the vectors in S_G. Fix an integer vector $\mathbf{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{Z}^{n+1}$.

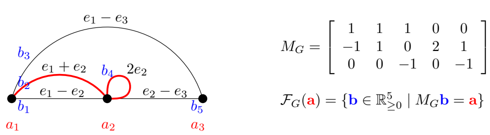


Fig. 1: A signed graph G on three vertices and the roots associated with each edge. The Kostant partition function $K_G(\mathbf{a})$ counts the number of ways of obtaining $\mathbf{a}$ as a nonnegative integer combination of the roots associated to G.

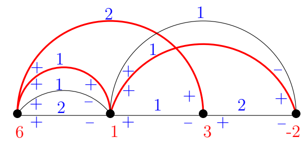


Fig. 2: A nonnegative integer flow on a signed graph G with excess flow vector $\mathbf{a} = (6, 1, 3, -2)$. The flows on the edges are in blue. The $+$ and $-$ signs denote the sign of the incidences of the edges.

An **a-flow** $\mathbf{f}_G$ on G is a vector $\mathbf{f}_G = (b_i)_{i\in[N]}$, $b_i \in \mathbb{R}_{\geq 0}$ such that $M_G\mathbf{f}_G = \mathbf{a}$. That is, for all $1 \leq i \leq n+1$, we have

$$\sum_{e\in E(G),\mathrm{inc}(e,v)=-} b(e) + a_v = \sum_{e\in E(G),\mathrm{inc}(e,v)=+} b(e) + \sum_{e=(v,v,+)} b(e), \tag{4}$$

where $b(e_i) = b_i$, $\mathrm{inc}(e, v) = -$ if $e = (g, v, -)$, $g < v$, and $\mathrm{inc}(e, v) = +$ if $e = (g, v, +)$, $g < v$, or $e = (v, j, \epsilon)$, $v < j$, and $\epsilon \in \{+, -\}$. Where $\mathrm{inc}(e, v) = +$ (respectively, $\mathrm{inc}(e, v) = -$) means the edge e is positively (respectively, negatively) incident to vertex v.

Example 5 *Figure 2 shows a signed graph G with four vertices with flow assigned to each edge. The excess flow is* $\mathbf{a} = (6, 1, 3, -2)$

Call $b(e)$ the **flow** assigned to edge e of G. If the edge e is negative, one can think of $b(e)$ units of fluid flowing on e from its smaller to its bigger vertex. If the edge e is positive, then one can think of $b(e)$ units of fluid flowing away both from e's smaller and bigger vertex to infinity. The positive edge e is then a "leak" taking away $2b(e)$ units of fluid.

From the above explanation it is clear that if we are given an **a**-flow $\mathbf{f}_G$ such that $\sum_{e=(i,j,+),i\leq j} b(e) = y$, then $2y = \sum_{i=1}^{n+1} a_i$.

An **integer a-flow** $\mathbf{f}_G$ on G is an **a**-flow $\mathbf{f}_G = (b_i)_{i\in[N]}$, with $b_i \in \mathbb{Z}_{\geq 0}$. It is a matter of checking the definitions to see that for a signed graph G on the vertex set $[n+1]$ and vector $\mathbf{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{Z}^{n+1}$, the number of integer **a**-flows on G are given by the Kostant partition function, as stated in the next remark.

Remark 6 *Given a signed graph G on the vertex set $[n+1]$ and a vector $\mathbf{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{Z}^{n+1}$, the integer $\mathbf{a}$-flows are in bijection with ways of writing $\mathbf{a}$ as a nonnegative linear combination of the roots associated to the edges of G. Thus $\#\{$integer $\mathbf{a}$-flows on $G\} = K_G(\mathbf{a})$.*

Define the **flow polytope** $\mathcal{F}_G(\mathbf{a})$ associated to a signed graph G on the vertex set $[n+1]$ and the integer vector $\mathbf{a} = (a_1, \ldots, a_{n+1})$ as the set of all **a**-flows $\mathbf{f}_G$ on G i.e., $\mathcal{F}_G = \{\mathbf{f}_G \in \mathbb{R}^N_{\geq 0} \mid M_G\mathbf{f}_G = \mathbf{a}\}$.

Next, we give the main examples of the flow polytopes we study (see Figure 4):

Examples 7 *(i) Let G be the graph with vertices $\{1, 2\}$ and edges $(1, 2, -)$ with multiplicity m_{12}; and let $\alpha = (1, -1)$. Then $\mathcal{F}_G(1, -1)$ is an $(m_{12} - 1)$-dimensional simplex.*

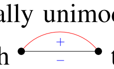
(ii) Let G be the signed graph with one vertex $\{1\}$ and loops $(1, 1, +)$ with multiplicity m_{11}; and let $\alpha = 2$. Then $\mathcal{F}_G(2)$ is an $(m_{11} - 1)$-dimensional simplex.

(iii) *Let $G = K_{n+1}$ be the complete graph with $n+1$ vertices (all edges $(i, j, -)$ $1 \le i < j \le n+1$) and $\alpha = e_1 - e_{n+1}$. Then $\mathcal{F}_{K_{n+1}}(e_1 - e_{n+1})$ is the type A_n Chan-Robbins-Yuen polytope or $CRYA_n$ [4, 5]. Such polytope is a face of the Birkhoff polytope of all $n \times n$ doubly stochastic matrices. It has dimension $\binom{n}{2}$, 2^{n-1} vertices, and Zeilberger [16] showed that its normalized volume is $\mathrm{vol}(CRYA_n) = \prod_{k=0}^{n-2} Cat(k)$ where $Cat(k) = \frac{1}{k+1}\binom{2k}{k}$ is the k^{th} Catalan number.*

(iv) *Let $G = K_n^D$ be the complete* signed *graph with n vertices (all edges $(i, j, \pm)$ $1 \le i < j \le n$) and $\alpha = 2e_1$. Then $CRYD_n = \mathcal{F}_{K_n^D}(2e_1)$ is a type D_n analogue of $CRYA_n$. We show it is integral with dimension $n(n-2)$ and $3^{n-1} - 2^{n-1}$ vertices. We conjecture (see Conjecture 3) that its normalized volume is $2^{(n-2)^2} \cdot \mathrm{vol}(CRYA_n)$.*

3 The vertices and the Ehrhart function of $\mathcal{F}_G(\mathbf{a})$

In [11] we characterize the vertices of the flow polytope $\mathcal{F}_G(\mathbf{a})$. As an application of our vertex characterization, we count the number of vertices of $CRYD_n$. We conclude the section by a simple expression for the Ehrhart function of flow polytopes. We give a short summary of these results.

If G is a graph with only negative edges, then for any integer vector $\mathbf{a}$ the vertices of $\mathcal{F}_G(\mathbf{a})$ are integral. This follows from the fact that the adjacency matrix of a signless graph is totally unimodular. Such a statement is not true for general signed graphs G. For example, if G is the graph  then $\mathcal{F}_G(1, 0)$ is a zero dimensional polytope with a vertex $(1/2, 1/2)$. However, we show, using our characterization of the vertices of $\mathcal{F}_G(\mathbf{a})$ that for special integer vectors $\mathbf{a}$ the vertices of $\mathcal{F}_G(\mathbf{a})$ are integral.

Theorem 8 *If $\mathbf{a} = (2, 0, \ldots, 0)$, then the vertices of $\mathcal{F}_G(\mathbf{a})$ are integer. In particular, the set of vertices of $\mathcal{F}_G(\mathbf{a})$ is a subset of the set of integer $\mathbf{a}$-flows on G.*

We note that the proof of Theorem 8 characterizes all vertices of $\mathcal{F}_G(2, 0, \ldots, 0)$ very concretely. This allows us to count the vertices of $CRYD_n$. Recall that $CRYA_n$ has 2^{n-1} vertices [5].

Proposition 9 *The polytope $CRYD_n$ has $3^{n-1} - 2^{n-1}$.*

From Remark 6 we see the first connection between flow polytopes $\mathcal{F}_G(\mathbf{a})$ and the Kostant partition function $K_G(\mathbf{a})$. This connection can be pushed further to obtain the *Ehrhart function* of $\mathcal{F}_G(\mathbf{a})$. Recall that given a convex polytope $\mathcal{P} \subset \mathbb{R}^N$, the number of lattice points of $t\mathcal{P}$ (the t^{th} dilate of $\mathcal{P}$) when t is a nonnegative integer is given the Ehrhart function $L_{\mathcal{P}}(t)$. If $\mathcal{P}$ has (rational) integral vertices then $L_{\mathcal{P}}(t)$ will be a (quasi) polynomial [3]. From the definition of the Ehrhart function it follows that $L_{\mathcal{F}_G(\mathbf{a})}(t) = K_G(t\mathbf{a})$ [1].

4 Reduction rules of the flow polytope $\mathcal{F}_G(\mathbf{a})$

In this section we propose an algorithmic way of triangulating the flow polytope $\mathcal{F}_G(\mathbf{a})$. This also yields a systematic way to calculate the volume of $\mathcal{F}_G(\mathbf{a})$ by summing the volumes of the simplices in the triangulation. This process is closely related to the triangulation of root polytopes by subdivision algebras, as studied by the first author in [9, 10].

Given a signed graph G on the vertex set $[n+1]$, if we have two edges incident to vertex i with opposite signs, *e.g.* $(a, i, -), (i, b, +)$ with flows p and q, we will add a new edge not incident to i, *e.g.* $(a, b, +)$, and discard one or both of the original edges to obtain graphs G_1, G_2, and G_3 respectively. We then

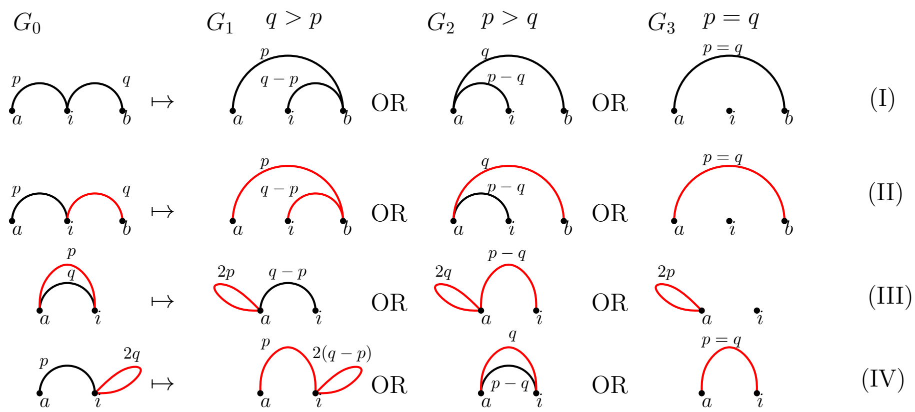


Fig. 3: Reduction rules from Equations (I)-(IV).

reassign flows to preserve the original excess flow on the vertices. We look at all possible cases and obtain the reduction rules (I)-(IV) in Figure 3. For instance, the signed graphs G_1, G_2, and G_3 in (I) have edges:

$$E(G_1) = E(G)\backslash\{(i,b,\pm)\} \cup \{(a,b,-)\}, \quad E(G_2) = E(G)\backslash\{(a,i,-)\} \cup \{(a,b,\pm)\}, \qquad \text{(I)}$$
$$E(G_3) = E(G)\backslash\{(a,i,-)\}\backslash\{(i,b,\pm)\} \cup \{(a,b,\pm)\}.$$

We say that G **reduces** to G_1, G_2, G_3 under the reduction rules defined by Equations (I)-(IV). Figure 3 shows these reduction rules graphically.

Proposition 10 *Given a signed graph G on the vertex set $[n+1]$, a vector $\mathbf{a} \in \mathbb{Z}^{n+1}$, and two edges e_1 and e_2 of G on which one of the reductions (I)-(IV) can be performed yielding the graphs G_1, G_2, G_3, then $\mathcal{F}_G(\mathbf{a}) = \mathcal{F}_{G_1}(\mathbf{a}) \bigcup \mathcal{F}_{G_2}(\mathbf{a})$, $\mathcal{F}_{G_1}(\mathbf{a}) \bigcap \mathcal{F}_{G_2}(\mathbf{a}) = \mathcal{F}_{G_3}(\mathbf{a})$, and $\mathcal{F}_{G_1}(\mathbf{a})^\circ \bigcap \mathcal{F}_{G_2}(\mathbf{a})^\circ = \emptyset$, where $\mathcal{P}^\circ$ denotes the interior of the polytope $\mathcal{P}$.*

5 Subdivision of the flow polytope $\mathcal{F}_G(\mathbf{a})$

In this section we use the reduction rules for signed graphs given in Section 4, following a specified order, to subdivide flow polytopes. The main result of this section is the Subdivision Lemma (Lemma 13). This lemma is key in all our pursuits: it lies at the heart of the relationship between flow polytopes and Kostant partition functions. It also is a tool for systematic subdivisions, and as such calculating volumes of particular flow polytopes.

In order to state the lemma, first we define the trees or equivalently compositions that are important for the subdivision (Sections 5.1 and 5.2). Then we define the order of application of reduction rules and state the Subdivision Lemma (Section 5.3). In the next section we use this lemma to compute volumes of flow polytopes for both signless graphs H and signed graphs G.

5.1 Noncrossing trees

The subdivisions mentioned above are encoded by bipartite trees with negative and positive edges that are noncrossing. We start by defining such trees.

A **negative bipartite noncrossing tree** T with left vertices $x_1, \dots, x_\ell$ and right vertices $x_{\ell+1}, \dots, x_{\ell+r}$ is a bipartite tree of negative edges that has no pair of edges $(x_p, x_{\ell+q}, -), (x_t, x_{\ell+u}, -)$ where $p < t$ and $q > u$. If L and R are the ordered sets $(x_1, \dots, x_\ell)$ and $(x_{\ell+1}, \dots, x_{\ell+r})$, let $\mathcal{T}^-_{L,R}$ be the set of such noncrossing bipartite trees. Note that $\#\mathcal{T}^-_{L,R} = \binom{\ell+r-2}{\ell-1}$, since they are in bijection with weak compositions of $\ell - 1$ into r parts. Namely, a tree T corresponds to the composition of indegrees of the right vertices: $(b_1, \dots, b_r)$, where b_i denotes the number of edges incident to $x_{\ell+i}$ in T. See Figure 5 (a) for an example of such a tree.

A **signed bipartite noncrossing tree** is a bipartite noncrossing tree T with negative $(\cdot, \cdot, -)$ and positive $(\cdot, \cdot, +)$ edges such that any right vertex is either incident to only negative edges or only positive edges. Let $\mathcal{T}^\pm_{L,R}(R^+)$ be the set of signed bipartite noncrossing tree with ordered left vertex set $L = (x_1, \dots, x_\ell)$, ordered right vertex set $R = (x_{\ell+1}, \dots, x_{\ell+r})$, and R^+ denoting the ordered set of right vertices incident to only positive edge (the ordering of R^+ is inherited from the ordering of R). Note that for fixed R^+, $\#\mathcal{T}^\pm_{L,R}(R^+) = \#\mathcal{T}^-_{L,R}$, and we can encode such trees with a signed composition $(b_1^\pm, b_2^\pm, \dots, b_r^\pm)$ indicating whether the incoming edges to each right vertex are all positive or all negative and where b_i denotes the number of edges incident to $x_{\ell+i}$ in T. See Figure 5 (b) for an example of such a tree. If both L and R are empty, the set $\mathcal{T}^\pm_{\varnothing,\varnothing}$ consists of one element: the empty tree.

5.2 Removing vertex i from a signed graph G

One of the points of the Subdivision Lemma is to start by a graph G on the vertex set $[n+1]$ and to subdivide the flow polytope of G into flow polytopes of graphs on a vertex set smaller than $[n+1]$. In this section we show the mechanics of this. We take a signed graph G and replace incoming and outgoing edges of a fixed vertex i by edges that avoid i and come from a noncrossing tree T. The outcome is a graph we denote by $G_T^{(i)}$ on the vertex set $[n+1]\backslash\{i\}$. To define this precisely we first introduce some notation:

Given a signed graph G and one of its vertices i, let $\mathcal{I}_i = \mathcal{I}_i(G)$ be the multiset of incoming edges to i (negative edges of the form $(\cdot, i, -)$). Let $\mathcal{O}_i = \mathcal{O}_i(G)$ be the multiset of outgoing edges from i (edges of the form $(\cdot, i, +)$ and $(i, \cdot, \pm)$). And let $\mathcal{O}_i^\pm$ be the signed refinement of $\mathcal{O}_i$. Note that $\#\mathcal{I}_i(G) = indeg_G(i)$, the indegree of vertex i.

Assign an ordering to the sets $\mathcal{I}_i$ and $\mathcal{O}_i$ and consider a tree $T \in \mathcal{T}^\pm_{\mathcal{I}_i,\mathcal{O}_i}(\mathcal{O}_i^+)$. For each tree-edge (e_1, e_2) of T where $e_1 = (r, i, -) \in \mathcal{I}_i$ and $e_2 \in \mathcal{O}_i$ ($e_2 = (i, s, \pm)$ or $(t, i, \pm)$), let $edge(e_1, e_2)$ be the following signed edge: $edge(e_1, e_2) = (r, s, \pm)$ if $e_2 = (i, s, \pm)$ or $edge(e_1, e_2) = (r, t, +)$ if $e_2 = (t, i, +)$. Note that if $e_1 = (r, i, -)$ and $e_2 = (r, i, +)$ then we allow $e(e_1, e_2)$ to be be the loop $(r, r, +)$.

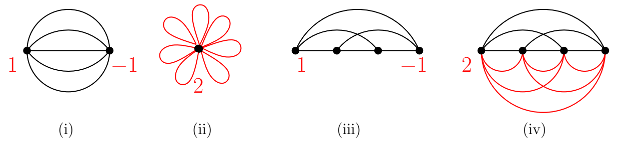
The graph $G_T^{(i)}$ is then defined as the graph obtained from G by removing the vertex i and all the edges incident to i from G and adding the edges $\{edge(e_1, e_2) \mid (e_1, e_2) \in E(T)\}$. See Figure 9 for examples of $G_T^{(i)}$.

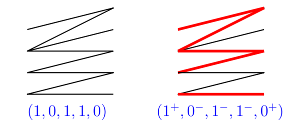
Remark 11 *If T is given by a weak composition of $\#\mathcal{I}_i - 1$ into $\#\mathcal{O}_i$ parts, say $(b_e)_{e \in \mathcal{O}_i}$. Then:*

(i) we record this composition by labeling the edges e in $\mathcal{O}_i$ of G with the corresponding part b_e. We can view this labeling as assigning a flow $b(e) = b_e$ to edges e of G.

(ii) The $b_e + 1$ edges $(\cdot, e)$ in T will correspond to $b_e + 1$ edges $edge(\cdot, e)$ in $G_T^{(i)}$. We think of these $b_e + 1$ edges as one edge coming from the original edge e in G, and b_e new edges.

The following is an easy consequence of the construction of $G_T^{(i)}$.





(a) $T_1 \in \mathcal{T}^-_{4,5}$ (b) $T_2 \in \mathcal{T}^{\pm}_{4,5}(1,3,5)$

Fig. 4: Graphs whose flow polytopes are: (i) and (ii) are simplices and (iii),(iv),(v) give instances of $CRY\,A_n$, $CRY\,D_n$ and $CRY\,C_n$.

Fig. 5: Examples of bipartite non-crossing trees.

Proposition 12 *Given a graph G on the vertex set $[n+1]$, the outgoing edges of vertex j of the graph $G^{(i)}_T$ on the vertex set $[n+1]\backslash\{i\}$ built above are: $\mathcal{O}_j(G^{(i)}_T) = \mathcal{O}_j(G) \cup \{$new edges $(k,j,+) \mid k < i < j\}$ if $j > i$ and $\mathcal{O}_j(G^{(i)}_T) = \mathcal{O}_j(G)$ otherwise.*

Next, we give a subdivision of the flow polytope $\mathcal{F}_G$ of a signed graph G in terms of flow polytopes $\mathcal{F}_{G^{(i)}_T}$ of graphs $G^{(i)}_T$.

5.3 Subdivision Lemma

In this subsection we are ready to state the Subdivision Lemma. The idea is that we want to subdivide the flow polytope of graph G on the vertex set $[n+1]$. To do this, we apply the reduction rules (I)-(IV) to incoming and outgoing edges of a vertex i in G with *zero flow*. We have to specify in which order we do the reduction at a given vertex i, since at any given stage there might be several choices of pairs of edges to reduce. First we fix a linear order $\theta_{\mathcal{I}}$ on the multiset $\mathcal{I}_i(G)$ of incoming edges to vertex i, and a linear order $\theta_{\mathcal{O}}$ on the multiset $\mathcal{O}_i(G)$ of outgoing edges from vertex i. Recall that $\mathcal{O}_i(G)$ also includes edges $(a,i,+)$ where $a < i$. We choose the pair of edges to reduce in the following way: we pick the first available edge from $\mathcal{I}_i(G)$ and from $\mathcal{O}_i(G)$ according to the orders $\theta_{\mathcal{I}}$ and $\theta_{\mathcal{O}}$. At each step of the reduction, one outcome will have one fewer incoming edge and the other outcome will have one fewer outgoing edge. In each outcome, when we choose the next pair of edges to reduce we pick the next edge from $I_i(G)$ and from $\mathcal{O}_i(G)$ that is still available.

The Subdivision Lemma shows that when we follow this order to repeatedly apply reductions to a vertex with zero flow, we obtain outcomes without this vertex; and the outcomes are encoded by signed bipartite noncrossing trees.

Lemma 13 (Subdivision Lemma) *Let G be a signed graph on the vertex set $[n+1]$ and $\mathcal{F}_G(\mathbf{a})$ be its flow polytope for $\mathbf{a} \in \mathbb{Z}^{n+1}$ with $a_i = 0$. Fix linear orders $\theta_{\mathcal{I}}$ and $\theta_{\mathcal{O}}$ on $\mathcal{I}_i(G)$ and $\mathcal{O}_i(G)$ respectively. If we apply the reduction rules to edges incident to vertex i following the linear orders, then the flow polytope subdivides as:*

$$\mathcal{F}_G(\mathbf{a}) = \bigcup_{T \in \mathcal{T}^{\pm}_{L,R}(R^+)} \mathcal{F}_{G^{(i)}_T}(a_1, \ldots, a_{i-1}, \hat{a}_i, a_{i+1}, \ldots, a_n, a_{n+1}), \tag{5}$$

where $G^{(i)}_T$ is as defined in Section 5.2; and $\mathcal{T}^{\pm}_{L,R}(R^+)$ is the set of signed trees with $L = \theta_{\mathcal{I}}(\mathcal{I}_i)$, $R = \theta_{\mathcal{O}}(\mathcal{O}_i)$ and $R^+ = \theta_{\mathcal{O}}(\mathcal{O}^+_i)$.

See Figure 7 for an example of a subdivision using the descending order and where the outcomes are indexed by signed bipartite trees.

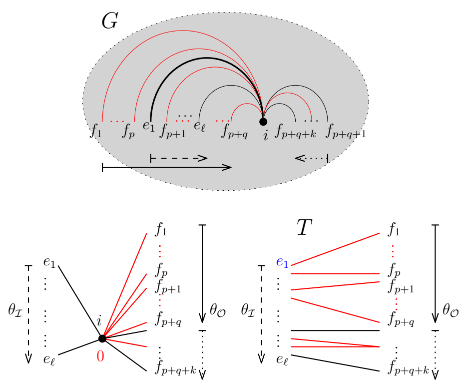


Fig. 6: Setting of Lemma 13 for edges incident to vertex i. We fix total orders $\theta_{\mathcal{I}}$ and $\theta_{\mathcal{O}}$ on $\mathcal{I}_i(G)$ and $\mathcal{O}_i(G)$ respectively. The resulting bipartite trees are in $\mathcal{T}^{\pm}(L,R)^{R^{"}}$ where $L = \theta_{\mathcal{I}}(\mathcal{I})$, $R = \theta_{\mathcal{O}}(\mathcal{O})$ and $R^+ = \theta_{\mathcal{O}}(\mathcal{O}^+)$.

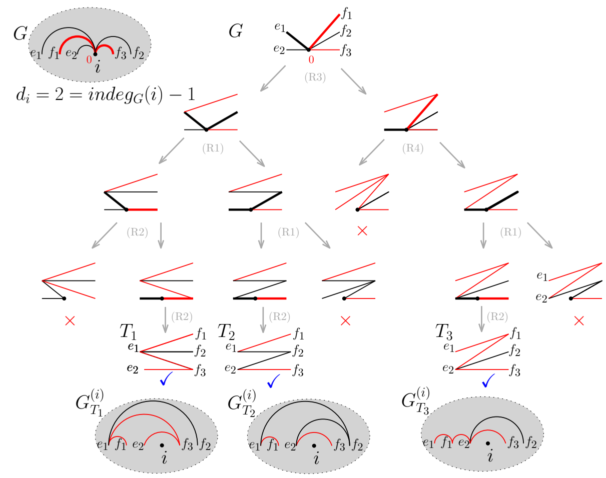


Fig. 7: Example of a subdivision (the selected edges to reduce are **bold**). The outcomes indicated by × are bad outcomes since they are priori lower dimensional. The final outcomes indicated by ✓ are indexed by signed trees in $\mathcal{T}^{\pm}_{\{e_1,e_2\},\{f_1,f_2,f_3\}}(f_1)$ or equivalently the compositions $(0^-,0^-,1^+)$, $(0^-,1^-,0^+)$, and $(1^-,0^-,0^+)$.

In the next section, we apply Lemma 13 to compute the volume of the flow polytope $\mathcal{F}_G(\mathbf{a})$ where G is a signed graph and $\mathbf{a} = (2,0,\ldots,0)$, the highest root of the root system C_{n+1}. As a motivation and to highlight differences, we first use a special case of the Subdivision Lemma, as done by Postnikov and Stanley [13, 14], to compute the volume of the polytope $\mathcal{F}_H(1,0,\ldots,0,-1)$ where H is a graph with only negative edges.

6 Volume of flow polytopes

In this section we use the Subdivision Lemma (Lemma 13) on flow polytopes $\mathcal{F}_H(1,0,\ldots,0,-1)$, where H is a graph consisting of only negative edges, and on $\mathcal{F}_G(2,0,\ldots,0)$, where G is a signed graph, to prove the formulae for their volumes given in Theorem 1 ([13, 14]) and Theorem 2, respectively. For the case of signed graphs G, we will motivate and introduce the notion of dynamic Kostant partition function, which specializes to Kostant partition functions in the case of signless graphs.

6.1 A correspondence between certain integer flows and simplices in a triangulation of $\mathcal{F}_H(e_1 - e_{n+1})$

Let H be a graph on the vertex set $[n+1]$ and *only* negative edges, and $\mathcal{F}_H(1,0,\ldots,0,-1)$ be its flow polytope where $(1,0,\ldots,0,-1) \in \mathbb{Z}^{n+1}$. We apply Lemma 13 successively to vertices $2,3,\ldots,n$. At the end we obtain the subdivision:

$$\mathcal{F}_H(1,0,\ldots,0,-1) = \bigcup_{T_n^-} \cdots \bigcup_{T_2^-} \mathcal{F}_{(\cdots(H^{(2)}_{T_2^-})^{(3)}_{T_3^-}\cdots)^{(n)}_{T_n^-}}(1,-1), \tag{6}$$

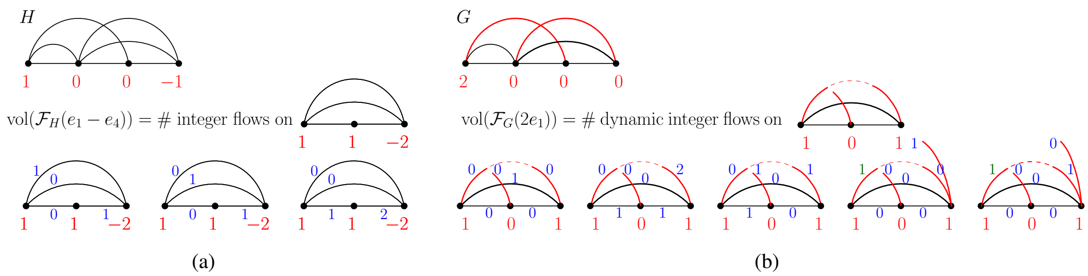


Fig. 8: Examples of Theorems 1 and 2. (a) Given a graph H with negative edges then $\mathrm{vol}\mathcal{F}_H(1,0,0,-1) = K_H(0,1,1,-2) = 3$ this is $\#\{$integer flows on H excess flow $(0,d_2,d_3,d_4)=(0,1,1,-2)\}$. (b) Given signed graph G then $\mathrm{vol}\mathcal{F}_G(2,0,0,0) = K_G^{\mathrm{dyn}}(0,1,0,1) = 5$ this is $\#\{$ dynamic integer flows on G with excess flow $(0,d_2,d_3,d_4)=(0,1,0,1)\}$.

where T_i^- are noncrossing trees with only negative edges. Then $H_n := (\cdots(H^{(2)}_{T_2^-})^{(3)}_{T_3^-}\cdots)^{(n)}_{T_n^-}$ is a graph consisting of two vertices, 1 and $n+1$ and $\#E(H)-n+1$ edges between them. Then $\mathcal{F}_{H_n}(1,-1)$ is an $(\#E(H)-n)$-dimensional simplex with normalized unit volume. Therefore, $\mathrm{vol}(\mathcal{F}_{H_n}(1,0,\ldots,0,-1))$ is the number of choices of bipartite noncrossing trees $T_2^-,\ldots,T_n^-$ where T_{i+1}^- encodes a composition of $\#\mathcal{I}_{i+1}(H_i)-1$ with $\#\mathcal{O}_{i+1}(H_i)$ parts. The next result by Postnikov and Stanley [13, 14] shows that this number is also the number of certain integer flows on H. See Figure 9 (a) for an example of such a subdivision and the correspondence with an integer flow.

Theorem 1 *[[13, 14]] Given a loopless (signless) graph H on the vertex set $[n+1]$, let $d_i = indeg_H(i)-1$ for $i \in \{2,\ldots,n\}$. Then, the normalized volume $\mathrm{vol}(\mathcal{F}_H(1,0,\ldots,0,-1)$ of the flow polytope associated to graph H is $\mathrm{vol}(\mathcal{F}_H(1,0,\ldots,0,-1)) = K_H(0,d_2,\ldots,d_n,-\sum_{i=2}^n d_i)$, where K_H is the Kostant partition function of H.*

Example 14 (Application of Theorem 1) *The flow polytope $\mathcal{F}_H(1,0,0,-1)$ for the negative graph H in Figure 8 (a) has normalized volume $K_H(0,1,1,-2)=3$.*

We now look at computing the normalized volume of $\mathcal{F}_G(\mathbf{a})$ where G is a signed graph and $\mathbf{a}=2e_1$.

6.2 A correspondence between certain dynamic integer flows and simplices in a triangulation of $\mathcal{F}_G(2e_1)$

Let G be a signed graph on the vertex set $[n+1]$ and $\mathbf{a}=(2,0,\ldots,0)$. In order to subdivide the polytope $\mathcal{F}_G(\mathbf{a})$, we follow the same first steps as in the previous case. Mainly:

We apply Lemma 13 successively to vertices $2,3,\ldots,n+1$. At the end we obtain:

$$\mathcal{F}_G(2,0,\ldots,0) = \bigcup_{T_{n+1}} \cdots \bigcup_{T_2} \mathcal{F}_{(\cdots(G^{(2)}_{T_2})^{(3)}_{T_3}\cdots)^{(n+1)}_{T_{n+1}}}(2), \tag{7}$$

where T_i are signed noncrossing trees. In this case, $G_{n+1} := (\cdots(G^{(2)}_{T_2})^{(3)}_{T_3}\cdots)^{(n+1)}_{T_{n+1}}$ is a graph consisting of one vertex with $\#E(G)-n$ positive loops. Thus, $\mathcal{F}_{G_{n+1}}(2)$ is an $(\#E(G)-n-1)$-dimensional simplex with normalized unit volume. Therefore, $\mathrm{vol}(\mathcal{F}_G(2,0,\ldots,0))$ is the number of choices of signed

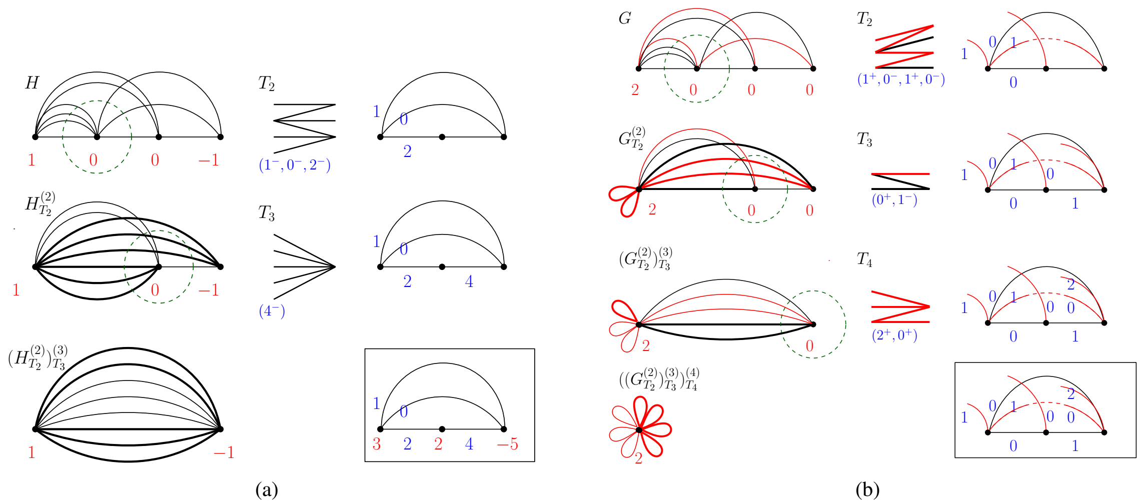


Fig. 9: Example of the subdivision to find the volume of (a) $\mathcal{F}_H(1,0,0,-1)$ for H with only negative edges and of (b) $\mathcal{F}_G(2,0,0,0)$ for signed G. The subdivision is encoded by noncrossing trees T_{i+1} that are equivalent to compositions $(b_1,\ldots,b_r)$ of $\#\mathcal{I}_{i+1}(H_i)-1$ ($\#\mathcal{I}_{i+1}(G_i)-1$) with $\#\mathcal{O}_{i+1}(H_i)$ ($\#\mathcal{O}_{i+1}(G_i)$) parts. These trees or compositions are recorded by the integer (dynamic) flow on $H\backslash\{1\}$ ($G\backslash\{1\}$) in the box with excess flow $(d_2,d_3,-d_2-d_3)=(3,2,-5)$ where $d_i=indeg_i(H)$ ($(d_2,d_3,d_4)=(2,1,1)$ where $d_i=indeg_i(G)$).

noncrossing bipartite trees $T_2,T_3,\ldots,T_{n+1}$ where T_{i+1} encodes a composition of $\#\mathcal{I}_{i+1}(G_i)-1$ with $\#\mathcal{O}_{i+1}(G_i)$ parts. However, instead of a correspondence between $(G;(T_2,T_3,\ldots,T_{n+1}))$ and the usual integer flows on G, there is a correspondence with a special kind of integer flow on G that we call **dynamic integer flow**. See Figure 9 (b) for an example of such a subdivision and the correspondence with such an integer flow.

Next, we motivate the need of these new integer flows. Let $G_i := (\cdots(G^{(2)}_{T_2})\cdots)^{(i)}_{T_i}$. The tree T_{i+1} is given by a signed composition $(b_e^{(i+1)})_{e\in\mathcal{O}_{i+1}(G_i)}$ of $\#\mathcal{I}_{i+1}(G_i)-1$ into $\#\mathcal{O}_{i+1}(G_i)$ parts. And by Remark 11 (i), we can encode the composition by assigning a flow $b(e)=b_e^{(i+1)}$ for $e\in\mathcal{O}_{i+1}(G_i)$ to G_i. However, iterating Proposition 12 we get $\mathcal{O}_{i+1}(G_i)\supseteq\mathcal{O}_{i+1}(G)$. If the graphs only have negative edges, these sets are equal but for signed graphs these sets are not the same (see Figure 9 (b)). Thus, we cannot encode the compositions as flows on a *fixed* graph G but rather on a graph G *and* additional positive edges added according to the flows assigned to previous positive edges. This is what we mean by dynamic flow. The next definition makes this precise.

Definition 15 (Dynamic integer flow) *Given a signed graph G, we regard its edge $e=(i,j,+)$ as two positive* half-edges *that still have "memory" of being together. Thus, we assign nonnegative* integer *flows $b_\ell(e)$ and $b_r(e)$ to the left and right halves of the positive edge, starting at $b_\ell(e)$. Once we assign $b_\ell(e)$ units of flow,* we add $b_\ell(e)$ new right positive half-edges e' incident to j *that can also be assigned nonnegative integer flows $b_r(e')$. When we assign a nonnegative integer flow to a right positive half-edges no edges are added.*

An analogue of Equation (4) *still holds:*

$$\sum_{e\in E(G),inc(e,v)=-} b(e) + a_v = \sum_{e\in E(G),inc(e,v)=+} (b_\ell(e) + b_r(e)) + \sum_{e',\text{new right pos. half edges}} b_r(e'), \quad (8)$$

where a_v is the excess flow at vertex v and $inc(e, v) = -$ if $e = (g, v, -)$, $g < v$, and $inc(e, v) = +$ if $e = (g, v, +)$, $g < v$, or $e = (v, j, \pm)$, $v < j$,. We call these integer $\mathbf{a}$*-flows* **dynamic**.

Definition 16 (dynamic Kostant partition function) *Given a signed graph G on the vertex set $[n + 1]$ and $\mathbf{a}$ a vector in $\mathbb{Z}^{n+1}$, the* **dynamic Kostant partition function** *$K^{dyn}_G(\mathbf{a})$ is the number of integer dynamic $\mathbf{a}$-flows in G.*

Example 17 *For the signed graph G in Figure 8 (b) with the positive edges $e = (1, 3, +)$ and $e' = (2, 4, +)$, we give its five integer dynamic flows with excess flow $(0, 1, 0, 1)$ where we add $b_\ell(e') = 0$ right half edges in three of them and $b_\ell(e') = 1$ on the other two of them.*

The dynamic Kostant partition function $K^{\mathrm{dyn}}_G(\mathbf{a})$ has the following generating series,

Proposition 18 *If $\mathbf{x}^\mathbf{a} = x_1^{a_1} x_2^{a_2} \cdots x_{n+1}^{a_{n+1}}$ then the generating series of the dynamic Kostant partition function is*

$$\sum_{\mathbf{a}\in\mathbb{Z}^{n+1}} K^{dyn}_G(\mathbf{a})\mathbf{x}^\mathbf{a} = \prod_{(i,j,-)\in E(G)} (1 - x_i x_j^{-1})^{-1} \prod_{(i,j,+)\in E(G)} (1 - x_i - x_j)^{-1}. \quad (9)$$

We are now ready to state and prove an application of the technique we developed.

Theorem 2 *Given a loopless signed graph G on the vertex set $[n + 1]$, let $d_i = indeg_G(i)$ for $i \in \{2, \ldots, n, n + 1\}$. The normalized volume $\mathrm{vol}(\mathcal{F}_G)$ of the flow polytope associated to graph G is*

$$\mathrm{vol}(\mathcal{F}_G(2, 0, \ldots, 0)) = K^{dyn}_G(0, d_2, \ldots, d_n, d_{n+1}).$$

Example 19 (Application of Theorem 2) *The flow polytope $\mathcal{F}_G(2, 0, 0, 0)$ for the signed graph G in Figure 8 (b) has normalized volume 5. This is the number of dynamic integer flows on G with excess flow $(0, d_2, d_3, d_4) = (0, 1, 0, 1)$ where $d_i = indeg_G(i) - 1$.*

7 The volumes of the (signed) Chan-Robbins-Yuen polytopes

Recall the definition of the polytope $CRYA_n$ in Examples 7 (iii) which was defined and studied in [4, 5]. By Theorem 1, its volume is given by $K_{K_{n+1}}(0, 1, 2, \ldots, n - 2, -\binom{n-1}{2})$. Zeilberger computed in [16] this value, and thus the volume of the polytope, using the *Morris identity* [12, Thm. 4.13].

Corollary 20 ([16], volume $CRYA_n$) *For $n \geq 1$, $\mathrm{vol}(CRYA_n) = \prod_{k=0}^{n-2} Cat(k)$, where $Cat(k) = \frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number.*

Recall the definition of the polytope $CRYD_n$ in Examples 7 (iv). By Theorem 2, its volume is given by $K^{dyn}_{K^D_n}(0, 1, 2, \ldots, n - 1)$. Calculating $\mathrm{vol}(CRYD_n)$ through the dynamic flow or direct volume computation suggests the following conjecture:

Conjecture 3 (volume $CRYD_n$) *For $n \geq 1$, $\mathrm{vol}(CRYD_n) = 2^{(n-2)^2} \prod_{k=0}^{n-2} Cat(k)$, where $Cat(k) = \frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number.*

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