Cuts and Flows of Cell Complexes

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Abstract. We study the vector spaces and integer lattices of cuts and flows of an arbitrary finite CW complex, and their relationships to its critical group and related invariants. Our results extend the theory of cuts and flows in graphs, in particular the work of Bacher, de la Harpe and Nagnibeda. We construct explicit bases for the cut and flow spaces, interpret their coefficients topologically, and describe sufficient conditions for them to be integral bases of the cut and flow lattices. Second, we determine the precise relationships between the discriminant groups of the cut and flow lattices and the higher critical and cocritical groups; these are expressed as short exact sequences with error terms corresponding to torsion (co)homology. As an application, we generalize a result of Kotani and Sunada to give bounds for the complexity, girth, and connectivity of a complex in terms of Hermite's constant.

Resumé. Nous étudions les espaces vectoriels et les réseaux entiers des coupures et flots d'un CW-complexe arbitraire fini, et leur relations avec son groupe critique et invariants similaires. Nos résultats développent la théorie des coupures et flots dans les graphes, en particulier le travail de Bacher, de la Harpe et Nagnibeda. Nous construisons des bases explicites pour les espaces des coupures et des flots, donnons une description topologique de leurs coefficients, et décrivons conditions suffisants pour qu'ils soient des bases entières des réseaux des coupures et des flots. De plus, nous déterminons les relations précises entre les groupes discriminantes des réseaux, et les groupes critical et cocritical; ces relations prennent la forme des suites exactes courtes, avec termes correspondant à la torsion (co)homologie. Comme application, nous généralisons un résultat de Kotani et Sunada sur bornes pour la complexité, la circonférence, et la connectivité d'un CW-complexe en termes de la constante d'Hermite.

Keywords: cut lattice, flow lattice, critical group, spanning forest, cell complex

1   Introduction

This paper is about vector spaces, integer lattices of cuts and flows, and finite group invariants associated with a finite cell complex.

By way of background, the critical group of a graph is a finite abelian group whose order is the number of spanning forests. The definition was introduced independently in several different settings, including

arithmetic geometry Lorenzini [1991], physics Dhar [1990], and algebraic geometry Bacher et al. [1997] (where it is also known as the Picard group or Jacobian group). It has received considerable recent attention for its connections to discrete dynamical systems, tropical geometry, and linear systems of curves; see, e.g., Baker and Norine [2007], Biggs [1999], Bond and Levine [2011], Haase et al. [2012].

In previous work, the authors studied cellular generalizations of the graph-theoretic concepts of spanning trees Duval et al. [2009, 2011a] and the critical group Duval et al. [2011b]. To summarize, a cellular spanning tree of a d-dimensional CW-complex Σ is a subcomplex $\Upsilon \subseteq \Sigma$ generated by facets corresponding to a column basis of the cellular boundary matrix $\partial : C_d(\Sigma) \to C_{d-1}(\Sigma)$. The critical group $K(\Sigma)$ is the torsion part of the cokernel of the combinatorial Laplacian $\partial \partial^*$, and its order is a weighted enumeration of the cellular spanning trees of Σ. Moreover, the action of the critical group on cellular $(d-1)$-cochains gives a model of discrete flow on Σ, generalizing the chip-firing and sandpile models; see, e.g., Biggs [1999], Dhar [1990].

The lattices $\mathcal{C}$ and $\mathcal{F}$ of integral cuts and flows of a graph were first defined in Bacher et al. [1997], in which the authors regarded a graph as an analogue of a Riemann surface and interpreted the discriminant groups $\mathcal{C}^\sharp/\mathcal{C}$ and $\mathcal{F}^\sharp/\mathcal{F}$ respectively as the Picard group of divisors and as the Jacobian group of holomorphic forms. In particular, they showed that the critical group $K(G)$ is isomorphic to both $\mathcal{C}^\sharp/\mathcal{C}$ and $\mathcal{F}^\sharp/\mathcal{F}$. Similar definitions and results appear in Biggs [1999].

Here, we define the cut and flow spaces and lattices of a cell complex Σ by

$$
\begin{aligned}
\mathrm{Cut}(\Sigma) &= \mathrm{im}_{\mathbb{R}} \, \partial^*, & \mathrm{Flow}(\Sigma) &= \ker_{\mathbb{R}} \partial, \\
\mathcal{C}(\Sigma) &= \mathrm{im}_{\mathbb{Z}} \, \partial^*, & \mathcal{F}(\Sigma) &= \ker_{\mathbb{Z}} \partial.
\end{aligned}
$$

In topological terms, cut- and flow-vectors are cellular coboundaries and cycles, respectively. Equivalently, the vectors in $\mathrm{Cut}(\Sigma)$ support sets of facets whose deletion increases the codimension-1 Betti number, and the vectors in $\mathrm{Flow}(\Sigma)$ support nontrivial rational homology classes. In the language of matroid theory, cuts and flows correspond to cocircuits and circuits, respectively, of the *cellular matroid* represented by the columns of ∂. Indeed, every cellular spanning tree $\Upsilon \subseteq \Sigma$ gives rise to a natural basis for each of the cut and flow spaces, whose elements are supported on fundamental cocircuits and circuits of Υ, respectively. In the graph case the coefficients of these basis vectors are ± 1; in the general case, they are (up to sign) the cardinalities of homology groups of cellular spanning trees obtained from Υ by matroid basis exchange. Under certain conditions, these vector space bases are in fact integral bases for the cut and flow lattices (Theorem 4.3).

The idea of studying cuts and flows of matroids goes back to Tutte [1965]. The recent work Su and Wagner [2010] defines cuts and flows of a regular matroid (i.e., one represented by a totally unimodular matrix M); when M is the boundary matrix of a cell complex, this is the case where the torsion coefficients are all trivial. Su and Wagner's definitions coincide with ours; their focus, however, is on recovering the structure of a matroid from the metric data of its flow lattice.

As we will see, the groups $\mathcal{C}^\sharp/\mathcal{C}$ and $\mathcal{F}^\sharp/\mathcal{F}$ are not necessarily isomorphic to each other. Their precise relationship involves several other groups: the critical group $K(\Sigma)$, a dually defined *cocritical group* $K^*(\Sigma)$, and the *cutflow group* $\mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F})$. We show (Theorem 5.5) that the critical and cocritical groups are respectively isomorphic to the discriminant groups of the cut lattice and flow lattice, and that the cutflow group mediates between them with an "error term" given by homology. The sizes of the critical and cocritical groups are respectively torsion-weighted enumerators for cellular spanning trees and for relatively acyclic subcomplexes. As an application of our theory, we generalize a theorem of

Kotani and Sunada [2000] to obtain a geometric bound for the girth and complexity of a cellular matroid (Theorem 6.2).

This is an extended abstract of the full article Duval et al. [2012], to which the reader is referred for the proofs of all results stated herein.

2 Preliminaries

We assume that the reader is familiar with the basic topology of cell complexes. In general, we adopt the notation of Hatcher [2002] for chain groups, (co)homology, etc. Throughout the paper, Σ will denote a finite CW complex of dimension d. We adopt the convention that Σ has a unique cell of dimension -1 (as though it were an abstract simplicial complex); this will allow our results to specialize correctly to the case $d = 1$ (i.e., that Σ is a graph). We write Σ_i for the set of i-dimensional cells in Σ, and $\Sigma_{(i)}$ for the i-dimensional skeleton of Σ, i.e., $\Sigma_{(i)} = \Sigma_i \cup \Sigma_{i-1} \cup \cdots \cup \Sigma_0$. A cell of dimension d is called a *facet*. Unless otherwise stated, every d-dimensional subcomplex $\Gamma \subseteq \Sigma$ is assumed to have a full codimension-1 skeleton, i.e., $\Gamma_{(d-1)} = \Sigma_{(d-1)}$. Accordingly, for simplicity of notation, we will often make no distinction between the subcomplex Γ itself and its set Γ_d of facets. For a coefficient ring R, we say that Σ is *R-acyclic in codimension one* if $\tilde{H}_{d-1}(\Sigma; R) = 0$. For a graph ($d = 1$), both $\mathbb{Q}$- and $\mathbb{Z}$-acyclicity in codimension one are equivalent to connectedness. The ith reduced Betti number is $\tilde{\beta}_i(\Sigma) = \dim \tilde{H}_i(\Sigma; \mathbb{Q})$, and the ith torsion coefficient $\mathbf{t}_i(\Sigma)$ is the cardinality of the torsion subgroup $\mathbf{T}(\tilde{H}_i(\Sigma; \mathbb{Z}))$.

A *cellular spanning forest* (CSF) of Σ is a subcomplex $\Upsilon \subseteq \Sigma$ such that $\Upsilon_{(d-1)} = \Sigma_{(d-1)}$ and

$$\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0, \tag{1a}$$

$$\operatorname{rank} \tilde{H}_{d-1}(\Upsilon; \mathbb{Z}) = \operatorname{rank} \tilde{H}_{d-1}(\Sigma; \mathbb{Z}), \quad \text{and} \tag{1b}$$

$$|\Upsilon_d| = |\Sigma_d| - \tilde{\beta}_d(\Sigma) + \tilde{\beta}_{d-1}(\Sigma). \tag{1c}$$

These conditions generalize the definition of a spanning forest of a graph G: respectively, it is acyclic, connected, and has $n - c$ edges, where n and c are the numbers of vertices and components of G. (By "spanning forest," we mean a maximal acyclic subgraph, not merely an acyclic subgraph containing all vertices.) Just as in the graphic case, any two of the conditions (1a), (1b), (1c) together imply the third. An equivalent definition is that a subcomplex $\Upsilon \subseteq \Sigma$ is a cellular spanning forest if and only if its d-cells correspond to a column basis for the cellular boundary matrix $\partial = \partial_d(\Sigma)$. In the case that Σ is $\mathbb{Q}$-acyclic in codimension one, this definition specializes to our earlier definition of a cellular spanning tree [Duval et al., 2011a, Definition 2.2].

The *complexity* of Σ is

$$\tau(\Sigma) = \tau_d(\Sigma) = \sum_{\text{CSFs } \Upsilon \subseteq \Sigma} |\mathbf{T}(\tilde{H}_{d-1}(\Upsilon; \mathbb{Z}))|^2. \tag{2}$$

When $d = 1$, this is just the number of spanning forests. More generally, the complexity can be calculated using a generalization of the matrix-tree theorem, as we now describe. Define the ith *up-down*, *down-up* and *total* Laplacian operators of Σ by

$$L_i^{\mathrm{ud}} = \partial_{i+1}\partial_{i+1}^* \colon C_i(\Sigma) \to C_i(\Sigma), \qquad L_i^{\mathrm{du}} = \partial_i^*\partial_i \colon C_i(\Sigma) \to C_i(\Sigma), \qquad L_i^{\mathrm{tot}} = L_i^{\mathrm{ud}} + L_i^{\mathrm{du}}.$$

(These are discrete versions of the Laplacian operators on differential forms of a Riemannian manifold. The interested reader is referred to Eckmann [1945] and Dodziuk and Patodi [1976] for their origins in differential geometry and, e.g., Denham [2001], Friedman [1998], Merris [1994] for more recent appearances in combinatorics.) The *cellular matrix-tree theorem* [Duval et al., 2011a, Theorem 2.8] states that if Σ is $\mathbb{Q}$-acyclic in codimension one and $L_{\tilde{\Upsilon}}$ is the submatrix of $L_{d-1}^{\mathrm{ud}}(\Sigma)$ obtained by deleting the rows and columns corresponding to the facets of a $(d-1)$-spanning tree Υ, then

$$\tau(\Sigma) = \frac{|\mathbf{T}(\tilde{H}_{d-2}(\Sigma; \mathbb{Z}))|^2}{|\mathbf{T}(\tilde{H}_{d-2}(\Upsilon; \mathbb{Z}))|^2} \det L_{\tilde{\Upsilon}}.$$

One of our first results is that the condition $\tilde{H}_{d-1}(\Sigma; \mathbb{Q}) = 0$ can be dropped; see equation (3.3) below. Closely related results, also applicable to all cell complexes, appear in Catanzaro et al. [2012] and Lyons [2009].

A *lattice* $\mathcal{L}$ is a discrete subgroup of a finite-dimensional vector space V; that is, it is the set of integer linear combinations of some basis of V. (For background, see, e.g., [Artin, 1991, Chapter 12], [Godsil and Royle, 2001, Chapter 14], [Hungerford, 1980, Chapter IV].) Every lattice $\mathcal{L} \subseteq \mathbb{R}^n$ is isomorphic to $\mathbb{Z}^r$ for some integer $r \leq n$, called the *rank* of $\mathcal{L}$. The elements of $\mathcal{L}$ span a vector space denoted by $\mathcal{L} \otimes \mathbb{R}$. For $\mathcal{L} \subseteq \mathbb{Z}^n$, the *saturation* of $\mathcal{L}$ is defined as $\hat{\mathcal{L}} = (\mathcal{L} \otimes \mathbb{R}) \cap \mathbb{Z}^n$. An *integral basis* of $\mathcal{L}$ is a set of linearly independent vectors $v_1, \ldots, v_r \in \mathcal{L}$ such that $\mathcal{L} = \{c_1 v_1 + \cdots + c_r v_r \colon c_i \in \mathbb{Z}\}$. Fixing the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$, we define the *dual lattice* of $\mathcal{L}$ by

$$\mathcal{L}^\sharp = \{v \in \mathcal{L} \otimes \mathbb{R} \colon \langle v, w \rangle \in \mathbb{Z} \ \forall w \in \mathcal{L}\}.$$

Note that $\mathcal{L}^\sharp$ can be identified with the dual $\mathbb{Z}$-module $\mathcal{L}^* = \mathrm{Hom}(\mathcal{L}, \mathbb{Z})$, and that $(\mathcal{L}^\sharp)^\sharp = \mathcal{L}$. A lattice is called *integral* if it is contained in its dual; for instance, any subgroup of $\mathbb{Z}^n$ is an integral lattice. The *discriminant group* (or *determinantal group*) of an integral lattice $\mathcal{L}$ is $\mathcal{L}^\sharp / \mathcal{L}$; its cardinality can be calculated as $\det M^T M$, for any matrix M whose columns form an integral basis of $\mathcal{L}$.

Many of our results of the paper may be expressed in the language of matroids, with which we assume the reader is familiar (for a general reference on matroids, see, e.g., Oxley [1992]). We adopt the following notation. If $\mathcal{M}$ is a matroid on ground set E and B is a basis of $\mathcal{M}$, then for every $e \in B$ the *fundamental bond* (or *fundamental cocircuit*) of e with respect to B is

$$\mathsf{bo}(B, e) = \{f \in E \colon B \setminus \{e\} \cup \{f\} \text{ is a basis of } \mathcal{M}$$

and for every $e \notin B$ the *fundamental circuit* is

$$\mathsf{ci}(B, e) = \{f \in E \colon B \cup \{e\} \setminus \{f\} \text{ is a basis of } \mathcal{M}.$$

The *cellular matroid* of Σ is the matroid $\mathcal{M}(\Sigma)$ represented over $\mathbb{R}$ by the columns of the boundary matrix ∂. Thus the ground set of $\mathcal{M}(\Sigma)$ naturally corresponds to the d dimensional cells Σ_d, and $\mathcal{M}(\Sigma)$ records which sets of columns of ∂ are linearly independent. If Σ is a graph, then $\mathcal{M}(\Sigma)$ is its usual graphic matroid, while if Σ is a simplicial complex then $\mathcal{M}(\Sigma)$ is its simplicial matroid (see Cordovil and Lindström [1987]). The bases of $\mathcal{M}(\Sigma)$ are the collections of facets of cellular spanning forests of Σ. If r is the rank function of the matroid $\mathcal{M}(\Sigma)$, then for each set of facets $B \subseteq \Sigma_d$, we have $r(B) = \mathrm{rank}\, \partial_B$, where ∂_B is the submatrix consisting of the columns indexed by the facets in B. Moreover, we have

$$r(\Sigma) := r(\Sigma_d) = \mathrm{rank}\, \mathcal{M}(\Sigma) = \mathrm{rank}\, \partial = |\Sigma_d| - \tilde{\beta}_d(\Sigma)$$

by the definition of Betti number.

A set of facets $B \subseteq \Sigma_d$ is called a *cut* if deleting B from Σ increases its codimension-one homology, i.e., $\tilde{\beta}_{d-1}(\Sigma \setminus B) > \tilde{\beta}_{d-1}(\Sigma)$. A cut B is a *bond* if $r(\Sigma \setminus B) = r(\Sigma) - 1$, but $r((\Sigma \setminus B) \cup \sigma) = r(\Sigma)$ for every $\sigma \in B$. That is, a bond is a minimal cut. In matroid terminology, a bond of Σ is precisely a cocircuit of $\mathcal{M}(\Sigma)$, i.e., a minimal set that meets every basis of $\mathcal{M}(\Sigma)$. Equivalently, a bond is the complement of a flat of rank $r(\Sigma) - 1$.

It is important to point out that the cut and flow spaces and lattices of a complex Σ are not matroid invariants, i.e., they are not determined by the cellular matroid $\mathcal{M}(\Sigma)$. (See Su and Wagner [2010] for more on this subject.) Below is a table collecting some of the standard terminology from linear algebra, graph theory, and matroid theory, along with the analogous concepts that we will be using for cell complexes.

Linear algebra	Graph	Matroid	Cell complex
Column vectors	Edges	Ground set	Facets
Independent set	Acyclic subgraph	Independent set	Acyclic subcomplex
Min linear dependence	Cycle	Circuit	Circuit
Basis	Spanning forest	Basis	CSF
Set meeting all bases	Disconnecting set	Codependent set	Cut
Min set meeting all bases	Bond	Cocircuit	Bond
Rank	# edges in spanning forest	Rank	# facets in CSF

Here "codependent" means dependent in the dual matroid.

3 Enumerating Cellular Spanning Forests

Our first result generalizes the simplicial and cellular matrix-tree theorems of Duval et al. [2009] and Duval et al. [2011a] (where we required that Σ be $\mathbb{Q}$-acyclic in codimension one). Closely related results have been obtained independently Catanzaro et al. [2012] and Lyons [2009].

Definition 3.1 *Let Σ be a d-dimensional cell complex with rank r. Let $\Gamma \subseteq \Sigma$ be a subcomplex of dimension less than or equal to $d - 1$ such that $\Gamma_{(d-2)} = \Sigma_{(d-2)}$. We say that Γ is* relatively acyclic *if the inclusion map $i \colon \Gamma \to \Sigma$ induces isomorphisms $i_* \colon \tilde{H}_k(\Gamma; \mathbb{Q}) \to \tilde{H}_k(\Sigma; \mathbb{Q})$ for all $k < d$.*

By the long exact sequence for relative homology, Γ is relatively acyclic if and only if $\tilde{H}_d(\Sigma; \mathbb{Q}) \to \tilde{H}_d(\Sigma, \Gamma; \mathbb{Q})$ is an isomorphism and $\tilde{H}_k(\Sigma, \Gamma; \mathbb{Q}) = 0$ for all $k < d$. These conditions can occur only if $|\Gamma_{d-1}| = |\Sigma_{d-1}| - r$. This quantity may be zero (in which case the only relatively acyclic subcomplex is $\Sigma_{(d-2)}$). A relatively acyclic subcomplex is precisely the complement of a $(d-1)$-cobase (a basis of the matroid represented over $\mathbb{R}$ by the rows of the boundary matrix ∂) in the terminology of Lyons [2009]. Two special cases are worth noting. First, if $d = 1$, then a relatively acyclic complex consists of one vertex in each connected component. Second, if $\tilde{H}_{d-1}(\Sigma; \mathbb{Q}) = 0$, then Γ is relatively acyclic if and only if it is a cellular spanning forest of $\Sigma_{(d-1)}$.

For a matrix M, denote by $M_{A,B}$ the submatrix with rows A and columns B.

Proposition 3.2 *Let Σ be a d-dimensional cell complex, let $\Upsilon \subseteq \Sigma$ be a cellular spanning forest, and let $\Gamma \subseteq \Sigma$ be a relatively acyclic $(d-1)$-subcomplex. Then*

$$\mathbf{t}_{d-1}(\Upsilon) \, \mathbf{t}_{d-1}(\Sigma, \Gamma) = \mathbf{t}_{d-1}(\Sigma) \, \mathbf{t}_{d-1}(\Upsilon, \Gamma).$$

The proof uses the following observation: the maximal nonsingular square submatrices of ∂ are precisely those whose columns correspond to a cellular spanning tree Υ and whose rows correspond to a

relatively acyclic subcomplex Γ, and in this case the determinant of such a matrix is (up to sign) the cardinality of the relative complex (Υ, Γ).

Expanding $\det L_\Gamma$ with the Binet-Cauchy formula and using the fact that $|\det \partial_{\Gamma, \Upsilon}| = \mathbf{t}_{d-1}(\Upsilon, \Gamma)$ (where Υ is a cellular spanning tree and Γ is a subcomplex generated by r ridges) implies the following formula for the complexity of Σ.

Proposition 3.3 *Let Σ be a d-dimensional cell complex and let $\Gamma \subseteq \Sigma$ be a relatively acyclic $(d-1)$-dimensional subcomplex, and let L_Γ be the restriction of $L_{d-1}^{\mathrm{ud}}(\Sigma)$ to the $(d-1)$-cells of Γ. Then*

$$\tau_d(\Sigma) = \frac{\mathbf{t}_{d-1}(\Sigma)^2}{\mathbf{t}_{d-1}(\Sigma, \Gamma)^2} \det L_\Gamma.$$

As an example of the usefulness of homological techniques, suppose that $\tilde{H}_{d-1}(\Sigma; \mathbb{Z})$ is purely torsion (a frequent case). Then the relative homology sequence of the pair (Σ, Γ) gives rise to the exact sequence

$$0 \to \mathbf{T}(\tilde{H}_{d-1}(\Sigma; \mathbb{Z})) \to \mathbf{T}(\tilde{H}_{d-1}(\Sigma, \Gamma; \mathbb{Z})) \to \mathbf{T}(\tilde{H}_{d-2}(\Gamma; \mathbb{Z})) \to \mathbf{T}(\tilde{H}_{d-2}(\Sigma; \mathbb{Z})) \to 0$$

which implies that $\mathbf{t}_{d-1}(\Sigma)/\mathbf{t}_{d-1}(\Sigma, \Gamma) = \mathbf{t}_{d-2}(\Sigma)/\mathbf{t}_{d-2}(\Gamma)$. Thus, Proposition 3.3 becomes the formula $\tau_d(\Sigma) = \frac{\mathbf{t}_{d-2}(\Sigma)^2}{\mathbf{t}_{d-2}(\Gamma)^2} \det L_\Gamma$, which was one of the original versions of the cellular matrix-tree theorem [Duval et al., 2011a, Theorem 2.8(2)] (see also Catanzaro et al. [2012] and Lyons [2009]).

4   Bases of the Cut and Flow Spaces

As before, let Σ be a cell complex of dimension d and rank r; that is, every cellular spanning forest of Σ has r facets), and identify cellular chains and cochains by the standard inner product. We wish to construct combinatorially meaningful bases for the cut space $\mathrm{im}\,\partial_d^*$ and the flow space $\ker \partial_d$. We first recall the construction in the case of a graph.

There are two natural ways to construct bases of the cut space of a graph. First, if G is a graph on vertex set V and R is a set of ("root") vertices, one in each connected component, then the rows of ∂ corresponding to the vertices $V \setminus R$ form a basis for $\mathrm{Cut}_1(G)$. This observation generalizes easily to cell complexes: a set of r rows of the top-dimensional boundary matrix forms a row basis if and only if the corresponding set of $(d-1)$-cells is the complement of a relatively acyclic $(d-1)$-subcomplex.

Second, for every spanning tree of a graph the signed characteristic vectors of its fundamental bonds form a basis of its cut space (see [Godsil and Royle, 2001, Chapter 14]) In the more general setting of a cell complex Σ, it is relatively straightforward to show that each bond in Σ supports a one-dimensional subspace of $\mathrm{Cut}(\Sigma)$.

Theorem 4.1 *Let Σ be a d-dimensional cell complex with top boundary map ∂, and let $L = L_d^{\mathrm{du}}(\Sigma) = \partial^* \partial$. Let $\Upsilon = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ be a cellular spanning tree, and $\sigma = \sigma_i \in \Upsilon$. Then:*

   1. *The vector*

$$\bar{\chi}(\Upsilon, \sigma_i) = \sum_{j=1}^{r} (-1)^j (\det L_{\Upsilon \setminus \sigma_i, \Upsilon \setminus \sigma_j}) L\sigma_j \; \in \; C_d(\Sigma)$$

   spans the space of all cut-vectors with support contained in the fundamental bond $\mathsf{bo}(\Upsilon, \sigma_i)$.

   2. *The set $\{\bar{\chi}(\Upsilon, \sigma_1), \ldots, \bar{\chi}(\Upsilon, \sigma_r)\}$ is a vector space basis for $\mathrm{Cut}(\Sigma)$.*

3. *The coefficients of* $\bar{\chi}(\Upsilon, \sigma_i)$ *have the following interpretation. Let*

$$\mu = \mu_\Upsilon = \mathbf{t}_{d-1}(\Upsilon) \sum_\Gamma \frac{\mathbf{t}_{d-1}(\Sigma, \Gamma)^2}{\mathbf{t}_{d-1}(\Sigma)^2}.$$

Then

$$\bar{\chi}(\Upsilon, \sigma) = \sum_{\rho \in \mathsf{bo}(\Upsilon, \sigma)} (\det L_{\Upsilon \setminus \sigma \cup \rho, \Upsilon}) \rho = \mu \sum_{\rho \in \mathsf{bo}(\Upsilon, \sigma)} (\pm \mathbf{t}_{d-1}(\Upsilon \setminus \sigma \cup \rho)) \rho.$$

The vectors $\chi(\Upsilon, \sigma) = \bar{\chi}(\Upsilon, \sigma)/\mu$ are the cellular analogues of signed characteristic vectors of bonds in graphs. (Note that if indeed $d = 1$, then all the torsion coefficients are 1; μ is just the number of vertices of Σ; and for any edge σ in Υ, the vector $\chi(\Upsilon; \sigma)$ is the usual signed characteristic vector of the fundamental bond $\mathsf{bo}(\Upsilon, \sigma)$.)

Torsion plays a role in the characteristic vectors of bonds, even when Σ is a simplicial complex. For example, let Σ be the the complete 2-dimensional simplicial complex on 6 vertices and let Υ be the triangulation of $\mathbb{RP}^2$ obtained by identifying opposite faces in an icosahedron. Then Υ is a cellular spanning forest of Σ (and in fact Σ has twelve spanning forests of this kind). For any facet $\sigma \in \Upsilon$, we have $\mathsf{bo}(\Upsilon, \sigma) = \Sigma_2 \setminus \Upsilon_2 \cup \{\sigma\}$, and the entries of the calibrated cut-vector include both ± 2 (in position σ) as well as ± 1's (in positions $\Sigma \setminus \Upsilon$).

The analogous theorem for the flow space is as follows.

Theorem 4.2 *Let* Σ *be a* d-*dimensional cell complex with top boundary map* ∂. *For every circuit* C *in the cellular matroid, the space of flow vectors supported on a subset of* C *is one-dimensional, spanned by*

$$\varphi(C) = \sum_{\sigma \in C} \pm \mathbf{t}_{d-1}(\Delta \setminus \sigma) \, \sigma.$$

Moreover, for every cellular spanning forest $\Upsilon \subseteq \Sigma$, *the set* $\{\varphi(\mathsf{ci}(\Upsilon, \sigma)) : \sigma \notin \Upsilon\}$ *is a vector space basis for* $\mathrm{Flow}(\Sigma)$.

The argument is easier for the flow space than for the cut space; for instance, the explicit formula for $\varphi(C)$ is essentially a calculation using Cramer's rule.

In the graph case, these combinatorial bases of the cut and flow spaces are in fact integral bases of the lattices $\mathcal{C}$ and $\mathcal{F}$ respectively. In the cellular case, the possibility of torsion requires additional assumptions. Specifically:

Theorem 4.3 *Let* Υ *be a cellular spanning forest of* Σ.

1. *If* $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ *is torsion-free, then* $\{\chi(\Upsilon, \sigma) : \sigma \in \Upsilon\}$ *is an integral basis for the cut lattice* $\mathcal{C}(\Sigma)$.

2. *If* $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z}) = \tilde{H}_{d-1}(\Sigma; \mathbb{Z})$, *then* $\{\hat{\varphi}(\Upsilon, \sigma) : \sigma \notin \Upsilon\}$ *is an integral basis for the flow lattice* $\mathcal{F}(\Sigma)$, *where* $\hat{\varphi}$ *denotes* φ *divided by the g.c.d. of its coefficients.*

5 Groups and Lattices

In this section, we define the critical, cocritical, and cutflow groups of a cell complex. We identify the relationships between these groups and to the discriminant groups of the cut and flow lattices. The

graphical case was studied in detail in Bacher et al. [1997] and Biggs [1999], and is presented concisely in [Godsil and Royle, 2001, Chapter 14]. Throughout the section, all (co)chain and (co)homology groups are assume to have coefficients in $\mathbb{Z}$; in particular, we identify both $C_d(\Sigma; \mathbb{Z})$ and $C^d(\Sigma; \mathbb{Z})$ with $\mathbb{Z}^n$.

Definition 5.1 *The* critical group *of Σ is*

$$K(\Sigma) := \mathbf{T}(\ker \partial_{d-1} / \operatorname{im} \partial_d \partial_d^*) = \mathbf{T}(\operatorname{coker}(\operatorname{im} \partial_d \partial_d^*)).$$

Definition 5.2 *The* cutflow group *of Σ is $\mathbb{Z}^n / (\mathcal{C}(\Sigma) \oplus \mathcal{F}(\Sigma))$.*

In order to define the cocritical group of a cell complex, we need the notion of an *acyclization.* of Σ, which is a $(d+1)$-dimensional complex Ω such that $\Omega_{(d)} = \Sigma$ and $\tilde{H}_{d+1}(\Omega; \mathbb{Z}) = \tilde{H}_d(\Omega; \mathbb{Z}) = 0$. Algebraically, this construction corresponds to finding an integral basis for $\ker \partial_d(\Sigma)$ and declaring its elements to be the columns of $\partial_{d+1}(\Omega)$ (so in particular $|\Omega_{(d+1)}| = \tilde{\beta}_d(\Sigma)$). The definition of acyclization and the universal coefficient theorem for cohomology together imply that $\tilde{H}^{d+1}(\Omega; \mathbb{Z}) = 0$; that is, $\partial_{d+1}^*(\Omega)$ is surjective.

Definition 5.3 *The* cocritical group $K^*(\Sigma)$ *is*

$$K^*(\Sigma) := C_{d+1}(\Omega; \mathbb{Z}) / \operatorname{im} \partial_{d+1}^* \partial_{d+1} = \operatorname{coker} L_{d+1}^{\mathrm{du}}.$$

It is not immediate that the group $K^*(\Sigma)$ is independent of the choice of Ω; we will prove this independence as part of Theorem 5.5. For the moment, it is at least clear that $K^*(\Sigma)$ is finite, since $\operatorname{rank} \partial_{d+1}^* = \operatorname{rank} L_{d+1}^{\mathrm{du}} = \operatorname{rank} C_{d+1}(\Omega; \mathbb{Z})$. In the special case of a graph, the cocritical group coincides with the discriminant group of the lattice generated by the columns of the "intersection matrix" defined by Kotani and Sunada Kotani and Sunada [2000]. (See also [Biggs, 2007, Sections 2, 3].)

Remark 5.4 *As in Duval et al. [2011b], one can define critical and cocritical groups in every dimension by*

$$K_i(\Sigma) = \mathbf{T}(C_i(\Sigma; \mathbb{Z}) / \operatorname{im} \partial_{i+1} \partial_{i+1}^*), \qquad K_i^*(\Sigma) = \mathbf{T}(C_i(\Sigma; \mathbb{Z}) / \operatorname{im} \partial_i^* \partial_i).$$

If the cellular chain complexes of Σ and Ψ are algebraically dual (for example, if Σ and Ψ are Poincaré dual cell structures on a compact orientable d-manifold), then $K_i(\Psi) = K_{d-i}^*(\Sigma)$ for all i.

Our main theorem states that the critical and cocritical groups are isomorphic to the discriminant groups of the cut and flow lattices respectively, and the cutflow group mediates between the critical and cocritical groups, with an "error term" given by torsion. Specifically:

Theorem 5.5 *Let Σ be a cell complex of dimension d with n facets. Then there are short exact sequences*

$$0 \to \mathbb{Z}^n / (\mathcal{C} \oplus \mathcal{F}) \to \mathcal{C}^\sharp / \mathcal{C} \cong K(\Sigma) \to \mathbf{T}(\tilde{H}^d(\Sigma; \mathbb{Z})) \to 0$$

and

$$0 \to \mathbf{T}(\tilde{H}_{d-1}(\Sigma; \mathbb{Z})) \to \mathbb{Z}^n / (\mathcal{C} \oplus \mathcal{F}) \to \mathcal{F}^\sharp / \mathcal{F} \cong K^*(\Sigma) \to 0.$$

In fact, the error terms $\mathbf{T}(\tilde{H}^d(\Sigma; \mathbb{Z}))$ and $\mathbf{T}(\tilde{H}_{d-1}(\Sigma; \mathbb{Z}))$ are in fact isomorphic, by a special case of the universal coefficient theorem for cohomology [Hatcher, 2002, p. 205, Corollary 3.3].

Corollary 5.6 *If $\tilde{H}_{d-1}(\Sigma; \mathbb{Z})$ is torsion-free, then the groups $K(\Sigma)$, $K^*(\Sigma)$, $\mathcal{C}^\sharp / \mathcal{C}$, $\mathcal{F}^\sharp / \mathcal{F}$, and $\mathbb{Z}^n / (\mathcal{C} \oplus \mathcal{F})$ are all isomorphic to each other.*

Corollary 5.6 includes the case that Σ is a graph, as studied in Bacher et al. [1997] and Biggs [1999]. It also includes the combinatorially important family of Cohen-Macaulay simplicial complexes, as well as cellulations of compact orientable manifolds.

Example 5.7 *Suppose that $\tilde{H}_d(\Sigma; \mathbb{Z}) = \mathbb{Z}$ and that $\tilde{H}_{d-1}(\Sigma; \mathbb{Z})$ is torsion-free. Then the flow lattice is generated by a single element, and it follows from Corollary 5.6 that $K(\Sigma) \cong K^*(\Sigma) \cong \mathcal{F}^\sharp/\mathcal{F}$ is a cyclic group. For instance, if Σ is homeomorphic to a cellular sphere or torus, then the critical group is cyclic of order equal to the number of facets. (The authors had previously proved this fact for simplicial spheres [Duval et al., 2011b, Theorem 3.7], but this approach using the cocritical group makes the statement more general and the proof transparent.)*

Example 5.8 *Let Σ be the standard cellulation $e^0 \cup e^1 \cup e^2$ of the real projective plane, whose cellular chain complex is*

$$\mathbb{Z} \xrightarrow{\partial_2 = 2} \mathbb{Z} \xrightarrow{\partial_1 = 0} \mathbb{Z}.$$

Then $\mathcal{C} = \operatorname{im} \partial_2^* = 2\mathbb{Z}$, $\mathcal{C}^\sharp = \frac{1}{2}\mathbb{Z}$, and $K(\Sigma) = \mathcal{C}^\sharp/\mathcal{C} = \mathbb{Z}_4$. Meanwhile, $\mathcal{F} = \mathcal{F}^\sharp = \mathcal{F}^\sharp/\mathcal{F} = K^*(\Sigma) = 0$. The cutflow group is $\mathbb{Z}_2$. Note that the rows of the Theorem 5.5 are not split in this case.

Example 5.9 *Let $a, b \in \mathbb{Z} \setminus \{0\}$. Let Σ be the cell complex whose cellular chain complex is*

$$\mathbb{Z} \xrightarrow{\partial_2 = [a\ b]} \mathbb{Z} \xrightarrow{\partial_1 = 0} \mathbb{Z}.$$

Topologically, Σ consists of a vertex e^0, a loop e^1, and two facets of dimension 2 attached along e^1 by maps of degrees a and b. Then

$$\mathcal{C}^\sharp/\mathcal{C} = \mathbb{Z}_\tau, \qquad \mathbb{Z}^2/(\mathcal{C} \oplus \mathcal{F}) = \mathbb{Z}_{\tau/g}, \qquad \mathcal{F}^\sharp/\mathcal{F} = \mathbb{Z}_{\tau/g^2},$$

where $\tau = a^2 + b^2$ and $g = \gcd(a, b)$. Note that $\tau = \tau_2(\Sigma)$ is the complexity of Σ (see equation (2)) and that $g = |\tilde{H}_1(\Sigma; \mathbb{Z})|$. The short exact sequence for $K^*(\Sigma)$ of Theorem 5.5 is in general not split (for example, if $a = 6$ and $b = 2$).

For a connected graph, the cardinality of the critical group equals the number of spanning trees. In the cellular case, Examples 5.8 and 5.9 both indicate that $K(\Sigma) \cong \mathcal{C}^\sharp/\mathcal{C}$ should have cardinality equal to the complexity $\tau(\Sigma)$. Indeed, in Theorem 4.2 of Duval et al. [2011b], the authors proved that $|K(\Sigma)| = \tau(\Sigma)$ whenever Σ has a cellular spanning tree Υ such that $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z}) = \tilde{H}_{d-1}(\Sigma; \mathbb{Z}) = 0$ (in particular, Σ must be not merely $\mathbb{Q}$-acyclic, but actually $\mathbb{Z}$-acyclic, in codimension one). We prove that this condition holds true for any cell complex. Together with Theorem 5.5, we obtain:

Theorem 5.10 *Let Σ be a d-dimensional cell complex and let $\mathbf{t} = \mathbf{t}_{d-1}(\Sigma) = |\mathbf{T}(\tilde{H}_{d-1}(\Sigma; \mathbb{Z}))|$. Then*

$$|\mathcal{C}^\sharp/\mathcal{C}| = |K(\Sigma)| = \tau_d(\Sigma),$$
$$|\mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F})| = \tau_d(\Sigma)/\mathbf{t}, \quad and$$
$$|\mathcal{F}^\sharp/\mathcal{F}| = |K^*(\Sigma)| = \tau_d(\Sigma)/\mathbf{t}^2.$$

Dually, we can interpret the cardinality of the cocritical group as enumerating cellular spanning forests by relative torsion (co)homology:

Theorem 5.11 *Let Ω be an acyclization of Σ. Then*

$$|K^*(\Sigma)| = \sum_\Upsilon |\tilde{H}^{d+1}(\Omega, \Upsilon; \mathbb{Z})|^2 = \sum_\Upsilon |\tilde{H}_d(\Omega, \Upsilon; \mathbb{Z})|^2$$

with the sums over all cellular spanning forests $\Upsilon \subseteq \Sigma$.

Note that the groups $\tilde{H}^{d+1}(\Omega, \Upsilon; \mathbb{Z})$ and $\tilde{H}_d(\Omega, \Upsilon; \mathbb{Z})$ are all finite, by definition of acyclization.

Remark 5.12 *Let $\tau^*(\Sigma) = \sum_\Upsilon |\tilde{H}_d(\Omega, \Upsilon; \mathbb{Z})|^2$, as in Theorem 5.11. Then combining Theorems 5.10 and Theorems 5.11 gives*

$$|\mathcal{C}^\sharp/\mathcal{C}| = |K(\Sigma)| = \tau(\Sigma) \quad = \tau^*(\Sigma) \cdot \mathbf{t}^2,$$
$$|\mathcal{F}^\sharp/\mathcal{F}| = |K^*(\Sigma)| = \tau^*(\Sigma) \quad = \tau(\Sigma)/\mathbf{t}^2,$$
$$|\mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F})| = \tau(\Sigma)/\mathbf{t} = \tau^*(\Sigma) \cdot \mathbf{t},$$

highlighting the duality between the cut and flow lattices.

6   Bounds on combinatorial invariants from lattice geometry

Let $n \geq 1$ be an integer. The *Hermite constant* γ_n is defined as the maximum value of

$$\left(\min_{x \in \mathcal{L} \setminus \{0\}} \langle x, x \rangle \right) (|\mathcal{L}^\sharp/\mathcal{L}|)^{-1/n} \tag{3}$$

over all lattices $\mathcal{L} \subseteq \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ is the standard inner product. The Hermite constant arises both in the study of quadratic forms and in sphere packing; see [Lagarias, 1995, Section 4]. It is known that γ_n is finite for every n, although the precise values are known only for $1 \leq n \leq 8$ and $n = 24$ Cohn and Kumar [2009].

As observed in Kotani and Sunada [2000], if $\mathcal{L} = \mathcal{F}$ is the flow lattice of a connected graph, then the shortest vector in $\mathcal{F}$ is the characteristic vector of a cycle of minimum length; therefore, the numerator in equation (3) is the girth of G. Meanwhile, $|\mathcal{F}^\sharp/\mathcal{F}|$ is the number of spanning trees. We now generalize this theorem to cell complexes.

Definition 6.1 *Let Σ be a cell complex. The* girth *and the* connectivity *are defined as the cardinalities of, respectively the smallest circuit and the smallest cocircuit of the cellular matroid of Σ.*

Theorem 6.2 *Let Σ be a cell complex of dimension d with girth g and connectivity k, and top boundary map of rank r. Let $b = \operatorname{rank} \tilde{H}_{d-1}(\Sigma; \mathbb{Z})$. Then*

$$k\tau(\Sigma)^{-1/r} \leq \gamma_r \qquad and \qquad g\tau^*(\Sigma)^{-1/b} \leq \gamma_b.$$

Proof: Every nonzero vector of the cut lattice (resp., the flow lattice) contains a cocircuit (resp., a circuit) in its support. Therefore,

$$\min_{x \in \mathcal{C} \setminus \{0\}} \langle x, x \rangle \geq k \qquad and \qquad \min_{x \in \mathcal{F} \setminus \{0\}} \langle x, x \rangle \geq g.$$

Meanwhile, $|\mathcal{C}^\sharp/\mathcal{C}| = \tau$ and $|\mathcal{F}^\sharp/\mathcal{F}| = \tau^*$ by Theorem 5.10. The desired inequalities now follow from applying the definition of Hermite's constant to the cut and flow lattices respectively. $\square$

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