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Type A molecules are Kazhdan-Lusztig

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Abstract. Let (W, S) be a Coxeter system. A W-graph is an encoding of a representation of the corresponding Iwahori-Hecke algebra. Especially important examples include the W-graph corresponding to the action of the Iwahori-Hecke algebra on the Kazhdan-Lusztig basis as well as this graph's strongly connected components (cells). In 2008, Stembridge identified some common features of the Kazhdan-Lusztig graphs ("admissibility") and gave combinatorial rules for detecting admissible W-graphs. He conjectured, and checked up to $n = 9$, that all admissible A_n-cells are Kazhdan-Lusztig cells. The current paper provides a possible first step toward a proof of the conjecture. More concretely, we prove that the connected subgraphs of A_n-cells consisting of simple (i.e. directed both ways) edges do fit into the Kazhdan-Lusztig cells.

Résumé. Soit (W, S) un système de Coxeter. Un W-graphe est un objet qui décrit certaines représentations de l'algèbre de Iwahori-Hecke. Des exemples particulièrement importants sont les W-graphes correspondant à l'action de l'algèbre de Iwahori-Hecke sur la base de Kazhdan-Lusztig ainsi que ses composantes fortement connexes (cellules). En 2008, Stembridge a identifié quelques caractéristiques communes des graphes de Kazhdan-Lusztig et a donné une caractérisation combinatoire de tous ces W-graphes. Il a conjecturé, et a vérifié jusqu'à $n = 9$, que toutes ces A_n-cellules sont des cellules de Kazhdan-Lusztig. Le présent article fournit la premirè étape d'une démonstration possible de cette conjecture. Plus concrètement, nous montrons que les sous-graphes connexes de A_n-cellules composés d'arêtes s'insèrent dans les cellules de Kazhdan-Lusztig.

Keywords: Iwahori-Hecke algebra, W-graphs, W-molecules, dual equivalence graphs, Kazhdan-Lusztig cells

1 Introduction

Let (W, S) be a Coxeter system. A W-graph is a graph with additional structure that encodes a representation of the corresponding Iwahori-Hecke algebra. Kazhdan and Lusztig (1979) introduced such graphs for the regular representation, and showed that the strongly connected components (called "cells") also yield representations. Stembridge identified several common features of the Kazhdan-Lusztig graphs, namely, they are bipartite, (nearly) edge-symmetric, and their edge weights are non-negative integers (collectively he called these properties "admissibility"). He proceeded to describe, via four combinatorial rules, when an admissible graph is a W-graph (Stembridge (2008a)). One hopes that the characterization will allow one to construct the Kazhdan-Lusztig cells without having to compute Kazhdan-Lusztig polynomials (a notoriously difficult task). A piece of evidence suggesting that the definition of a general admissible W-cell approximates a Kazhdan-Lusztig cell is a more recent result of Stembridge that there are only finitely many admissible W-cells for each W (Stembridge (2012)).

There are no known examples of admissible A_n-cells besides the Kazhdan-Lusztig cells (Stembridge experimentally checked it up to $n = 9$). A possible strategy of proof is as follows:

1. An A_n-cell is a strongly connected directed graph. Consider the subgraphs which are connected via two-sided edges (of course these are strongly connected on their own, but a cell may contain several of them). The subgraphs satisfy combinatorial rules slightly weaker than those satisfied by a cell; a graph satisfying these rules is called *a molecule*. The first step is to show that any A_n-molecule *is Kazhdan-Lusztig*, i.e. it appears in the Kazhdan-Lusztig graph.

2. It is known that a Kazhdan-Lusztig A_n-cell is connected via two-sided edges, and these edges are well understood (they are called *dual Knuth moves*). The second step is to prove that no cell may have multiple molecules. The fact that no two Kazhdan-Lusztig A_n molecules may be connected inside a cell has been experimentally checked for $n \leqslant 12$ (Stembridge (2011)).

3. The last part is to prove that there can be only one A_n-graph with a given underlying molecule. For Kazhdan-Lusztig molecules this has been checked for $n \leqslant 13$ (Stembridge (2011)).

In this paper we complete the first part of the above program, namely, we prove that any A_n-molecule is Kazhdan-Lusztig. Together with the above computations, this result implies that all A_n-cells up to $n = 12$ are Kazhdan-Lusztig. The main ingredient of the proof is the axiomatization of graphs on tableaux generated by dual Knuth moves (Assaf (2008)). Five of the axioms follow easily from the molecules axioms, but the last one presents a challenge.

The paper is structured as follows. Section 2 introduces W-molecules. Section 3 discusses Assaf's dual equivalence graphs and relates them to molecules. The last section contains an outline of the proof of the main theorem that the simple part of a type A molecule is a dual equivalence graph.

2 Molecules

This section summarizes the required W-molecules terminology as described by Stembridge (2008a,b).

Let (W, S) be a simply-laced Coxeter system. A significant part of this section extends to multiply-laced types; see the above two papers. The papers are mostly concerned with W-graphs, i.e. graphs that encode certain representations of the corresponding Iwahori-Hecke algebra. It turns out that the simple (i.e. directed both ways) edges of these graphs are much easier to understand than other edges. For example, there is a very explicit description of them for the case of cells arising in the Kazhdan-Luztig W-graph (we will give it in Section 3.1). Thus we consider subgraphs connected by simple edges. These subgraphs are not W-graphs (i.e. they do not encode representations), but they satisfy certain combinatorial rules which are slightly weaker than Stembridge's W-graph rules. We begin this paper by formalizing the definitions and presenting the rules.

2.1 Definitions

An *(admissible) S-labeled graph* is a tuple $G = (V, m, \tau)$, where V is a set (vertices), $m : V \times V \to \mathbb{Z}^{\geqslant 0}$, and $\tau : V \to 2^S$ such that

1. as a directed graph (with edges given by pairs of vertices with non-zero m value), G is bipartite,

2. if $\tau(u) \subseteq \tau(v)$ then $m(u, v) = 0$,

3. if $\tau(u)$ and $\tau(v)$ are incomparable, then $m(u, v) = m(v, u)$.

The function τ is referred to as the τ-invariant.

By a *simple edge* we mean a pair of vertices (v_1, v_2) such that neither $m(v_1, v_2)$ nor $m(v_2, v_1)$ are 0 (in the graphs that we consider both weights will be 1). By an *arc* $v_1 \to v_2$ we mean a pair of vertices (v_1, v_2) such that $m(v_1, v_2) \neq 0$, but $m(v_2, v_1) = 0$. Notice if $u \to v$ is an arc, then $\tau(u) \supset \tau(v)$. If (u, v) is a simple edge then $\tau(u)$ and $\tau(v)$ are incomparable, and $m(u, v) = m(v, u)$.

A simple edge (u, v) *activates* a bond $i - j$ in the Coxeter graph if precisely one of $\tau(u)$ and $\tau(v)$ contains i, and precisely the other one contains j.

For distinct $i, j \in S$, a path $u \to v_1 \to v_2 \to \cdots \to v_{r-1} \to v$ in G is *alternating of type* (i, j) if

- $i, j \in \tau(u)$ and $i, j \notin \tau(v)$,
- $i \in \tau(v_k), j \notin \tau(v_k)$ for odd k,
- $i \notin \tau(v_k), j \in \tau(v_k)$ for even k.

Let $N^r_{ij}(G; u, v)$ denote the weighted count of such paths:

$$N^r_{ij}(G; u, v) := \sum_{v_1, \ldots, v_{r-1}} m(u, v_1) m(v_1, v_2) \ldots m(v_{r-1}, v).$$

Definition 2.1 *An S-labeled graph is called* a molecular graph *if it satisfies*

(SR) If (u, v) is a simple edge then $m(u, v) = m(v, u) = 1$.

(CR) If $u \to v$ is an edge, i.e. $m(u, v) \neq 0$, then every $i \in \tau(u) \setminus \tau(v)$ is bonded to every $j \in \tau(v) \setminus \tau(u)$.

(BR) Suppose $i - j$ is a bond in the Coxeter graph of (W, S). Any vertex u with $i \in \tau(u)$ and $j \notin \tau(u)$ is adjacent to precisely one edge which activates $i - j$.

(LPR2) For any $i, j \in S$ for any $u, v \in V$ with $i, j \in \tau(u)$, $i, j \notin \tau(v)$ and $\tau(v) \setminus \tau(u) \neq \varnothing$, we have

$$N^2_{ij}(G; u, v) = N^2_{ji}(G; u, v).$$

(LPR3) Let $k, i, j, l \in S$ be a copy of A_4 in the Coxeter graph: $k - i - j - l$. For any $u, v \in V$ with $i, j \in \tau(u)$, $i, j \notin \tau(v)$, $k, l \notin \tau(u)$, $k, l \in \tau(v)$, we have

$$N^3_{ij}(G; u, v) = N^3_{ji}(G; u, v).$$

The rules are called, respectively, simplicity rule, compatibility rule, bonding rule, and local polygon rules.

Definition 2.2 *A molecular graph is called* a molecule *if there is a path of simple edges between any pair of vertices.*

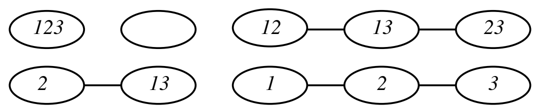
Example 2.3 *It is easy to classify all the S_4 molecules. Because of admissibility, a vertex whose τ-invariant is $\varnothing$ cannot be connected to any other vertex by a simple edge. Similarly for a vertex whose τ-invariant is $\{1, 2, 3\}$.*

Suppose we have a vertex v_1 whose τ-invariant is $\{1\}$. By BR, it is connected by a simple edge to a vertex v_2 whose τ-invariant contains 2, but not 1. By CR, $3 \notin \tau(v_2)$, and hence $\tau(v_2) = \{2\}$. By BR, v_2

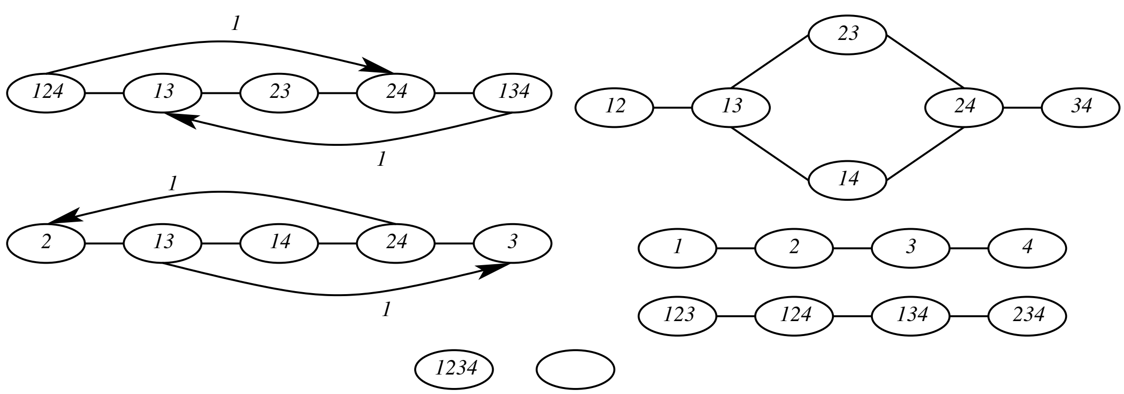
is connected by a simple edge to a vertex v_3 whose τ-invariant contains 3, but not 2. We already know $v_3 \neq v_1$. By BR, $\tau(v_3) = \{3\}$. There are no other simple edges possible, and this is a complete molecule. The same analysis works for v_1 having τ-invariants of $\{3\}, \{1,2\}, \{2,3\}$.

Suppose we have a vertex v_1 whose τ-invariant is $\{2\}$. By BR, it is connected by a simple edge to a vertex v_2 whose τ-invariant contains 1, but not 2. The case of $\tau(v_2) = \{1\}$ was described above, so the only choice is $\tau(v_2) = \{1,3\}$. This yields a complete molecule. The same argument works for v_1 having τ-invariant of $\{1,3\}$.

This completes the classification:



Example 2.4 *It takes some more work to classify the S_5 molecules (see the paper of Stembridge (2008a))*



The *simple part* of a molecule is the graph formed by erasing all the arcs. We usually view it as an undirected graph. A *morphism of molecules* $\varphi : M \to N$ is a map between the vertex sets which

1. is a graph morphism of the simple parts,
2. preserves τ-invariants.

Notice that a morphism does not need to respect arcs, aside from ones whose weights are determined by the local polygon rules from the simple edges.

2.2 Restriction

Let $J \subseteq S$ and let W_J be the corresponding parabolic subgroup.

Let $M = (V, m, \tau)$ be a W-molecular graph. The *W_J-restriction* of M is $N = (V, m', \tau')$, with

1. for all $v \in V$, $\tau'(v) = \tau(v) \cap J$,
2. for all $u, v \in V$,

$$m'(u,v) = \begin{cases} 0, & \text{if } \tau'(u) \subseteq \tau'(v), \\ m(u,v), & \text{otherwise.} \end{cases}$$

The W_J-restriction of M is a W_J-molecular graph. A W_J*-submolecule* of M is a W_J-molecule (i.e. component connected by simple edges) of the W_J-restriction of M. There is a natural inclusion map of a W_J-submolecule into the original molecular graph. Sometimes, abusing notation, we refer to the image of this map as a W_J-submolecule. The sense in which we use the word should be clear from the context.

3 Dual equivalence graphs

This section summarizes the relevant definitions and results of Assaf (2008); they are restated and slightly specialized to make the similarity with W-molecules more apparent.

Fix $n \in \mathbb{Z}^{>0}$. Let (W, S) be a Coxeter system of type A_n. Identify S in a natural way with $\{1, \dots, n\}$. Define a_i to be the edge of the Coxeter graph (throughout the paper we will refer to these edges as *bonds*) which links i and $i + 1$. Then $B := \{a_1, \dots, a_{n-1}\}$ is the set of all edges of the Coxeter graph. For examples with small n we will use the notation $a, b, c, \dots$ instead.

Definition 3.1 *A* signed colored graph of type $n + 1$ *is a tuple* (V, E, τ, β)*, where* (V, E) *is a finite undirected simple graph,* $\tau : V \to 2^S$*, and* $\beta : E \to 2^B$.

Denote by E_i the set of edges with label i (i.e. such that the corresponding value of β contains i); we call these i*-colored* edges. This is a slight reindexing from Assaf's original definition; in the original E_i was the set of edges whose label contains $i - 1$.

3.1 "Standard" dual equivalence graphs

We start by constructing a family, indexed by partitions, of signed colored graphs.

Let λ be a partition of $n + 1$. Let $SYT(\lambda)$ be the set of standard Young tableaux of shape λ. The left descent set of a tableau T is

$$\tau(T) := \{1 \leqslant i \leqslant n : i \text{ is located in a higher row than } i + 1 \text{ in } T\}.$$

The left descent sets are shown in red in Example 3.2.

The set of vertices of our graph is $V := SYT(\lambda)$.

By a diagonal of a tableau we mean a $NW - SE$ diagonal. A *dual Knuth move* is the exchange of i and $i + 1$ in a standard tableau, provided that either $i - 1$ or $i + 2$ lies (necessarily strictly) between the diagonals containing i and $i + 1$. This corresponds to dual Knuth moves on the symmetric group via the "content reading word" (reading each diagonal from top to bottom, and concatenating in order of increasing height of the diagonals). The set of edges of our graph is the set of pairs of tableaux related by a dual Knuth move:

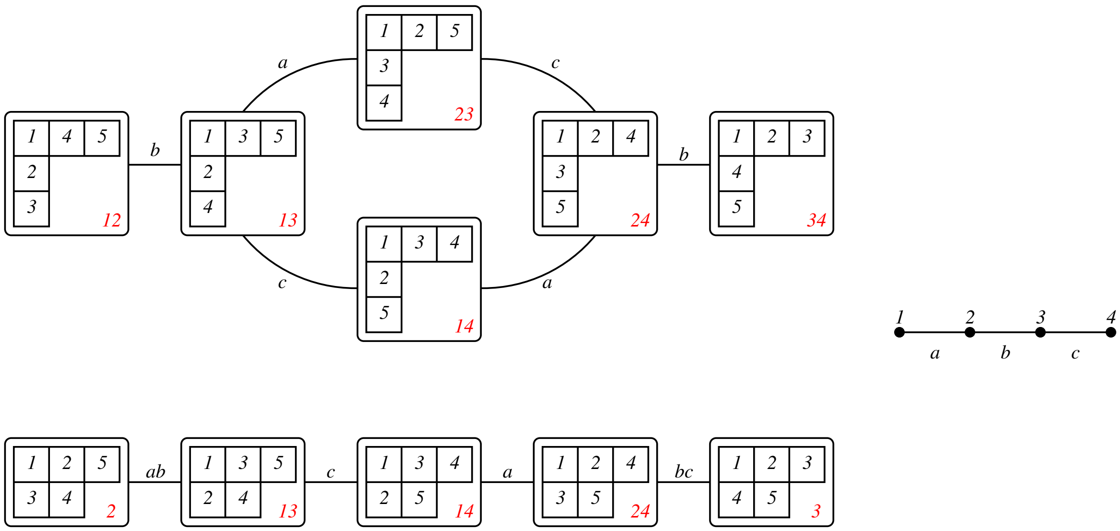
$$E := \{(T, U) : T \text{ and } U \text{ are related by a dual Knuth move}\}.$$

A dual Knuth move between tableaux T and U *activates the bond* a_i if i lies in precisely one of $\tau(T)$ and $\tau(U)$, and $i + 1$ lies precisely in the other. Denote this condition by $T \overset{a_i}{-} U$. For $(T, U) \in E$, let

$$\beta(T, U) := \{a_i \in B : T \overset{a_i}{-} U\}.$$

The graph $G_\lambda := (V, E, \tau, \beta)$ is a signed colored graph of type $n + 1$.

Example 3.2 *Two standard dual equivalence graphs (corresponding to the shapes* 311 *and* 32*).*



The values of τ *are shown in red.*

3.2 Axiomatics

A vertex w of a signed colored graph is said to *admit an* i*-neighbor* if precisely one of i and $i+1$ lies in $\tau(w)$.

Definition 3.3 *A* dual equivalence graph of type $n+1$ *is a signed colored graph* (V, E, τ, β) *such that for any* $1 \leqslant i < n$*:*

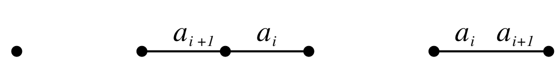
1. *For* $w \in V$*,* w *admits an* i*-neighbor if and only if there exists* $x \in V$ *which is connected to* w *by an edge of color* i*. In this case* x *must be unique*

2. *Suppose* (w, x) *is an edge of color* i*. Then* $i \in \tau(w)$ *iff* $i \notin \tau(x)$*,* $i+1 \in \tau(w)$ *iff* $i+1 \notin \tau(x)$*, and if* $h < i-1$ *or* $h > i+2$ *then* $h \in \tau(w)$ *iff* $h \in \tau(x)$*.*

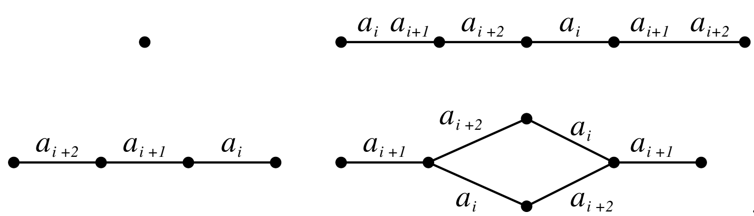
   In other words, going along an i *colored edge switches* i *and* $i+1$ *in the* τ *value, and does not affect anything except* $i-1, i, i+1$*, and* $i+2$*.*

3. *Suppose* (w, x) *is an edge of color* i*. If* $i-1 \in \tau(w)\Delta\tau(x)$ *then (*$i-1 \in \tau(w)$ *iff* $i+1 \in \tau(w)$*), where* Δ *is the symmetric difference. If* $i+2 \in \tau(w)\Delta\tau(x)$ *then (*$i+2 \in \tau(w)$ *iff* $i \in \tau(w)$*).*

4. *If* $i < n-2$*, consider the subgraph on all the vertices and edges labeled* a_i *or* a_{i+1}*. Each of its connected components has the form:*



If $i < n-3$, consider the subgraph on all the vertices and edges labeled a_i, a_{i+1}, or a_{i+2}. Each of its connected components has the form:



5. *Suppose $(w,x) \in E_i, (x,y) \in E_j$, and $|i-j| \geqslant 3$. Then there exists $v \in V$ such that $(w,v) \in E_j, (v,y) \in E_i$.*

6. *Consider a connected component of the subgraph on all the vertices and edges of colors $\leqslant i$. If we erase all the i-colored edges it breaks down into several components. Any two these were connected by an i-colored edge.*

A *weak dual equivalence graph* is a signed colored graph satisfying $1-5$ of the above.

Proposition 3.4 *The graph G_λ (the standard dual equivalence graph) is actually a dual equivalence graph. Moreover, $\{G_\lambda\}_\lambda$ is a complete collection of isomorphism class representatives of dual equivalence graphs.*

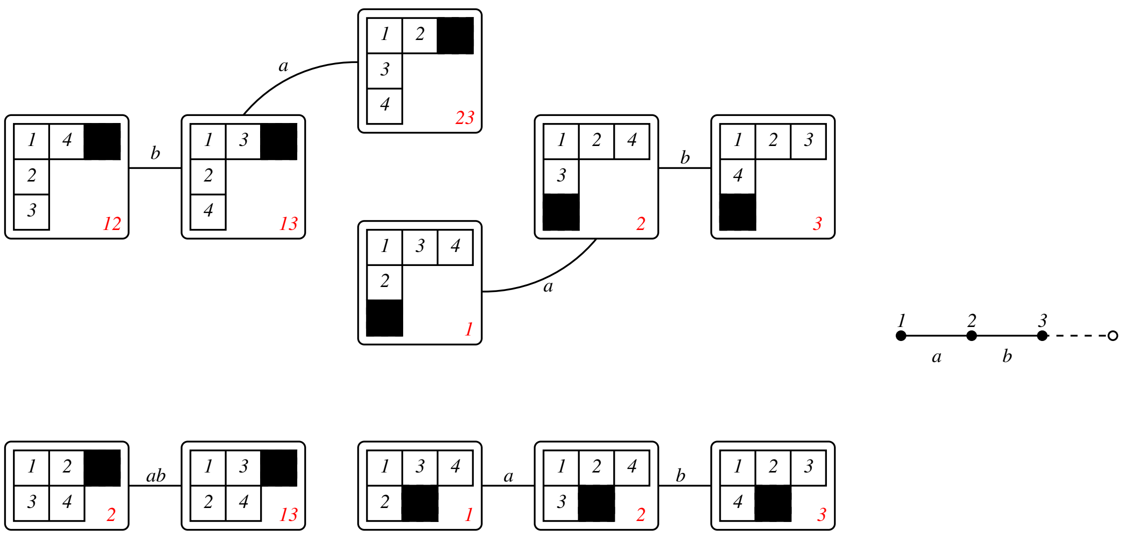
Proof: *The references are to the paper of Assaf (2008). The first statement is Proposition 3.5. The second is a combination of Theorem 3.9 and Proposition 3.11.* □

3.3 Restriction

Suppose G is a signed colored graph of type $n+1$. For $0 \leqslant k < n+1$, a $(k+1)$-*restriction* of G consists of the same vertex set V, the edges colored $\leqslant k-1$, the τ function post-composed with intersection with $\{1,\ldots,k\}$, and the β function post-composed with restriction to $\{a_1,\ldots,a_{k-1}\}$ (see Example 3.5). The $(k+1)$-restriction of G is a signed colored graph of type $k+1$. The property of being a dual equivalence graph (or a weak dual equivalence graph) is preserved by restriction. By a $(k+1)$-*component of* G we mean either the connected component of the restriction, or the induced subgraph of G on vertices corresponding to such connected component. It should be clear from the context which of these we are talking about.

The n-components of G_λ are obtained by fixing the position of $n+1$ in the tableau. Such a component is isomorphic to G_μ, where μ if formed from λ by erasing the outer corner which contained $n+1$. Here is what it looks like on the above examples:

Example 3.5



The condition of being a weak dual equivalence graph is already quite powerful. The following lemma is relevant to us. It essentially says that a weak dual equivalence graph with a nice restriction property is necessarily a cover of a dual equivalence graph.

Lemma 3.6 *Suppose G is a weak dual equivalence graph of type $n+1$. Suppose moreover that each n-component is a dual equivalence graph. Then there is a surjective morphism $\varphi : G \to G_\lambda$ for some partition λ of $n+1$, which restricts to an isomorphism on the n-components.*

Let $C \cong G_\mu$ be an n-component. Then for any partition $\nu \neq \mu$ of n with $\nu \subset \lambda$, there exists a unique n-component $D \cong G_\mu$ with which is connected to C by an $(n-1)$-colored edge. Also, two n-components which are both isomorphic to G_μ are not connected by an $(n-1)$-colored edge.

Proof: *The references are again to the paper of Assaf (2008). The existence of the morphism is shown in Theorem 3.14. Its surjectivity follows by Remark 3.8. The fact that it restricts to an isomorphism on the n-components follows from the proof of Theorem 3.14. The covering properties from the second paragraph are shown in Corollary 3.15, though the last one is not explicitly mentioned.* □

3.4 Molecules and dual equivalence graphs

Proposition 3.7 *The simple part of an A_n molecule, with $\beta(u,v) = \{$bonds activated by the edge $(u,v)\}$ is a weak dual equivalence graph.*

Proof: *Axioms (1), (2), (3) follow directly from SR, BR, and CR. The S_4 and S_5 molecules have been computed by Stembridge (2008a) (see Examples 2.3 and 2.4). This shows that (4) is satisfied. The axiom (5) is a weaker version of the local polygon rule.* □

Consider the graph G_λ from section 3.1. It is clear that (viewed as a weighted directed graph with all edges pointing both ways and having weight 1) it is an admissible S-labeled graph for the A_n root system.

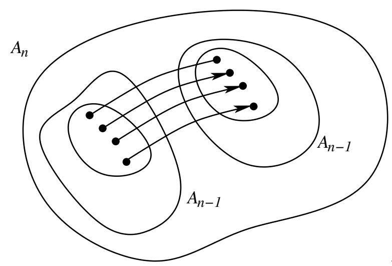
It is well known that it forms the simple part of an A_n-molecule (the left Kazhdan-Lusztig cell) which we call $\overline{G_\lambda}$.

Definition 3.8 *An A_n molecule is a* Kazhdan-Lusztig molecule *if it is isomorphic to $\overline{G_\lambda}$, i.e. if its simple part is a dual equivalence graph.*

4 Main theorem

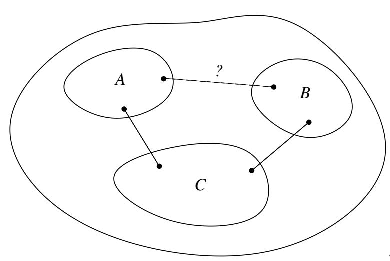
In this section we show that any A_n molecule is Kazhdan-Lusztig. The proof will proceed by induction on n, so the preliminary results will start with an A_n molecule whose A_{n-1} submolecules are Kazhdan-Lusztig.

The first of these results states that if two such A_{n-1} molecules are connected by a simple edge, then the connecting A_{n-2} submolecules are isomorphic and there is a "cabling" of edges (possibly arcs) of weight 1 between these A_{n-2} molecules:



Lemma 4.1 *Let M be an A_n molecule whose A_{n-1} submolecules are Kazhdan-Lusztig. Suppose A and B are two such submolecules which are joined by a simple edge (in M), namely there exist $x \in A$, $y \in B$ such that the edge $x - y$ is simple. Let A' (resp. B') be the A_{n-2} submolecule of M containing x (resp. y). Then there is an isomorphism ψ between A' and B' such that $\psi(x) = y$. Moreover, if $n \in \tau(x)$ then $m(z \to \psi(z)) = 1$ for all $z \in A'$.*

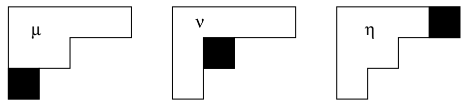
The second preliminary result shows that if, out of three A_{n-1} submolecules, two pairs (satisfying some conditions) are connected by simple edges, then the third pair is also connected by a simple edge:



The conditions will later be removed to show that any pair of A_{n-1} submolecules of an A_n molecule is connected by a simple edge.

Lemma 4.2 *Let M be an A_n-molecule whose A_{n-1} submolecules are Kazhdan-Lusztig. By Proposition 3.6, there is a surjective morphism $\varphi : M \to \overline{G_\lambda}$ for some partition λ of $n + 1$. Let A, B, C be A_{n-1}*

submolecules of M such that A and B are both connected to C by simple edges. Then $A \cong \overline{G_\mu}$, $B \cong \overline{G_\nu}$, $C \cong \overline{G_\eta}$, for some partitions formed by deleting outer corners of λ. The three partitions have to be different by Proposition 3.6. Suppose moreover that the deleted corner for η was the highest of the three, namely:



Then A and B are connected by a simple edge.

We can now finish the proof of the theorem.

Theorem 4.3 *Any A_n molecule is Kazhdan-Lusztig.*

Proof: *We know that the simple part of an A_n molecule is a weak dual equivalence graph. It remains to show that it satisfies the axiom (6), namely that any two A_{n-1} submolecules are connected by a simple edge.*

Proceed by induction on n, the case $n = 1$ being trivial. Let M be an A_n molecule. By inductive assumption, all A_{n-1} molecules are Kazhdan-Lusztig. So, according to 3.6 there is a covering $M \to \overline{G_\lambda}$, for some partition λ of $n + 1$.

Choose two of these A_{n-1} submolecules, A and Z. Choose a path of simple edges between them which goes through the fewest number of molecules. If it does not go through other molecules, then we are done. Suppose that is not so. Let A, B, C be the first three molecules on the path (it may happen that $Z = C$). The partitions μ, ν, η corresponding to $A, B,$ and C are formed by removing an outer corner of λ; they are all distinct by Proposition 3.6.

Again using Proposition 3.6, consider the following string of A_{n-1} submolecules connected by simple edges: $A - B - C - A' - B'$, with $A \cong A'$, $B \cong B'$, and some of these possibly equalities. Out of μ, ν, and η choose the partition which is formed by removing the highest box of λ. In the above string, choose a copy of the corresponding A_{n-1} submolecule with submolecules attached on both sides (for example, if $\lambda \setminus \mu$ was highest of the three, then we should choose A'). Then the triple consisting of this submolecule and the two adjacent ones satisfies the condition of the Lemma 4.2 (in the example, it would be the triple $C - A' - B'$). Applying it we get that $A' = A$, and $B' = B$. But then A is connected to C, contradicting our assumption that the path went through a minimal number of molecules.

So any two A_{n-1} molecules are connected by an edge, finishing the proof. □

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