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Gog, Magog and Schützenberger II: left trapezoids

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Abstract. We are interested in finding an explicit bijection between two families of combinatorial objects: Gog and Magog triangles. These two families are particular classes of Gelfand-Tsetlin triangles and are respectively in bijection with alternating sign matrices (ASM) and totally symmetric self complementary plane partitions (TSSCPP). For this purpose, we introduce left Gog and GOGAm trapezoids. We conjecture that these two families of trapezoids are equienumerated and we give an explicit bijection between the trapezoids with one or two diagonals.

Résumé. Nous nous intéressons ici à trouver une bijection explicite entre deux familles d'objets combinatoires: les triangles Gog et Magog. Ces deux familles d'objets sont des classes particulières des triangles de Gelfand-Tsetlin et sont respectivement en bijection avec les matrices à signes alternants (ASMs) et les partitions planes totalement symétriques auto-complémentaires (TSSCPPs). Pour ce faire, nous introduisons les Gog et les GOGAm trapèzes gauches. Nous conjecturons que ces deux familles de trapèzes sont équipotents et nous donnons une bijection explicite entre ces trapèzes à une et deux lignes.

Keywords: Gog, Magog triangles and trapezoids, Schützenberger Involution, alternating sign matrices, totally symmetric self complementary plane partitions

1 Introduction

This paper is a sequel to [1], to which we refer for more on the background of the Gog-Magog problem (see also [2] and [3] for a thorough discussion). It is a well known open problem in bijective combinatorics to find a bijection between alternating sign matrices and totally symmetric self complementary plane partitions. One can reformulate the problem using so-called Gog and Magog triangles, which are particular species of Gelfand-Tsetlin triangles. In particular, Gog triangles are in simple bijection with alternating sign matrices of the same size, while Magog triangles are in bijection with totally symmetric self complementary plane partitions. In [4], Mills, Robbins and Rumsey introduced trapezoids in this problem by cutting out k diagonals on the right (with the conventions used in the present paper) of a triangle of size n, and conjectured that Gog and Magog trapezoids of the same size are equienumerated. Zeilberger [7] proved this conjecture, but no explicit bijection is known, except for $k = 1$ (which is a relatively easy problem) and for $k = 2$, this bijection being the main result of [1]. In this last paper a new class of triangles and trapezoids was introduced, called GOGAm triangles (or trapezoids), which are in bijection with the Magog triangles by the Schützenberger involution acting on Gelfand-Tsetlin triangles.

In this paper we introduce a new class of trapezoids by cutting diagonals of Gog and GOGAm triangles on the left instead of the right. We conjecture that the left Gog and GOGAm trapezoids of the same shape are equienumerated, and give a bijective proof of this for trapezoids composed of one or two diagonals. Furthermore we show that our bijection is compatible with the previous bijection between right trapezoids. It turns out that the bijection we obtain for left trapezoids is much simpler than the one of [1] for right trapezoids.Finally we can also consider rectangles (intersections of left and right trapezoids). For such rectangles we also conjecture that Gog and GOGAm are equienumerated.

Our results are presented in this paper as follows. In section 2 we give some elementary definitions about Gelfand-Tsetlin triangles, Gog and GOGAm triangles, and then define left and right Gog and GOGAm trapezoids and describe their minimal completion. Section 3 is devoted to the formulation of a conjecture on the existence of a bijection between Gog and GOGAm trapezoids of the same size. We end this paper by section 4 where we give a bijection between $(n, 2)$ left Gog and GOGAm trapezoids and we show how its work on an example. Finally, we consider another combinatorial object; rectangles.

2 Basic definitions

We start by giving definitions of our main objects of study. We refer to [1] for more details.

2.1 Gelfand-Tsetlin triangles

Definition 2.1 *A Gelfand-Tsetlin triangle of size n is a triangular array $X = (X_{i,j})_{n \geqslant i \geqslant j \geqslant 1}$ of positive integers*

$$\begin{array}{ccccccccccc}
X_{n,1} & & X_{n,2} & & \cdots & & \cdots & & X_{n,n-1} & & X_{n,n} \\
& X_{n-1,1} & & X_{n-1,2} & & \cdots & & \iddots & & X_{n-1,n-1} & \\
& & \ddots & & & & \iddots & & \iddots & & \\
& & & \ddots & & \iddots & & \iddots & & & \\
& & & & X_{2,1} & & X_{2,2} & & & & \\
& & & & & X_{1,1} & & & & &
\end{array} \tag{1}$$

such that

$$X_{i+1,j} \leqslant X_{i,j} \leqslant X_{i+1,j+1} \quad \text{for } n-1 \geqslant i \geqslant j \geqslant 1. \tag{2}$$

The set of all Gelfand-Tsetlin triangles of size n is a poset for the order such that $X \leqslant Y$ if and only if $X_{ij} \leqslant Y_{ij}$ for all i, j. It is also a lattice for this order, the infimum and supremum being taken entrywise: $\max(X, Y)_{ij} = \max(X_{ij}, Y_{ij})$.

2.2 Gog triangles and trapezoids

Definition 2.2 *A Gog triangle of size n is a Gelfand-Tsetlin triangle such that*

1. *its rows are strictly increasing;*

$$X_{i,j} < X_{i,j+1}, \qquad j < i \leqslant n-1 \tag{3}$$

2. *and such that*

$$X_{n,j} = j, \qquad 1 \leqslant j \leqslant n. \tag{4}$$

Here is an example of Gog triangle of size $n = 5$.

$$\begin{array}{ccccccccc}
1 & & 2 & & 3 & & 4 & & 5 \\
& 1 & & 3 & & 4 & & 5 & \\
& & 1 & & 4 & & 5 & & \\
& & & 2 & & 4 & & & \\
& & & & 3 & & & &
\end{array} \tag{5}$$

It is immediate to check that the set of Gog triangles of size n is a sublattice of the Gelfand-Tsetlin triangles.

Definition 2.3 *A (n, k) right Gog trapezoid (for $k \leqslant n$) is an array of positive integers $X = (X_{i,j})_{n \geqslant i \geqslant j \geqslant 1; i-j \leqslant k-1}$ formed from the k rightmost SW-NE diagonals of some Gog triangle of size n.*

Below is a $(5, 2)$ right Gog trapezoid.

$$\begin{array}{cccccc}
& & & 4 & & 5 \\
& & 4 & & 5 & \\
& 3 & & 4 & & \\
1 & & 3 & & & \\
& 2 & & & &
\end{array} \tag{6}$$

Definition 2.4 *A (n, k) left Gog trapezoid (for $k \leqslant n$) is an array of positive integers $X = (X_{i,j})_{n \geqslant i \geqslant j \geqslant 1; k \geqslant j}$ formed from the k leftmost NW-SE diagonals of a Gog triangle of size n.*

A more direct way of checking that a left Gelfand-Tsetlin trapezoid is a left Gog trapezoid is to verify that its rows are strictly increasing and that its SW-NE diagonals are bounded by $1, 2, \ldots, n$ as it is shown in the figure below which represents a $(5, 2)$ left Gog trapezoid.

$$\begin{array}{cccccccccccccc}
& & & \color{blue}{1} & & & & \color{blue}{2} & & & & & & \\
1 & \color{blue}{\leqslant} & & & 2 & \color{blue}{\leqslant} & & & & \color{blue}{3} & & & & \\
& & 1 & & & & 3 & \color{blue}{\leqslant} & & & & \color{blue}{4} & & \\
& & & & 2 & & & & 3 & \color{blue}{\leqslant} & & & & \color{blue}{5} \\
& & & & & & 2 & & & & 4 & \color{blue}{\leqslant} & & \\
& & & & & & & & 4 & & & & &
\end{array} \tag{7}$$

There is a simple involution $X \to \tilde{X}$ on Gog triangles given by

$$\tilde{X}_{i,j} = n + 1 - X_{i,i+1-j} \tag{8}$$

which exchanges left and right trapezoids of the same size. This involution corresponds to a vertical symmetry of associated ASMs.

2.2.1 Minimal completion

Since the set of Gog triangles is a lattice, given a left (resp. a right) Gog trapezoid, there exists a smallest Gog triangle from which it can be extracted. We call this Gog triangle the *canonical completion* of the left (resp. the right) Gog trapezoid. Their explicit value is computed in the next Proposition.

Proposition 2.5

1. *Let X be a (n, k) right Gog trapezoid, then its canonical completion satisfies*

$$X_{ij} = j \quad \textit{for} \quad i \geqslant j + k. \tag{9}$$

2. *Let X be a (n, k) left Gog trapezoid, then its canonical completion satisfies*

$$X_{i,j} = \max(X_{i,k} + j - k, X_{i-1,k} + j - k - 1, \ldots, X_{i-j+k,k}) \quad \textit{for} \quad j \geqslant k. \tag{10}$$

Proof: The first case (right trapezoids) is trivial, the formula for the second case (left trapezoids) is easily proved by induction on $j - k$. □

For example, the completion of the $(5, 2)$ left Gog trapezoid in (7)

$$\begin{array}{ccccccccc} 1 & & 2 & & {\color{red}3} & & {\color{red}4} & & {\color{red}5} \\ & 1 & & 3 & & {\color{red}4} & & {\color{red}5} & \\ & & 2 & & 3 & & {\color{red}4} & & \\ & & & 2 & & 4 & & & \\ & & & & 4 & & & & \end{array} \tag{11}$$

Remark that the supplementary entries of the canonical completion of a left Gog trapezoid depend only on its rightmost NW-SE diagonal.

The right trapezoids defined above coincide (modulo easy reindexations) with those of Mills, Robbins, Rumsey [4], and Zeilberger [7]. They are in obvious bijection with the ones in [1] (actually the Gog trapezoids of [1] are the canonical completions of the right Gog trapezoids defined above).

2.3 GOGAm triangles and trapezoids

Definition 2.6 *A GOGAm triangle of size n is a Gelfand-Tsetlin triangle such that $X_{nn} \leqslant n$ and, for all $1 \leqslant k \leqslant n-1$, and all $n = j_0 > j_1 > j_2 \ldots > j_{n-k} \geqslant 1$, one has*

$$\left(\sum_{i=0}^{n-k-1} X_{j_i+i,j_i} - X_{j_{i+1}+i,j_{i+1}} \right) + X_{j_{n-k}+n-k,j_{n-k}} \leqslant k \tag{12}$$

It is shown in [1] that GOGAm triangles are exactly the Gelfand-Tsetlin triangles obtained by applying the Schützenberger involution to Magog triangles. It follows that the problem of finding an explicit bijection between Gog and Magog triangles can be reduced to that of finding an explicit bijection between Gog and GOGAm triangles. In the sequel, Magog triangles will not be considered anymore.

Definition 2.7 *A (n,k) right GOGAm trapezoid (for $k \leqslant n$) is an array of positive integers $X = (x_{i,j})_{n \geqslant i \geqslant j \geqslant 1; i-j \leqslant k-1}$ formed from the k rightmost SW-NE diagonals of a GOGAm triangle of size n.*

Below is a $(5,2)$ right GOGAm trapezoid.

$$
\begin{array}{cccccc}
 & & & 2 & & 4 \\
 & & 2 & & 4 & \\
 & 2 & & 4 & & \\
1 & & 4 & & & \\
 & 3 & & & &
\end{array}
\tag{13}
$$

Definition 2.8 *A (n,k) left GOGAm trapezoid (for $k \leqslant n$) is an array of positive integers $X = (x_{i,j})_{n \geqslant i \geqslant j \geqslant 1; k \geqslant j}$ formed from the k leftmost NW-SE diagonals of a GOGAm trapezoid of size n.*

Below is a $(5,2)$ left GOGAm trapezoid.

$$
\begin{array}{cccccc}
1 & & 1 & & & \\
 & 1 & & 2 & & \\
 & & 1 & & 2 & \\
 & & & 2 & & 3 \\
 & & & & 3 &
\end{array}
\tag{14}
$$

2.3.1 Minimal completion

The set of GOGAm triangles is not a sublattice of the Gelfand-Tsetlin triangles, nevertheless, given a right (resp. a left) GOGAm trapezoid, we shall see that there exists a smallest GOGAm triangle which extends it. We call this GOGAm triangle the *canonical completion* of the left (resp. the right) GOGAm trapezoid.

Proposition 2.9

1. *Let X be a (n,k) right GOGAm trapezoid, then its canonical completion is given by*

$$X_{ij} = 1 \quad \textit{for } n \geqslant i \geqslant j + k. \tag{15}$$

2. *Let X be a (n,k) left GOGAm trapezoid, then its canonical completion is given by*

$$X_{i,j} = X_{i-j+k,k} \quad \textit{for } n \geqslant i \geqslant j \geqslant k \tag{16}$$

in other words, the added entries are constant on SW-NE diagonals

Proof: In both cases, the completion above is the smallest Gelfand-Tsetlin triangle containing the trapezoid, therefore it is enough to check that if X is a (n,k) right or left GOGAm trapezoid, then its completion, as indicated in the proposition 2.9, is a GOGAm triangle. The claim follows from the following lemma.

Lemma 2.10 *Let X be a GOGAm triangle.*

- **i)** *The triangle obtained from X by replacing the entries on the upper left triangle $(X_{ij}, n \geqslant i \geqslant j+k)$ by 1 is a GOGAm triangle.*
- **ii)** *Let $n \geqslant m \geqslant k \geqslant 1$. If X is constant on each partial SW-NE diagonal $(X_{i+l,k+l}; n-i \geqslant l \geqslant 0)$ for $i \geqslant m+1$ then the triangle obtained from X by replacing the entries $(X_{m+l,k+l}; n-m \geqslant l \geqslant 1)$ by $X_{m,k}$ is a GOGAm triangle.*

Proof: It is easily seen that the above replacements give a Gelfand-Tsetlin triangle. Both proofs then follow by inspection of the formula (12), which shows that, upon making the above replacements, the quantity on the left cannot increase. □

End of proof of Proposition 2.9. The case of right GOGAm triangles is dealt with by part **i)** of the preceding Lemma. The case of left trapezoids follows by replacing successively the SW-NE partial diagonals as in part **ii)** of the Lemma. □

For example, the completion of the $(5,2)$ left GOGAm trapezoid in (14) is as follows.

$$\begin{array}{ccccccccc} 1 & & 1 & & 2 & & 2 & & 3 \\ & 1 & & 2 & & 2 & & 3 & \\ & & 1 & & 2 & & 3 & & \\ & & & 2 & & 3 & & & \\ & & & & 3 & & & & \end{array} \tag{17}$$

3 Results and conjectures

Theorem 3.1 (Zeilberger [7]) *For all $k \leqslant n$, the (n,k) right Gog and GOGAm trapezoids are equienumerated*

Actually Zeilberger proves this theorem for Gog and Magog trapezoids, but composing by the Schützenberger involution yields the above result. In [1] a bijective proof is given for $(n,1)$ and $(n,2)$ right trapezoids.

Conjecture 3.2 *For all $k \leqslant n$, the (n,k) left Gog and GOGAm trapezoids are equienumerated.*

In the next section we will give a bijective proof of this conjecture for $(n,1)$ and $(n,2)$ trapezoids.

Remark that the right and left Gog trapezoids of shape (n,k) are equienumerated (in fact a simple bijection between them was given above).

If we consider left GOGAm trapezoids as GOGAm triangles, using the canonical completion, then we can take their image by the Schützenberger involution and obtain a subset of the Magog triangles, for each (n,k). It seems however that this subset does not have a simple direct characterization. This shows that GOGAm triangles and trapezoids are a useful tool in the bijection problem between Gog and Magog triangles.

4 Bijections between Gog and GOGAm left trapezoids

4.1 $(n,1)$ *left trapezoids*

The sets of $(n,1)$ left Gog trapezoids and of $(n,1)$ left GOGAm trapezoids coincide with the set of sequences $X_{n,1},\dots,X_{1,1}$ satisfying $X_{j,1} \leqslant n-j+1$ (note that these sets are counted by Catalan numbers). Therefore the identity map provides a trivial bijection between these two sets.

4.2 $(n,2)$ *left trapezoids*

In order to treat the $(n,2)$ left trapezoids we will recall some definitions from [1].

4.2.1 *Inversions*

Definition 4.1 *An inversion in a Gelfand-Tsetlin triangle is a pair (i,j) such that $X_{i,j} = X_{i+1,j}$.*

For example, the Gog triangle in (18) contains three inversions, $(2,2)$, $(3,1)$, $(4,1)$, the respective equalities being in red on this picture.

$$\begin{array}{ccccccccc}
1 & & 2 & & 3 & & 4 & & 5 \\
 & 1 & & 3 & & 4 & & 5 & \\
 & & 1 & & 4 & & 5 & & \\
 & & & 2 & & 4 & & & \\
 & & & & 3 & & & &
\end{array} \tag{18}$$

Definition 4.2 *Let $X = (X_{i,j})_{n \geqslant i \geqslant j \geqslant 1}$ be a Gog triangle and let (i,j) be such that $1 \leqslant j \leqslant i \leqslant n$.*

An inversion (k,l) covers (i,j) if $i = k+p$ and $j = l+p$ for some p with $1 \leqslant p \leqslant n-k$.

The entries (i,j) covered by an inversion are depicted with " $+$ " on the following picture.

$$\begin{array}{ccccccccc}
\circ & & + & & + & & \circ & & + \\
 & \circ & & + & & \circ & & + & \\
 & & \circ & & \circ & & + & & \\
 & & & \circ & & \circ & & & \\
 & & & & \circ & & & &
\end{array} \tag{19}$$

4.2.2 Standard procedure

The basic idea for our bijection is that for any inversion in the Gog triangle we should subtract 1 to the entries covered by this inversion, scanning the inversions along the successive NW-SE diagonals, starting from the rightmost diagonal, and scanning each diagonals from NW to SE. We call this the *standard procedure*. This procedure does not always yield a Gelfand-Tsetlin triangle, but one can check that it does so if one starts from a Gog triangle corresponding to a permutation matrix in the correspondance between alternating sign matrices and Gog triangles. Actually the triangle obtained is also a GOGAm triangle.

Although we will not use it below, it is informative to make the following remark.

Proposition 4.3 *Let X be the canonical completion of a left (n, k) Gog trapezoid. The triangle Y obtained by applying the standard procedure to the $n-k+1$ rightmost NW-SE diagonals of X is a Gelfand-Tsetlin triangle such that $Y_{i+l,k+l} = X_{i,k}$ for $n-i \geqslant l \geqslant 1$.*

Proof: The Proposition is proved easily by induction on the number $n-k+1$. □

For example, applied to the $(5, 2)$ left Gog trapezoid in (7), this yields

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 3 & & 4 \\ & 1 & & 3 & & 3 & & 4 & \\ & & 2 & & 3 & & 4 & & \\ & & & 2 & & 4 & & & \\ & & & & 4 & & & & \end{array} \tag{20}$$

Like in [1] the bijection between left Gog and GOGAm trapezoids will be obtained by a modification of the Standard Procedure.

4.2.3 Characterization of $(n, 2)$ GOGAm trapezoids

The family of inequalities (12) simplifies in the case of $(n, 2)$ GOGAm trapezoids, indeed if we identify such a trapezoid with its canonical completion, then most of the terms in the left hand side are zero, so that these inequalities reduce to

$$X_{i,2} \leqslant n-i+2 \tag{21}$$

$$X_{i,2} - X_{i-1,1} + X_{i,1} \leqslant n-i+1 \tag{22}$$

Remark that, since $-X_{i-1,1} + X_{i,1} \leqslant 0$, the inequality (22) follows from (21) unless $X_{i-1,1} = X_{i,1}$.

4.2.4 From Gog to GOGAm

Let X be a $(n, 2)$ left Gog trapezoid. We shall construct a $(n, 2)$ left GOGAm trapezoid Y by scanning the inversions in the leftmost NW-SE diagonal of X, starting from NW. Let us denote by $n > i_1 > \ldots > i_k \geqslant 1$ these inversions, so that $X_{i,1} = X_{i+1,1}$ if and only if $i \in \{i_1, \ldots, i_k\}$. We also put $i_0 = n$. We will construct a sequence of $(n, 2)$ left Gelfand-Tsetlin trapezoids $X = Y^{(0)}, Y^{(1)}, Y^{(2)}, \ldots, Y^{(k)} = Y$.

Let us assume that we have constructed the trapezoids up to $Y^{(l)}$, that $Y^{(l)} \leqslant X$, that $Y^{(l)}_{ij} = X_{ij}$ for $i \leqslant i_l$, and that inequalities (21) and (22) are satisfied by $Y^{(l)}$ for $i \geqslant i_l + 1$. This is the case for $l = 0$.

Let m be the largest integer such that $Y^{(l)}_{m,2} = Y^{(l)}_{i_{l+1}+1,2}$. We put

$$\begin{array}{lclll} Y^{(l+1)}_{i,1} & = & Y^{(l)}_{i,1} & \text{for} & n \geqslant i \geqslant m \quad \text{and} \quad i_{l+1} \geqslant i \\ Y^{(l+1)}_{i,1} & = & Y^{(l)}_{i+1,1} & \text{for} & m-1 \geqslant i > i_{l+1} \\ Y^{(l+1)}_{i,2} & = & Y^{(l)}_{i,2} & \text{for} & n \geqslant i \geqslant m+1 \quad \text{and} \quad i_{l+1} \geqslant i \\ Y^{(l+1)}_{i,2} & = & Y^{(l)}_{i,2} - 1 & \text{for} & m \geqslant i \geqslant i_{l+1} + 1. \end{array}$$

From the definition of m, and the fact that X is a Gog trapezoid, we see that this new triangle is a Gelfand-Tsetlin triangle, that $Y^{(l+1)} \leqslant X$, and that $Y^{(l+1)}_{ij} = X_{ij}$ for $i \leqslant i_{l+1}$. Let us now check that the trapezoid $Y^{(l+1)}$ satisfies the inequalities (21) and (22) for $i \geqslant i_{l+1} + 1$. The first series of inequalities, for $i \geqslant i_{l+1}+1$, follow from the fact that $Y^{(l)} \leqslant X$. For the second series, they are satisfied for $i \geqslant m+1$ since this is the case for $Y^{(l)}$. For $m \geqslant i \geqslant i_{l+1} + 1$, observe that

$$Y^{(l+1)}_{i,2} - Y^{(l+1)}_{i+1,1} + Y^{(l+1)}_{i+1,1} \leqslant Y^{(l+1)}_{i,2} = Y^{(l+1)}_{m,2} = Y^{(k)}_{m,2} - 1 \leqslant n - m + 1$$

by (21) for $Y(l)$, from which (22) follows.

This proves that $Y^{(l+1)}$ again satisfies the induction hypothesis. Finally $Y = Y^{(k)}$ is a GOGAm triangle: indeed inequalities (21) follow again from $Y^{(l+1)} \leqslant X$, and (22) for $i \leqslant i_k$ follow from the fact that there are no inversions in this range. It follows that the above algorithm provides a map from $(n, 2)$ left Gog trapezoids to $(n, 2)$ left GOGAm trapezoids.Observe that the number of inversions in the leftmost diagonal of Y is the same as for X, but the positions of these inversions are not the same in general.

4.2.5 Inverse map

We now describe the inverse map, from GOGAm left trapezoids to Gog left trapezoids.

We start from an $(n, 2)$ GOGAm left trapezoid Y, and construct a sequence
$Y = Y^{(k)}, Y^{(k-1)}, Y^{(k-2)}, \ldots, Y^{(0)} = X$ of intermediate Gelfand-Tsetlin trapezoids.
Let $n - 1 \geqslant \iota_1 > \iota_2 \ldots > \iota_k \geqslant 1$ be the inversions of the leftmost diagonal of Y, and let $\iota_{k+1} = 0$. Assume that $Y^{(l)}$ has been constructed and that $Y^{(l)}_{ij} = Y_{ij}$ for $i - j \geqslant \iota_{l+1}$. This is the case for $l = k$.

Let p be the smallest integer such that $Y^{(l)}_{\iota_l+1,2} = Y^{(l)}_{p,2}$. We put

$$\begin{array}{lclll} Y^{(l-1)}_{i,1} & = & Y^{(l)}_{i,1} & \text{for} & n \geqslant i \geqslant \iota_l + 1 \quad \text{and} \quad p \geqslant i \\ Y^{(l-1)}_{i,1} & = & Y^{(l)}_{i-1,1} & \text{for} & \iota_l \geqslant i \geqslant p \\ Y^{(l-1)}_{i,2} & = & Y^{(l)}_{i,2} & \text{for} & n \geqslant i \geqslant \iota_l + 2 \quad \text{and} \quad p - 1 \geqslant i \\ Y^{(l-1)}_{i,2} & = & X^{(l)}_{i,2} - 1 & \text{for} & \iota_l + 1 \geqslant i \geqslant p. \end{array}$$

It is immediate to check that if X is an $(n, 2)$ left Gog trapezoid, and Y is its image by the first algorithm then the above algorithm applied to Y yields X back, actually the sequence $Y^{(l)}$ is the same. Therefore in order to prove the bijection we only need to show that if Y is a $(n, 2)$ left GOGAm trapezoid then the algorithm is well defined and X is a Gog left trapezoid. This is a bit cumbersome, but not difficult, and very similar to the opposite case, so we leave this task to the reader.

4.2.6 A statistic

Observe that in our bijection the value of the bottom entry $X_{1,1}$ is unchanged when we go from Gog to GOGAm trapezoids. The same was true of the bijection in [1] for right trapezoids. Actually we make the following conjecture, which extends Conjecture 3.2 above.

Conjecture 4.4 *For each n, k, l the (n, k) left Gog and GOGAm trapezoids with bottom entry $X_{1,1} = l$ are equienumerated.*

4.3 An example

In this section we work out an example of the algorithm from the Gog trapezoid X to the GOGAm trapezoid Y by showing the successive trapezoids $Y^{(k)}$. At each step we indicate the inversion in green, as well as the entry covered by this inversion in shaded green, and the values of the parameters i_l, p. The algorithm also runs backwards to yield the GOGAm→Gog bijection.

$$\begin{array}{cccccccc}
1 & & & & & & & \\
 & 1 & & 2 & & & & \\
 & & 1 & & 4 & & & \\
 & & & 2 & & 4 & & \\
 & & & & 3 & & 4 & \\
 & & & & & 3 & & 4 \\
 & & & & & & 3 &
\end{array}$$

$X = Y^{(0)}$

$i_1 = 6$

$m = 7$

$$\begin{array}{cccccccc}
1 & & 1 & & & & & \\
 & 1 & & & & & & \\
 & & 1 & & 4 & & & \\
 & & & 2 & & 4 & & \\
 & & & & 3 & & 4 & \\
 & & & & & 3 & & 4 \\
 & & & & & & 3 &
\end{array}$$

$Y^{(2)}$

$i_2 = 5$

$m = 6$

$$
\begin{array}{cccccccccc}
1 & & 1 & & & & & & & Y^{(2)} \\
& 1 & & 1 & & & & & & \\
& & 1 & & 4 & & & & & i_3 = 2 \\
& & & 2 & & 4 & & & & m = 5 \\
& & & & 3 & & & & & \\
& & & & & 3 & & 4 & & \\
& & & & & & 3 & & &
\end{array}
$$

$$
\begin{array}{cccccccccc}
1 & & 1 & & & & & & & Y^{(3)} \\
& 1 & & 1 & & & & & & \\
& & 1 & & 3 & & & & & i_4 = 1 \\
& & & 1 & & 3 & & & & m = 2 \\
& & & & 2 & & 3 & & & \\
& & & & & 3 & & & & \\
& & & & & & 3 & & &
\end{array}
$$

$$
\begin{array}{cccccccccc}
1 & & 1 & & & & & & & \\
& 1 & & 1 & & & & & & \\
& & 1 & & 3 & & & & & \\
& & & 1 & & 3 & & & & Y^{(4)} = Y \\
& & & & 2 & & 3 & & & \\
& & & & & 3 & & 3 & & \\
& & & & & & 3 & & &
\end{array}
$$

4.4 Rectangles

Definition 4.5 *For (n, k, l) satisfying $k + l \leqslant n + 1$, a (n, k, l) Gog rectangle is an array of positive integers $X = (x_{i,j})_{n \geqslant i \geqslant j \geqslant 1; k \geqslant j; j+l \geqslant i+1}$ formed from the intersection of the k leftmost NW-SE diagonals and the l rightmost SW-NE diagonals of a Gog triangle of size n.*

Definition 4.6 *For (n, k, l) satisfying $k + l \leqslant n + 1$, a (n, k, l) GOGAm rectangle is an array of positive integers $X = (x_{i,j})_{n \geqslant i \geqslant j \geqslant 1; k \geqslant j; j+l \geqslant i+1}$ formed from the intersection of the k leftmost NW-SE diagonals and the l rightmost SW-NE diagonals of a GOGAm triangle of size n.*

Similarly to the case of trapezoids, one can check that a (n, k, l) Gog (resp. GOGAm) rectangle has a canonical (i.e. minimal) completion as a (n, k) left trapezoid, or as a (n, l) right trapezoid, and finally as a triangle of size n.

Conjecture 4.7 *For any (n, k, l) satisfying $k + l \leqslant n + 1$ the (n, k, l) Gog and GOGAm rectangles are equienumerated.*

As in the case of trapezoids, there is also a refined version of the conjecture with the statistic X_{11} preserved.

One can check, using standard completions, that our bijections, in [1] and in the present paper, restrict to bijections for rectangles of size $(n, k, 2)$ or $(n, 2, l)$. Furthermore, in the case of $(n, 2, 2)$ rectangles, the bijections coming from left and right trapezoids are the same.

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References

[1] P. Biane and H. Cheballah. Gog and magog triangles and the schützenberger involution. *Séminaire Lotharingien de Combinatoire*, B66d, apr 2012.

[2] D. M. Bressoud. *Proofs and Confirmations, The Story of the Alternating Sign Matrix Conjecture*. Cambridge University Press,Cambridge, 1999.

[3] H. Cheballah. *Combinatoire des matrices à signes alternants et des partitions planes*. PhD thesis, LIPN, Université Paris-Nord, 2011.

[4] W. Mills, D. Robbins, and H.Rumsey. Self complementary totally symmetric plane partitions. *J. Combin. Theory Ser. A*, 42, 1986.

[5] T. Sage-Combinat community. Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, 2008. `http://combinat.sagemath.org`.

[6] W. Stein et al. *Sage Mathematics Software (Version 5.0.1)*. The Sage Development Team, 2012. `http://www.sagemath.org`.

[7] D. Zeilberger. Proof of the alternating sign matrix conjecture. *Electronic J. Combin*, 3, 1996.