Samples of geometric random variables with multiplicity constraints

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We investigate the probability that a sample \( \Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n) \) of independent, identically distributed random variables with a geometric distribution has no elements occurring exactly \( j \) times, where \( j \) belongs to a specified finite ‘forbidden set’ \( A \) of multiplicities. Specific choices of the set \( A \) enable one to determine the asymptotic probabilities that such a sample has no variable occurring with multiplicity \( b \), or which has all multiplicities greater than \( b \), for any fixed integer \( b \geq 1 \).

Keywords: Geometric random variable, Mellin transform, Poisson transform, multiplicity

1 Introduction

We study samples of independent, identically distributed (i.i.d.) random variables with a geometric distribution. Specifically, let \( \Gamma_1, \Gamma_2, \Gamma_3, \ldots \) be i.i.d. geometric random variables with parameter \( p \), that is, \( \Pr(\Gamma_1 = j) = pq^{j-1}, j \in \mathbb{N} \), with \( p + q = 1 \). There is now an extensive literature on the combinatorics of geometric random variables and its applications in Computer Science. We are interested in the probability that a random sample of \( n \) such variables consists of elements whose multiplicities belong to specified sets. That is, we place restrictions on the number of times any element/letter can occur in the sample.

As a simple example, we may wish to consider a sample where none of the \( n \) elements occur exactly \( b \) times. In this case \( A = \{ b \} \). Another example of such a forbidden set is when a letter can occur only \( b \) times or more (or not at all), i.e., \( A = \{ 1, 2, \ldots, b-1 \} \), where \( b \geq 2 \). Note that we do not allow 0 in the forbidden set. Previously in (HK05; LP05), certain geometric samples with 0 in the forbidden set were studied under the names of ‘complete’ and ‘gap-free’ samples.

\[ p_n = 1 - \frac{T^*(0)}{\ln(1/q)} - \frac{2}{\ln(1/q)} \Re \left( \sum_{k=1}^{\infty} \exp\{\chi_k \ln(q/n)\} T^*(\chi_k) \right) + O(n^{-1}), \]

where we set \( \chi_k = \frac{2k\pi i}{\ln(1/q)} \), and where

\[ T^*(0) = \sum_{j \in A} p^j \sum_{n \geq 0} p^n q^n \frac{1}{n+j} \binom{n+j}{j} \]

and

\[ T^*(\chi_k) = \sum_{j \in A} \frac{p^j}{j!} \sum_{n \geq 0} p^n q^n \Gamma(n+j+\chi_k), \quad \text{for } k \in \mathbb{Z} \setminus \{0\}. \]

†This material is based upon work supported by the National Research Foundation under grant number 2053740

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2 Outline of the proof of Theorem [1]

We want a recursion on the variable $p_n$, the probability that the sample does not have letters which appear exactly $j$ times if $j$ is an element of the forbidden set $A \subset \mathbb{N}$. Let $B$ represent the set of all permitted samples. If we let $p_n = \mathbb{P}(\Gamma \in B)$ (the probability that the sample $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$ has no letter occurring exactly $j$ times for $j \in A$), then we can write

$$p_n = \sum_{j \geq 0} \mathbb{P}(\{\Gamma \in B \cap \{\sum_{\ell=1}^{n} I_{\Gamma_{\ell}} = j\}\}) = \sum_{j \geq 0} \mathbb{P}(\Gamma \in B) \sum_{\ell=1}^{n} I_{\Gamma_{\ell}} = j \cdot \mathbb{P}(\sum_{\ell=1}^{n} I_{\Gamma_{\ell}} = j),$$

where the indicator function $I$ takes values 1 for true and 0 for false. Using the law of total probability, and the memoryless property of geometric random variables, we obtain the recursion:

$$p_n = \sum_{j=0}^{n} p_{n-j} \binom{n}{j} p^j q^{n-j} - \sum_{j \in A} p_{n-j} \binom{n}{j} p^j q^{n-j}. \quad (2.1)$$

We would like to see how $p_n$ behaves asymptotically as $n \to \infty$. The Poissonisation technique we use can be seen in [JS98, Szp01], but we follow more specifically the process used in [HK05, JS97]. Namely, we consider the Poisson transform of the sequence $(p_n)$, analyse its asymptotics with Mellin transforms, then de-Poissonise to recover the asymptotics of $(p_n)$. To do this, we make use of the exponential generating function $P(z)$, which is the Poisson transform of $(p_n)$, given by $P(z) := \sum_{n \geq 0} p_n \frac{z^n}{n!} e^{-z}$. We use (2.1) to show that

$$P(z) - e^{-z} = \sum_{n \geq 1} p_n \frac{z^n}{n!} e^{-z} = P(qz) - e^{-pz} \sum_{j \in A} \binom{n}{j} q^j P(qz) - e^{-z}.$$

We thus have a functional equation of the form:

$$P(z) = P(qz) - e^{-pz} \sum_{j \in A} \binom{n}{j} q^j P(qz), \quad (2.2)$$

The technique we now use is the Mellin transform. A standard reference on Mellin transforms is [FGD95]. We define the function (see (2.2))

$$T(z) := e^{-pz} \sum_{j \in A} \binom{n}{j} q^j P(qz) \quad \left(= \sum_{n \geq 0} p_n \frac{z^n}{n!} \sum_{j \in A} \binom{n}{j} q^j z^{n+j} e^{-z}\right). \quad (2.3)$$

We note that the Mellin transform of $T(z)$ has a fundamental strip of at least $(-1, \infty)$. Now we find the Mellin transform of (2.3) to be

$$T^*(s) = \sum_{n \geq 0} p_n \frac{z^n}{n!} \sum_{j \in A} \frac{p^j}{j!} \mathcal{M}(z^{n+j} e^{-z}) = \sum_{n \geq 0} p_n \frac{z^n}{n!} \sum_{j \in A} \frac{p^j}{j!} \Gamma(n+j+s).$$

In particular we will make use of the values $T^*(0)$ and $T^*(\chi k)$, as given in (1.1) and (1.2). After iterating (2.2) we get

$$P(z) = P(qz) - T(z) = P(q^{m+1} z) - \sum_{j=0}^{m} T(q^j z),$$

for any $m \geq 0$ and thus in the limit as $m \to \infty$, $P(z) = 1 - \sum_{j=0}^{\infty} T(q^j z)$. We define $Q(z) := P(z) - 1 = -\sum_{j=0}^{\infty} T(q^j z)$. Then the corresponding Mellin transform is

$$Q^*(s) = -\sum_{j=0}^{\infty} q^{-j-s} T^*(s) = -\frac{T^*(s)}{1-q^{-s}} = \frac{T^*(s)}{q^{-s} - 1},$$

where $Q^*(s)$ exists in the fundamental strip $(-1, 0)$. The inverse Mellin transform is

$$Q(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Q^*(s) x^{-s} ds,$$
for any $-1 < c < 0$. The residue theorem can be used to evaluate this. This process (compare to (HK05)) results in

$$Q(x) \sim \sum_{\chi} \text{Res}_{s=\chi}(Q^*(s)x^{-s}).$$

Now, $x^{-s}Q^*(s) = x^{-s}T^*(s)/(q^{-s} - 1)$ has simple poles at $q^{-s} = 1$, i.e. at $\chi_k = 2k\pi i/\ln(1/q)$, $k \in \mathbb{Z}$, and the corresponding residues are

$$\lim_{s \to \chi_k} (s - \chi_k) x^{-s}T^*(s) = \frac{x^{-\chi_k}T^*(\chi_k)}{q^{-\chi_k}\ln(1/q)} = \left(\frac{q}{x}\right)^{\chi_k}\frac{T^*(\chi_k)}{\ln(1/q)}.$$

Of these quantities, all but the $k = 0$ term contribute oscillations of small amplitude. We need to use asymptotic de-Poissonisation to deduce that $p_n \sim P(n) = Q(n) + 1$. We consider the theorem given in Szpankowski, (Szp01) page 463), whose five conditions are met by choosing $\gamma_1(z) = 0$, $\gamma_2(z) = 1$, and $t(z) = -T(z)$. We can now deduce that

$$p_n = P(n) + O(n^{-1}) = Q(n) + 1 + O(n^{-1})$$

where

$$P(n) = Q(n) + 1 \sim 1 - \frac{T^*(0)}{\ln(1/q)} - \frac{2}{\ln(1/q)} \Re\left(\sum_{k=1}^{\infty} \exp\{\chi_k \ln(q/n)\}T^*(\chi_k)\right),$$

with $T^*(0)$ and $T^*(\chi_k)$ given in (1.1) and (1.2) respectively. This concludes the proof of Theorem 1.

3 The complementary set $\mathbb{N}\setminus A$ is finite

We consider now the complementary problem where the permitted set $B = \mathbb{N}\setminus A$ is finite. The de-Poissonisation method cannot be used in this case, but we can bound the probabilities by elementary means to show that $p_n \to 0$ for all finite sets $B$. (Consequently fluctuations are also absent in these cases.) Suppose the permitted set of multiplicities $B = \mathbb{N}\setminus A$ is finite with largest element $k$. The probability that all multiplicities belong to such a set $B$ is bounded above by the case when $B = \{0, 1, 2, ..., k\}$. Now samples where all multiplicities are at most $k$ are themselves a subset of the set of samples with the weaker restriction that there are at most $k$ ones.

The probability of exactly $j$ ones in a sample of length $n$ is $\binom{n}{j}p^j(1-p)^{n-j}$. Hence the probability of at most $k$ ones is

$$\sum_{j=0}^{k} \binom{n}{j}p^j(1-p)^{n-j} \leq (k+1)\binom{n}{k}(1-p)^{n-k} = O(n^k q^n),$$

which is exponentially small.

4 Further work

We aim to continue this work by considering other examples of forbidden sets $A$. In addition, we would like to reconsider the results of this paper from an urn model standpoint, as was used to obtain the results in [LP05].

Acknowledgements

We would like to thank Helmut Prodinger for the helpful advice he gave at the start of this project, relating to finding asymptotics from the recursion (1.1).

References


