Probe split graphs

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An undirected graph $G = (V, E)$ is a *probe split* graph if its vertex set can be partitioned into two sets, N (non-probes) and P (probes) where N is independent and there exists $E' \subseteq N \times N$ such that $G' = (V, E \cup E')$ is a split graph. Recently Chang et al. gave an $O(V^4(V + E))$ time recognition algorithm for probe split graphs. In this article we give $O(V^2 + VE)$ time recognition algorithms and characterisations by forbidden induced subgraphs both for the case when the partition into probes and non-probes is given, and when it is not given.

Keywords: probe graphs, probe split, probe interval, graph class

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In 1994, in the context of genome research, Zhang [15] introduced probe interval graphs. A graph is a probe interval graph if its vertex set can be partitioned into two sets, probes P and non-probes N, such that N is independent and new edges can be added between non-probes in such a way that the resulting graph is an interval graph. This definition can of course readily be generalized to some graph class $\mathcal{C}$: A graph is probe $\mathcal{C}$ if its vertex set can be partitioned into two sets, probes P and non-probes N, such that N is independent and new edges can be added between non-probes in such a way that the resulting graph is in $\mathcal{C}$.

In this manner probe interval [15, 10, 12, 11], probe chordal [5, 3], including the intersection of probe chordal with weakly chordal graphs [5], the intersection of trees with probe interval graphs [14], the intersection of 2-trees with probe interval graphs [13], and probe interval bigraphs [2] have been defined and investigated. Moreover, already in 1989 — before Zhang's article — Hertz [7] defined what he called *slim* graphs, which are in fact probe Meyniel graphs, and proved that these are perfect. Hoang and Maffray [8] used Hertz' construction to define probe Gallai graphs. In the recent article [3] several probe classes are discussed, among them probe split graphs.

Given $\mathcal{C}$, the new graph class of all probe $\mathcal{C}$ graphs clearly contains $\mathcal{C}$ as a subclass. And when $\mathcal{C}$ is a subclass of $\mathcal{D}$, then so is probe $\mathcal{C}$ a subclass of probe $\mathcal{D}$. The general questions for new graph classes,

1. the existence of an efficient recognition algorithm, and
2. a characterisation of the structure, possibly by forbidden induced subgraphs,

exist in the case of probe $\mathcal{C}$ graphs in two variants: one where the partition into probes and non-probes is given and a more general one where it is not given, resulting in four problems.

For the class of probe interval graphs, the first recognition algorithm for the partitioned case was given in [10], and one with $O(V + E \log V)$ running time in [11], while recently the unpartitioned case has been solved [1]. Probe chordal graphs (unpartitioned) can be recognized in $O(V^2E)$ time [3]. A characterisation by forbidden induced subgraphs is open for both classes. A characterisation by forbidden induced subgraphs is known for the intersection of trees with probe interval graphs [14], both partitioned and unpartitioned, but for 2-trees it already seems to be a difficult problem [13]. The forbidden induced subgraph characterisation for the intersection of trees with probe interval graphs leads to a polynomial recognition algorithm, both partitioned and unpartitioned. Probe interval bigraphs can be recognized in polynomial time [2], a characterisation by forbidden subgraphs is unknown. Concerning probe split graphs, in [3] a polynomial recognition algorithm is sketched that can be implemented to run in time $O(V^4(V+E))$, but the characterisation problem is not tackled.

From these results it is clear that neither recognizing nor characterising probe $\mathcal{C}$ graphs is a straightforward matter; in fact there is no positive characterisation result on any probe $\mathcal{C}$ class at all – the only positive result is on a subclass of probe interval graphs, the intersection of trees with probe interval graphs. In this article we will solve both the recognition and the characterisation problem for unpartitioned and partitioned probe split graphs.

All graphs considered are finite, undirected and simple. Given a graph H and a positive integer n, nH denotes the graph consisting of n disjoint copies of H. C_n, P_n and K_n denote the chordless cycle, chordless path, and complete graph, respectively, on n vertices. A chordless cycle on at least five vertices is called *hole*. Its complement is a *anti-hole*. An *odd* hole is a hole of odd length.

For two disjoint sets of vertices X and Y, $X \circledast Y$ ($X \circledcirc Y$) means that every vertex in X is adjacent (nonadjacent) to every vertex in Y. Occasionally, when such use is clear from context, we may use $\circledast, \circledcirc$ as graph-valued operators: For graphs X and Y, $X \circledcirc Y$ is the disjoint union of X and Y and $X \circledast Y$ results from $X \circledcirc Y$ by adding all possible edges with one endpoint in X and the other in Y. We often identify a subset of vertices with the subgraph induced by that subset, and vice versa. A set of vertices is called *independent* or *stable* if the vertices are pairwise non-adjacent and it is called a *clique* if they are pairwise adjacent. Adjacency of two vertices x, y in a graph $G = (V, E)$ is written $xy \in E$ or $x \sim y$. The *(open) neighbourhood* $N(v)$ of a vertex v is the set of its neighbours, the *closed neighbourhood* $N[v] = N(v) \cup \{v\}$ and the non-neighbourhood $\overline{N}(v) = V \setminus N[v]$. Given a set of vertices $X \subseteq V$, the subgraph induced by X is written $G[X]$. The complement of G is written $\overline{G}$.

Recall that a graph is a split graph [4, 6] iff it can be partitioned into a clique and an independent set, and that the split graphs are precisely the $(2K_2, C_4, C_5)$-free graphs.

Definition 1 A graph $G = (V, E)$ is a *probe split* graph if its vertex set V can be partitioned into two sets, N (non-probes) and P (probes) where N is independent and there exists $E' \subseteq N \times N$ such that $G' = (V, E \cup E')$ is a split graph.

As mentioned before, all split graphs are probe split. Bipartite graphs $G = (X \cup Y, E)$ are probe split as well: Take $N = X, P = Y$ and $E' = N \times N$.

Theorem 2 $G = (V, E)$ is a probe split graph iff its vertex set can be partitioned into three, possibly empty, sets C, S, I, such that C is a clique, S, I are independent sets and $C \circledast S$.

Proof: ($\Rightarrow$) Let $G = (V, E)$ be a probe split graph with probes P and non-probes N ($V = P \cup N$). Let $G' = (V, E \cup E')$, where $E' \subseteq N \times N$, be a split graph with clique K and independent set U. Now, let $C = K \cap P$, $S = K \cap N$ and $I = U$. In G clearly C is a clique, S and I are independent, and $C \circledast S$.

($\Leftarrow$) Let $G = (V, E)$ be a graph with a partition (C, S, I) of V into a clique C and independent sets S, I such that $C \circledast S$. Now, let $P = C \cup I$, $N = S$ and $E' = N \times N$, then obviously $G' = (V, E \cup E')$ is a split graph. Thus, G is probe split. □

From Th. 2 we see that probe split graphs are (1,2)–colourable and P_4–brittle (every P_4 has an endpoint in I) and therefore P_4–bipartite, perfectly orderable and perfect. (Definitions and references for all these classes can be found in ISGCI [9].)

Definition 3 A partition (C, S, I) of the vertex set of a graph G such that C is a clique, S and I are independent sets and $C \circledast S$ is called a *valid* probe split partition of G.

By Th. 2 a graph is probe split iff it has a valid probe split partition. This fact will be used often and without further reference.

1 Unpartitioned probe split graphs - An overview

As every induced subgraph of a probe split graph is again a probe split graph, probe split graphs can characterized by forbidden induced subgraphs. In this section we will outline the proof that a graph is a probe split graph iff it has no induced subgraph from Fig. 1 on page 210. This will also give rise to a polynomial recognition algorithm for unpartitioned probe split graphs. The long and technical proof follows in the next section.

Theorem 4 A graph is a probe split graph iff it has no induced subgraph isomorphic to a graph in Fig. 1.

Proof: ($\Rightarrow$) By Lem. 5.
($\Leftarrow$) The proof is given in this and the next section. □

Although the necessity of forbidding the graphs in Fig. 1 can easily be checked by brute force[(i)], the discussion in the next lemma will provide the reader with more insight in the structure of probe split graphs.

Lemma 5 The graphs in Fig. 1 are not probe split.

Proof: The key lies in the probe split partition from Th. 2. From this theorem we immediately get the following observations:

1. Every clique has at most one I-vertex, at most one S-vertex, the rest are C-vertices. In particular, every triangle contains at least one C-vertex.
2. Every C_4 consists either of alternating S and I vertices, or of two non-adjacent S-vertices, with the other two vertices being one C and one I-vertex.
3. Combining these two, every house has an I-vertex as the roof, with one neighbour a C-vertex. From this the other vertices follow.

[(i)] In fact, we verified by computer that the graphs in Fig. 1 are minimal non probe split graphs.

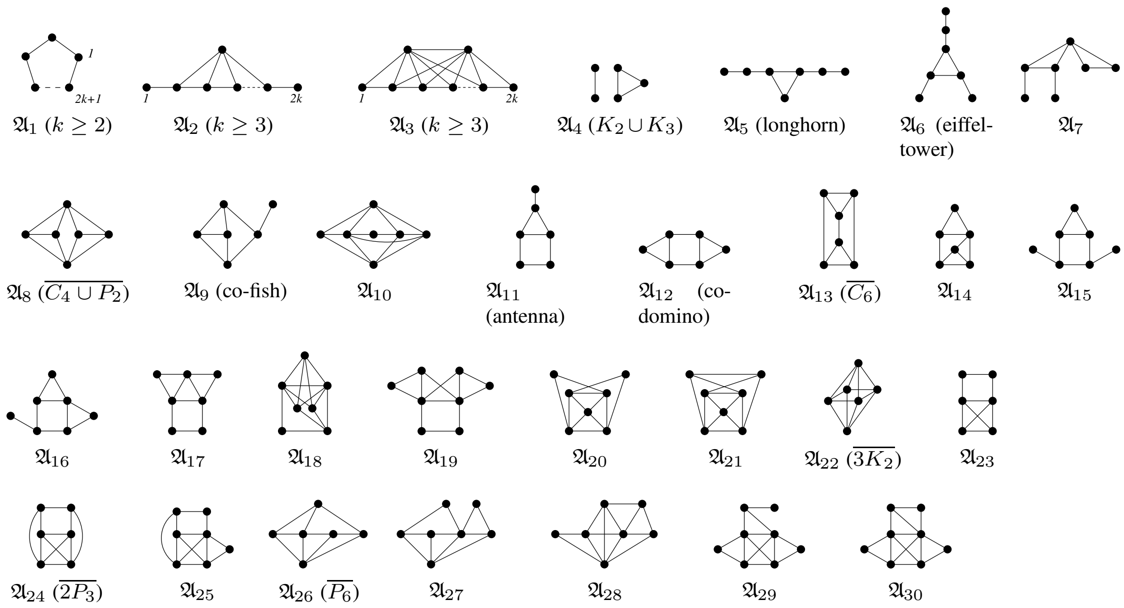


Fig. 1: Forbidden subgraphs for probe split graphs (continued on next page).

4. Every path on at least six vertices consists of alternating S and I vertices. If a path has even length, then precisely one of the end-vertices is an S-vertex.

5. Every P_5 either consists of alternating S and I-vertices, or the middle vertex is a C-vertex, its neighbours S- and the outer vertices I-vertices.

6. If the non-neighbourhood of a vertex contains an edge, then that vertex cannot be a C-vertex.

Obs. 1 gives $\mathfrak{A}_4$ ($K_2 \cup K_3$) as forbidden subgraph, and together with Obs. 4 $\mathfrak{A}_5$ (longhorn). If we take two triangles and unite one vertex of each triangle, we get a so-called butterfly. The vertex of degree four must be a C-vertex, while each wing consists of an S and an I-vertex. This gives $\mathfrak{A}_7$. Obs. 2 gives $\mathfrak{A}_8$ ($\overline{C_4 \cup P_2}$) to $\mathfrak{A}_{10}$, $\mathfrak{A}_{22}$ ($\overline{3K_2}$) and $\mathfrak{A}_{24}$ ($\overline{2P_3}$). Obs. 3 gives $\mathfrak{A}_{11}$ (antenna) to $\mathfrak{A}_{19}$, $\mathfrak{A}_{25}$ to $\mathfrak{A}_{28}$ and $\mathfrak{A}_{40}$. Obs. 4 gives $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_{35}$, and odd cycles of length at least 7. Additionally a C_5 is forbidden because it cannot consist solely of S and I vertices, but if it contains a C vertex, then its two non-neighbours in the C_5, which must be I-vertices, are adjacent to each other. This gives odd holes ($\mathfrak{A}_1$). Note that $\mathfrak{A}_{26}$ ($\overline{P_6}$) forbids odd anti-holes of lenght at least 7, so probe split graphs are (odd hole, odd anti-hole)-free. Obs. 5 gives $\mathfrak{A}_6$ (eiffeltower).

Now consider the graph formed by $\mathfrak{A}_{29}$ with the vertex of degree one deleted. Since every vertex of degree two has two different non-neighbours in the K_4, and the K_4 has at most one S and at most one I-vertex, it follows that the vertices of degree two must be I-vertices. This implies that the K_4 does not contain an I-vertex, and in particular that at least one of the lower two vertices of the K_4 must be a C-vertex. Hence every neighbour of the highest vertex of degree two must be adjacent to this C-vertex. This gives $\mathfrak{A}_{29}$, $\mathfrak{A}_{30}$, $\mathfrak{A}_{31}$.

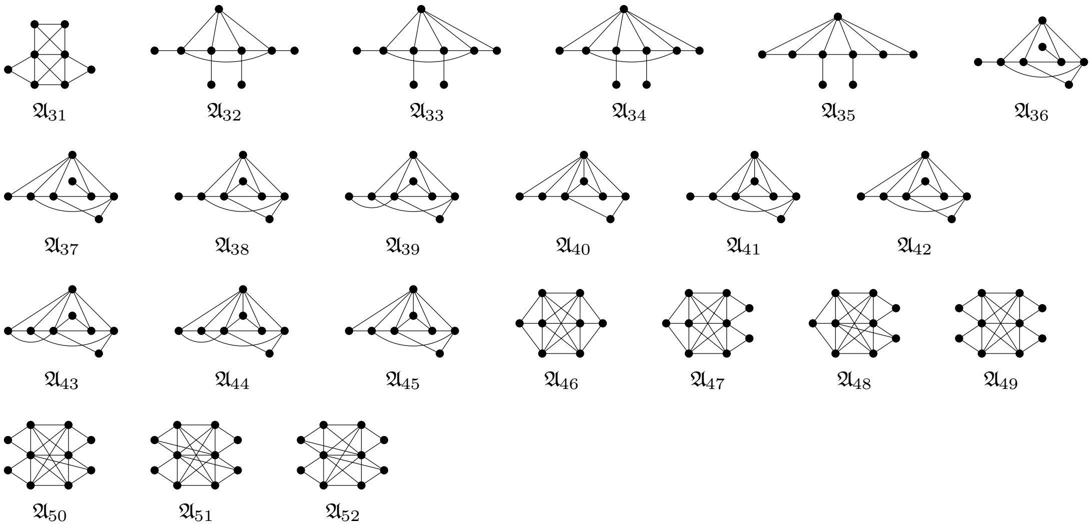


Fig. 1: Forbidden subgraphs for probe split graphs (continuation of previous page).

From Obs. 2 we have that a vertex v that is universal to a C_4 is a C-vertex. Vertices that are non-adjacent to v then must be I-vertices, and their neighbours in the C_4 are in S or C. This leads to $\mathfrak{A}_{20}$, $\mathfrak{A}_{21}$, $\mathfrak{A}_{32}$ to $\mathfrak{A}_{34}$, $\mathfrak{A}_{36}$ to $\mathfrak{A}_{39}$ and $\mathfrak{A}_{41}$ to $\mathfrak{A}_{45}$.

Finally to graphs $\mathfrak{A}_{46}$ to $\mathfrak{A}_{52}$: A vertex that is not part of the $P_3 \circledast P_3$ must be an I-vertex (Obs. 6 - it is easy to see that they cannot be S-vertices), and thus the P_3s do not contain any I-vertices. But every P_3 contains at least one non-C-vertex, i.c. an S-vertex. These S-vertices are adjacent. $\square$

The sufficiency of Th. 4 is proved inductively. Let G have no induced subgraphs isomorphic to a graph listed in Fig. 1. Set

$$B = \{v \in V(G) \mid \overline{N}(v) \text{ is not a stable set}\},$$
$$C = \{v \in V(G) \mid N(v) \text{ does not induce a split graph}\}.$$

Proposition 6 Assuming $G = (V, E)$ does not contain any graph listed in Fig. 1 as an induced subgraph, then

a) $B \cap C = \emptyset$;

b) C is a clique;

c) $G[B]$ is bipartite.

Proposition 7 Assuming $G = (V, E)$ does not contain any graph listed in Fig. 1 as an induced subgraph, then for every nontrivial (i.e. with at least two vertices) connected component $B' = (X \cup Y, E')$ of $G[B]$, $X \circledast C$ or $Y \circledast C$.

The basis of the induction is the case that $V = B \cup C$. In this case, the Prop. 7 states how to define a valid probe split partition $(C, S, B \setminus S)$ of G: For every non-trivial component $B' = (X \cup Y, E')$ of B, let $S(B') \in \{X, Y\}$ with $S(B') \circledast C$ and define $S = \bigcup_{B'} S(B')$.

Algorithm 1 Give a valid partition of $G = (V, E)$ iff G is a probe split graph.

Compute the subsets B and C
if $B \cap C \neq \emptyset$ or $G[B]$ is not bipartite or $G[C]$ is not a clique or there is a non-trivial connected component $B' = (X \cup Y, E')$ of $G[B]$ such that neither $X \circledast C$ nor $Y \circledast C$ **then**
FAIL
end if
if $V = B \cup C$ **then**
For each non-trivial connected component $B' = (X \cup Y, E')$ of $G[B]$, let $S(B') \in \{X, Y\}$ with $S(B') \circledast C$ (in case both $X \circledast C$ and $Y \circledast C$, choose $S(B') \in \{X, Y\}$ arbitrarily)
$S := \bigcup_{B'} S(B')$; $I := B \setminus S$
SUCCESS: return (C, S, I)
end if
if $V \neq B \cup C$ **then**
Choose an arbitrary vertex $v \in V \setminus (B \cup C)$
Recursively compute a valid partition (C', S', I') for $G - v$
Compute a valid partition (C, S, I) for G from (C', S', I')
SUCCESS: return (C, S, I)
end if

The induction step occurs when $V \neq B \cup C$. According to Prop. 8, we can remove a vertex $v \in V \setminus (B \cup C)$ and inductively build a probe split partition on a valid partition of $G - v$:

Proposition 8 Assuming $G = (V, E)$ does not contain any graph listed in Fig. 1 as an induced subgraph, then for every vertex $v \in V \setminus (B \cup C)$: If $G - v$ is a probe split graph, then so is G. Moreover, a valid partition of G can be obtained from a valid partition of $G - v$ in linear time.

Using Prop. 6, 7 and 8 we can give Alg. 1 to recognize probe split graphs.

Theorem 9 Algorithm 1 correctly recognizes probe split graphs, and can be implemented with running time $O(V^2 + VE)$.

Proof: Correctness follows from Lem. 5, Prop. 6, Prop. 7 and Prop. 8. It is obvious that calculating B and C and verifying Prop. 6 and Prop. 7 (first if-statement) can be done in the desired timebound. The next two if-statement form a recursive algorithm. Note that in the recursion we do not need to calculate B, C, nor to verify Prop. 6 and Prop. 7. Let $n = |V \setminus (B \cup C)|$, then the basis of the recursion (second if-statement) can be executed in time $T(0) = O(V + E)$, and the third if-statement takes $T(n) = T(n-1) + O(V + E)$ by Prop. 8. Solving the recurrence gives $O(V^2 + VE)$. □

2 Unpartitioned probe split graphs - The proof of Prop. 6, 7 and 8

In this section we prove Prop. 6, Prop. 7 and Prop. 8. We assume that $G = (V, E)$ has no induced subgraph from Fig. 1 and that $B, C \subseteq V$ are as defined in the previous section.

2.1 The proof of Proposition 6

Lemma 10 (Prop. 6.a) $B \cap C = \emptyset$.

Proof: Assume to the contrary that there exists a vertex $v \in B \cap C$. As $v \in B$, there exist adjacent vertices $x, y \in \overline{N}(v)$. As $v \in C$ and G is C_5-free, there exist vertices $v_1, v_2, v_4, v_4 \in N(v)$ such that $\{v_1, v_2, v_3, v_4\}$ induces a $2K_2$ with edges v_1v_2 and v_3v_4, or the $C_4 = (v_1, v_2, v_3, v_4, v_1)$.

We consider the first case. In this case, each of x, y is adjacent to at most two of the vertices v_1, v_2, v_3, v_4 otherwise x or y together with v_1, v_2, v_3, v_4 and v would induce a $\mathfrak{A}_{14}$ or $\mathfrak{A}_8$ ($\overline{C_4 \cup P_2}$). Moreover, x (respectively, y) cannot be adjacent to both v_1, v_2, or to both v_3, v_4. For, if x is adjacent to both v_1, v_2 then x, y, v_3, v_4 and v would induce a $\mathfrak{A}_4$ ($K_2 \cup K_3$) (if $y \not\sim v_3, v_4$), or y, x, v_1, v and v_3 or v_4 would induce a C_5 (if $y \sim v_3$ or $y \sim v_4$). So, let without loss of generality, x be nonadjacent to v_1 and v_3.

Now, if y is nonadjacent to all v_i, $i = 1, 2, 3, 4$, then $x, y, v, v_1, v_2, v_3, v_4$ induce a $\mathfrak{A}_{16}$ (if $x \sim v_2, v_4$), or $\{x, y, v, v_1, v_2\}$ or $\{x, y, v, v_3, v_4\}$ induce a $\mathfrak{A}_4$ ($K_2 \cup K_3$) (otherwise).

If y is adjacent to exactly one of the v_i, then we may assume without loss of generality that y is nonadjacent to v_1, v_2. Then x must be adjacent to v_2 otherwise x, y, v, v_1, v_2 would induce a $\mathfrak{A}_4$ ($K_2 \cup K_3$). Hence y and v_3 are nonadjacent otherwise x, y, v, v_2, v_3 would induce a C_5. It follows that y is adjacent to v_4, and x, y, v, v_2, v_4 induce a C_5 (if $x \not\sim v_4$), or x, y, v, v_1, v_2, v_4 induce a $\mathfrak{A}_{12}$ (co-domino).

Finally, let y be adjacent to (exactly) two of the v_i. If y is adjacent to v_1 and v_3 then x, y, v_1, v_2, v_3, v_4 and v induce a $\mathfrak{A}_{16}$ (if $x \not\sim v_2, v_4$), or x, y, v, v_1, v_4 or x, y, v, v_2, v_3 induce a C_5 (if $x \sim v_4$ or $x \sim v_2$). If y is adjacent to v_1 and v_4 then $x, y, v, v_1, v_2, v_3, v_4$ induce a $\mathfrak{A}_{16}$ (if $x \not\sim v_2, v_4$), or x, y, v, v_2, v_4 induce a C_5 (if $x \sim v_2, x \not\sim v_4$), or $x, y, v, v_1, v_2, v_3, v_4$ induce a $\mathfrak{A}_{12}$ (co-domino) (if $x \not\sim v_2, x \sim v_4$) or x, y, v, v_1, v_2, v_4 induce a $\mathfrak{A}_{13}$ ($\overline{C_6}$). The first case is settled.

We consider the case of the $C_4 = (v_1, v_2, v_3, v_4, v_1)$. In this case, each of x, y is adjacent to at least two of the vertices v_1, v_2, v_3, v_4 otherwise there would be a $\mathfrak{A}_4$ ($K_2 \cup K_3$) or a C_5, but to at most three of the v_i otherwise there would be a $\mathfrak{A}_{22}$ ($\overline{3K_2}$).

Let x be adjacent to v_1, v_2, v_3. If y is also adjacent to v_1, v_2, v_3 then x, y, v, v_1, v_3, v_4 induce a $\mathfrak{A}_8$ ($\overline{C_4 \cup P_2}$). If y is adjacent to v_1, v_2, v_4 or to v_2, v_3, v_4 then x, y, v, v_1, v_3, v_4 induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$). If y is adjacent to v_1, v_3, v_4 then x, y, v, v_2, v_4 induce a C_5. If y is adjacent to (exactly) v_1, v_2, or to v_2, v_3 then x, y, v, v_1, v_3, v_4 induce a $\mathfrak{A}_{14}$. If y is adjacent to (exactly) v_2, v_4 then x, y, v, v_1, v_2, v_4 induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$). If y is adjacent to v_1, v_4 (or to v_3, v_4) then x, y, v, v_2, v_4 induce a C_5. Thus, x, and by symmetry, y must be adjacent to exactly two of the v_i.

We first consider the case that at least one of x, y is adjacent to (exactly) two non-neighbours in $\{v_1, \ldots, v_4\}$. Without loss of generality, assume x is adjacent to v_1, v_3. If y is adjacent to v_2, v_4 then there is a $\mathfrak{A}_{21}$ and if y is adjacent to v_1, v_3 then x, y, v, v_1, v_3, v_4 induce a $\mathfrak{A}_8$ ($\overline{C_4 \cup P_2}$). If y is adjacent to v_1, v_2 or to v_2, v_3, then x, y, v, v_1, v_3, v_4 induce a $\mathfrak{A}_{14}$ and if y is adjacent to v_3, v_4 or to v_4, v_1, then x, y, v, v_1, v_2, v_3 induce a $\mathfrak{A}_{14}$.

Finally, consider the case that each of x, y is adjacent to two neighbours in $\{v_1, \ldots, v_4\}$. Without loss of generality, let x be adjacent to v_1, v_2. If y is adjacent to v_1, v_4 or to v_2, v_3, then x, y, v_2, v_3, v_4 or x, y, v_1, v_3, v_4, respectively, induce a C_5, and if y is adjacent to v_3, v_4, then x, y, v_1, v_2, v_3, v_4 induce a $\mathfrak{A}_{13}$ ($\overline{C_6}$). $\square$

Lemma 11 (Prop. 6.b) C is a clique.

Proof: Assume to the contrary that there exist two nonadjacent vertices $c_1 \neq c_2$ in C. By Prop. 6.a, $\overline{N}(c_1)$ and $\overline{N}(c_2)$ are independent. This implies that every vertex that is (not) adjacent to c_1 is also (not) adjacent to c_2 and vice versa. Thus, $N(c_1) = N(c_2)$. Now, as $c_1 \in C$ and because G is C_5-free, $G[N(c_1)]$ contains an induced $2K_2$ or C_4. But then $G[N(c_1) \cup \{c_2\}]$ contains a $\mathfrak{A}_8$ ($\overline{C_4 \cup P_2}$) or a $\mathfrak{A}_{22}$ ($\overline{3K_2}$). □

Lemma 12 (Prop. 6.c) $G[B]$ is bipartite.

Proof: Assume to the contrary that B does not induce a bipartite graph. As G is odd hole-free, B contains a triangle $v_1v_2v_3$. As $v_i \in B$, there exist adjacent vertices x_i, y_i in $\overline{N}(v_i)$, $i = 1, 2, 3$. In the following claims, i, j, k are arbitrarily chosen with $\{i, j, k\} = \{1, 2, 3\}$.

CLAIM 1 $N(v_i) \cap \{x_j, y_j, x_k, y_k\} \neq \emptyset$.

PROOF: Assume to the contrary that $N(v_i) \cap \{x_j, y_j, x_k, y_k\} = \emptyset$. As G has no induced $\mathfrak{A}_4$ ($K_2 \cup K_3$), $v_j \sim x_k$ or $v_j \sim y_k$, but not both otherwise v_i, v_k, x_k, y_k would induce a $2K_2$ in $N(v_j)$ and v_j would belong to C, contradicting Prop. 6.a. Likewise, v_k is adjacent to exactly one of x_j, y_j.

Let, without loss of generality, $v_j \sim x_k$ and $v_k \sim x_j$. Then $x_j \notin \{x_k, y_k\}$ and $x_k \notin \{x_j, y_j\}$. Moreover, $y_j \neq y_k$ otherwise v_j, v_k, x_j, y_j, x_k would induce a C_5 (if $x_j \not\sim x_k$) or $v_i, v_j, v_k, x_j, y_j, x_k$ would induce a $\mathfrak{A}_{12}$ (co-domino) (if $x_j \sim x_k$). Hence $y_j \not\sim y_k$ otherwise y_j, y_k, v_i, v_j, v_k would induce a $\mathfrak{A}_4$ ($K_2 \cup K_3$), and $y_j \not\sim x_k$, $y_k \not\sim x_j$ otherwise there would be a C_5 or a $\mathfrak{A}_{12}$ (co-domino). Now, $v_i, v_j, v_k, x_j, y_j, x_k, y_k$ induce a $\mathfrak{A}_{15}$ (if $x_j \sim x_k$) or a $\mathfrak{A}_5$ (longhorn) (otherwise). □

CLAIM 2 x_i, y_i, v_j, v_k do not induce a C_4.

PROOF: Assume to the contrary that x_i, y_i, v_j, v_k do induce a C_4, say $v_j \sim x_i$, $v_k \sim y_i$ and $v_j \not\sim y_i$, $v_k \not\sim x_i$. We first prove some observations.

$$N(v_i) \setminus N(v_j) \subseteq N(v_k) \setminus N(y_i). \tag{1}$$

Proof of (1): Let $v \in N(v_i) \setminus N(v_j)$. First assume $v \not\sim v_k$. Then $v \sim y_i$ or $v \sim x_i$ (otherwise there would be a $\mathfrak{A}_{11}$ (antenna)), but not both (otherwise there would be a $\mathfrak{A}_{13}$ ($\overline{C_6}$)). Now, v_i, x_i, y_i, v and v_j (if $v \sim y_i$) or v_k (if $v \sim x_i$) induce a C_5. Thus, $v \sim v_k$. Next, assume $v \sim y_i$. Then $v \sim x_i$, too (otherwise there would be a C_5). Hence $v_i, v_j, v_k, x_i, y_i, v$ induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$). Thus, $v \not\sim y_i$, and (1) follows.

By symmetry, we have:

$$N(v_i) \setminus N(v_k) \subseteq N(v_j) \setminus N(x_i). \tag{2}$$

We next show:

$$N(v_i) \setminus N(v_j) \neq \emptyset \Rightarrow \overline{N}(v_k) \subseteq \overline{N}(v_i). \tag{3}$$

Proof of (3): Let $v \in N(v_i) \setminus N(v_j)$. By (1), $v \sim v_k$, $v \not\sim y_i$. Consider a vertex $u \in \overline{N}(v_k)$, and assume $u \sim v_i$. Then, by (2), $u \sim v_j$, $u \not\sim x_i$. Moreover, $v \not\sim u$ otherwise v_i would belong to C, contradicting Prop. 6.a.

Now, if $u \not\sim y_i$ and $v \not\sim x_i$ then $v_i, v_j, v_k, x_i, y_i, u, v$ induce a $\mathfrak{A}_{17}$. If $u \sim y_i$ and $v \sim x_i$ then v_i, x_i, y_i, u, v induce a C_5. If $u \sim y_i$ and $v \not\sim x_i$ (or $u \not\sim y_i$ and $v \sim x_i$) then v_i, x_i, y_i, u, v and v_j (respectively, v_k) induce a $\mathfrak{A}_{11}$ (antenna). Thus, $u \not\sim v_i$, and (3) follows.

By symmetry, we have:

$$N(v_i) \setminus N(v_k) \neq \emptyset \Rightarrow \overline{N}(v_j) \subseteq \overline{N}(v_i). \tag{4}$$

Now, by Claim 1 and symmetry, v_i is adjacent to a vertex $v \in \{x_j, y_j\}$. By (1), $v \sim v_k$, $v \not\sim y_i$. In particular, $v \notin \{x_k, y_k\}$. By (3), x_k and y_k are nonadjacent to v_i, hence $v_j \sim x_k$ or $v_j \sim y_k$ otherwise there would be a $\mathfrak{A}_4$ ($K_2 \cup K_3$), but not both otherwise v_j would belong to C, contradicting Prop. 6.a.

Let, without loss of generality, $v_j \sim x_k$. Then $v \sim x_k$ otherwise v_i, v_k, v, x_k, y_k would induce a $\mathfrak{A}_4$ ($K_2 \cup K_3$) (if $v \not\sim y_k$) or v_i, v, v_j, x_k, y_k would induce a C_5 (if $v \sim y_k$). Now, $v_i, v_j, v_k, v, x_k, y_k$ induce a $\mathfrak{A}_9$ (co-fish) (if $v \not\sim y_k$) or a $\mathfrak{A}_{14}$ (if $v \sim y_k$). This final contradiction proves Claim 2. □

CLAIM 3 If $x_j, y_j \in N(v_i)$ then $N(v_k) \cap \{x_j, y_j\} \neq \emptyset$, $|N(v_k) \cap \{x_i, y_i\}| = 1$, and $N(v_j) \cap \{x_i, y_i\} = \emptyset$.

PROOF: Note first that x_i, y_i, x_j, y_j are pairwise distinct, since $x_j, y_j \in N(v_i)$. From $v_i \in B$ and Prop. 6.a it follows that $N(v_k) \cap \{x_j, y_j\} \neq \emptyset$. Let $v_k \sim x_j$, say.

Suppose that $|N(v_k) \cap \{x_i, y_i\}| \neq 1$. If $x_i \sim v_k$ and $y_i \sim v_k$ then $G[N(v_k)]$ contains an induced C_5, C_4, or $2K_2$, hence $v_k \in C$, contradicting Prop. 6.a. Thus, $N(v_k) \cap \{x_i, y_i\} = \emptyset$. Then $v_j \sim x_i$ or $v_j \sim y_i$ (otherwise v_i, v_j, v_k, x_i, y_i would induce a $\mathfrak{A}_4$ ($K_2 \cup K_3$)), but not both (otherwise v_j would belong to C, contradicting Prop. 6.a). Let $v_j \sim x_i$, say. Also, $x_j \sim x_i$ or $x_j \sim y_i$ (otherwise x_j, v_i, v_k, x_i, y_i would induce a $\mathfrak{A}_4$ ($K_2 \cup K_3$)), but not both (otherwise $v_i, v_j, v_k, x_i, y_i, x_j$ would induce a $\mathfrak{A}_{14}$). Now, if $x_j \sim x_i$ then $v_i, v_j, v_k, x_i, y_i, x_j$ induce a $\mathfrak{A}_9$ (co-fish), and if $x_j \sim y_i$ then v_i, v_j, x_i, y_i, x_j induce a C_5. It follows that $|N(v_k) \cap \{x_i, y_i\}| = 1$.

Let, without loss of generality, $x_i \sim v_k$. Next we suppose that $N(v_j) \cap \{x_i, y_i\} \neq \emptyset$. If $x_i \sim v_j$ then $x_i \not\sim x_j$ (otherwise v_k would belong to C, contradicting Prop. 6.a), and $y_i \not\sim x_j$ (otherwise $v_i, v_j, v_k, x_i, y_i, x_j$ would induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$) (if $y_i \sim v_j$) or x_i, y_i, x_j, v_i, v_j induce a C_5 (if $y_i \not\sim v_j$)). Now, if $y_j \not\sim x_i$ and $y_j \not\sim y_i$ then v_i, x_i, y_i, x_j, y_j induce a $\mathfrak{A}_4$ ($K_2 \cup K_3$). If $y_j \sim x_i$ and $y_j \sim y_i$ then $v_i, v_k, x_i, y_i, x_j, y_j$ induce a $\mathfrak{A}_{14}$. If $y_j \sim x_i$ and $y_j \not\sim y_i$ then $v_i, v_k, x_i, y_i, x_j, y_j$ induce a $\mathfrak{A}_9$ (co-fish), and finally, if $y_j \not\sim x_i$ and $y_j \sim y_i$ then x_i, y_i, x_j, y_j, v_k induce a C_5. Thus, $x_i \not\sim v_j$. By Claim 2, therefore, $y_i \not\sim v_j$, and it follows that $N(v_j) \cap \{x_i, y_i\} = \emptyset$. □

CLAIM 4 $|N(v_i) \cap \{x_j, y_j\}| \leq 1$

PROOF: Assume to the contrary that $x_j \sim v_i$ and $y_j \sim v_i$. By Claim 3, $N(v_j) \cap \{x_i, y_i\} = \emptyset$ and we may assume that $v_k \sim x_j$, $v_k \sim x_i$, and $v_k \not\sim y_i$.

By Claim 1, $N(v_j) \cap \{x_k, y_k\} \neq \emptyset$. If $x_k, y_k \in N(v_j)$ then, by Claim 3, $N(v_k) \cap \{x_j, y_j\} = \emptyset$, a contradiction. Thus, $|N(v_j) \cap \{x_k, y_k\}| = 1$, say $v_j \sim x_k$ and $v_j \not\sim y_k$. Moreover, $|N(v_i) \cap \{x_k, y_k\}| \leq 1$, otherwise, by Claim 3 again, $N(v_k) \cap \{x_i, y_i\} = \emptyset$, a contradiction.

Suppose $|N(v_i) \cap \{x_k, y_k\}| = 1$. Then by Claim 2, $v_i \sim x_k$. In particular, $x_k \notin \{x_i, y_i\}$, and $x_k \sim x_i$ or $x_k \sim y_i$ otherwise v_i, v_j, x_k, x_i, y_i would induce a $\mathfrak{A}_4$ ($K_2 \cup K_3$), but not both otherwise $v_i, v_j, v_k, x_k, x_i, y_i$ would induce a $\mathfrak{A}_{14}$. Now, x_k, y_i, x_i, v_k, v_i induce a C_5 (if $x_k \not\sim x_i$) or $v_i, v_j, v_k, x_k, x_i, y_i$ induce a $\mathfrak{A}_9$ (co-fish) (if $x_k \sim x_i$).

Thus, $N(v_i) \cap \{x_k, y_k\} = \emptyset$. Then $x_j \notin \{x_k, y_k\}$, and the same arguments above applied for $v_i, v_j, v_k, x_j, x_k, y_k$ show again that there exists a $\mathfrak{A}_4$ ($K_2 \cup K_3$), or a $\mathfrak{A}_{14}$, or a C_5, or a $\mathfrak{A}_9$ (co-fish). Claim 4 is proved. □

Now, as G has no induced $\mathfrak{A}_4$ ($K_2 \cup K_3$), and because of Claims 2 and 4, we have the following two cases with a suitable choice of i, j, k.

Case 1: $N(v_j) \cap \{x_i, y_i\} = N(v_k) \cap \{x_i, y_i\}$ *and* $N(v_k) \cap \{x_j, y_j\} = N(v_i) \cap \{x_j, y_j\}$.
Then we may assume that $N(v_j) \cap \{x_i, y_i\} = N(v_k) \cap \{x_i, y_i\} = \{x_i\}$ and $N(v_k) \cap \{x_j, y_j\} = N(v_i) \cap \{x_j, y_j\} = \{x_j\}$. It follows that $x_i \notin \{x_j, y_j\}$, $x_j \notin \{x_i, y_i\}$. As $v_k \in B$ and because of Prop. 6.a, $x_i \not\sim x_j$. Moreover, $x_i \not\sim y_j$ (otherwise x_i, y_j, x_j, v_i, v_j would induce a C_5), $x_j \not\sim y_i$ (otherwise x_j, y_i, x_i, v_j, v_i would induce a C_5). In particular, $y_i \neq y_j$.

Now, if $y_i \sim y_j$ then y_i, y_j, x_j, v_k, x_i induce a C_5. If $y_i \not\sim y_j$ then $v_i, v_j, v_k, x_i, y_i, x_j, y_j$ induce a $\mathfrak{A}_2$. Case 1 is settled.

Case 2: $N(v_j) \cap \{x_i, y_i\} = \emptyset$ *and* $N(v_k) \cap \{x_j, y_j\} = \emptyset$.
Then $|N(v_k) \cap \{x_i, y_i\}| = 1$, say $v_k \sim x_i$, and $|N(v_i) \cap \{x_j, y_j\}| = 1$, say $v_i \sim x_j$. It follows that $x_i \notin \{x_j, y_j\}$, $x_j \notin \{x_i, y_i\}$.

Suppose $x_i \sim y_j$. Then x_i, y_j, x_j, v_i, v_k induce a C_5 (if $x_i \not\sim x_j$) or $x_i, x_j, y_j, v_i, v_j, v_k$ induce a $\mathfrak{A}_{12}$ (co-domino) (if $x_i \sim x_j$). Thus, $x_i \not\sim y_j$. Likewise, $x_j \not\sim y_i$. In particular, $y_i \neq y_j$, and $y_i \not\sim y_j$ otherwise y_i, y_j, v_i, v_j, v_k would induce a $\mathfrak{A}_4$ ($K_2 \cup K_3$).

Now, $v_i, v_j, v_k, x_i, y_i, x_j, y_j$ induce a $\mathfrak{A}_5$ (longhorn) (if $x_i \not\sim x_j$) or a $\mathfrak{A}_{15}$ (otherwise). Case 2 is settled, completing the proof of Prop. 6.c. □

2.2 The proof of Proposition 7

Before we prove Prop. 7, we need some auxiliary results. The next three lemmas deal with the situation when

$$\text{for every } c \in C\text{: } c \circledast X \text{ or } c \circledast Y. \tag{5}$$

Let $c_1 \neq c_2 \in C$ such that

$$\begin{aligned} &c_1 \circledast X, \quad c_1 \not\circledast Y, \\ &c_2 \circledast Y, \quad c_2 \not\circledast X. \end{aligned}$$

Lemma 13 Assume $C \not\circledast X$ and $C \not\circledast Y$. Let $u \in X$ and $v \in Y$ such that $c_2 \not\sim u$ and $c_1 \not\sim v$. If u and v are adjacent, then, for all adjacent vertices $x \in X \setminus \{u\}$ and $y \in Y \setminus \{v\}$, $c_1 \not\sim y$ or $c_2 \not\sim x$.

Proof: For contradiction, assume there are adjacent vertices $x \in X \setminus \{u\}$, $y \in Y \setminus \{v\}$ such that $c_1 \sim y$ and $c_2 \sim x$. Then $x \sim v$ or $y \sim u$, but not both (otherwise c_1, c_2, u, v, x, y would induce a $\mathfrak{A}_{23}$ or a $\mathfrak{A}_{24}$ ($\overline{2P_3}$)). By symmetry, we may assume

$$x \sim v \quad \text{and} \quad y \not\sim u,$$

CLAIM 1 $x \circledast C$
PROOF: Assume $x \not\sim c$ for some $c \in C \setminus \{c_1, c_2\}$. By (5), $c \circledast Y$. Then u, v, c, c_1, y, x induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$) (if $c \sim u$), or u, v, c, c_1, c_2, x, y induce a $\mathfrak{A}_{18}$ (if $c \not\sim u$). □

As $x \in B$, there exist adjacent vertices $x', y' \in \overline{N}(x)$. As $x \circledast C$, we may assume that $x' \in X \setminus \{x\}$, $y' \in Y \setminus \{v, y\}$.

Suppose $y' \sim u$. Then x, y, c_1, c_2, u, y' induce a $\mathfrak{A}_{23}$ (if $y' \not\sim c_1$), or u, v, c_1, c_2, x, y' induce a $\mathfrak{A}_{14}$ (if $y' \sim c_1$). Thus,

$$y' \not\sim u.$$

In particular, $x' \neq u$.

Suppose $x' \sim y$. Then $x' \sim c_2$, otherwise x, x', y, c_1, c_2, v would induce a $\mathfrak{A}_{24}$ ($\overline{2P_3}$) (if $x' \sim v$), or x, x', y, c_1, c_2, u, v would induce a $\mathfrak{A}_{25}$ (if $x' \not\sim v$). Now, x', y, c_1, c_2, u, v induce a $\mathfrak{A}_{23}$ (if $x' \not\sim v$), or x, x', y, c_1, c_2, u, v induce a $\mathfrak{A}_{18}$ (if $x' \sim v$). Thus,

$$x' \not\sim y.$$

Therefore, $x' \sim c_2$ or $y' \sim c_1$ (otherwise x, y, c_1, c_2, x', y' would induce a $\mathfrak{A}_{23}$). Now, if $x' \sim c_2$ and $y' \sim c_1$ then x', y', c_1, c_2, u, v induce a $\mathfrak{A}_{23}$ (if $x' \not\sim v$), or $x, y, x', y', c_1, c_2, u, v$ induce a $\mathfrak{A}_{44}$ (if $x' \sim v$). If $x' \sim c_2$ and $y' \not\sim c_1$ then $x, x', y, y', c_1, c_2, u, v$ induce a $\mathfrak{A}_{40}$ (if $x' \not\sim v$), or a $\mathfrak{A}_{45}$ (if $x' \sim v$). If $x' \not\sim c_2$ and $y' \sim c_1$ then x, x', y', c_1, c_2, v induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$) (if $x' \sim v$), or $x, x', y', c_1, c_2, u, v$ induce a $\mathfrak{A}_{27}$ (if $x' \not\sim v$). □

Lemma 14 Assume $C \not\circledast X$ and $C \not\circledast Y$. Let $v_1v_2v_3v_4$, be an induced path in B connecting $v_1 \in X$ and $v_4 \in Y$. If $c_1 \not\sim v_4$ and $c_2 \not\sim v_1$ then $c_1 \not\sim v_2$ or $c_2 \not\sim v_3$.

Proof: For a contradiction, assume that $c_1 \sim v_2$ and $c_2 \sim v_3$. We will show that $v_2 \circledast C$ and $v_3 \circledast C$. As a first step we prove:

CLAIM 1 $v_2 \circledast C$ or $v_3 \circledast C$

PROOF: Suppose $v_2 \not\sim c_1'$ and $v_3 \not\sim c_2'$ for some $c_1', c_2' \in C \setminus \{c_1, c_2\}$. By (5), $c_1' \circledast X$ and $c_2' \circledast Y$. In particular, $c_1' \neq c_2'$. Moreover, $c_1' \sim v_4$ or $c_2' \sim v_1$ (otherwise $c_1, c_2, c_1', c_2', v_1, v_2, v_3, v_4$ would induce a $\mathfrak{A}_{46}$), but not both (otherwise $c_1', c_2', v_1, v_2, v_3, v_4$ would induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$)). Now $v_1, v_2, v_3, v_4, c_1', c_2'$, and c_1 (if $c_1' \not\sim v_4$) or c_2 (if $c_2' \not\sim v_1$) would induce a $\mathfrak{A}_{18}$. □

Let, without loss of generality,

$$v_2 \circledast C.$$

As $v_2 \in B$, there exist adjacent vertices $x, y \in \overline{N}(v_2)$. As $v_2 \circledast C$, we may assume that $x \in X \setminus \{v_1, v_3\}$, $y \in Y \setminus \{v_2\}$. Moreover,

$$x \not\sim v_4, \tag{6}$$

otherwise $v_1, v_2, v_3, v_4, c_1, c_2$ and x would induce a $\mathfrak{A}_{25}$ (if $c_2 \not\sim x$) or a $\mathfrak{A}_{28}$ (if $c_2 \sim x$). In particular, $y \neq v_4$.

Suppose $y \not\sim v_3$. Then $y \not\sim v_1$ otherwise $v_1, v_2, v_3, v_4, c_1, c_2$ and y would induce a $\mathfrak{A}_{25}$ (if $c_1 \not\sim y$) or a $\mathfrak{A}_{28}$ (if $c_1 \sim y$). Moreover, $c_1 \sim y$ or $c_2 \sim x$ otherwise c_1, c_2, x, y, v_2, v_3 would induce a $\mathfrak{A}_{23}$. But now $c_1, c_2, x, y, v_1, v_2, v_3, v_4$ induce a $\mathfrak{A}_{31}$ (if $c_1 \sim y$ and $c_2 \sim x$) or a $\mathfrak{A}_{30}$ (if $c_1 \not\sim y$ or $c_2 \not\sim x$). Thus,

$$y \sim v_3. \tag{7}$$

Hence also

$$y \sim c_1, \tag{8}$$

otherwise $v_1, v_2, v_3, c_1, c_2, y$ would induce a $\mathfrak{A}_{24}$ ($\overline{2P_3}$) (if $y \sim v_1$), or $v_1, v_2, v_3, c_1, c_2, x, y$ would induce a $\mathfrak{A}_{25}$ (if $y \not\sim v_1, x \not\sim c_2$) or a $\mathfrak{A}_{28}$ (if $y \not\sim v_1, x \sim c_2$).

We next show:

CLAIM 2 $v_3 \circledast C$

PROOF: Suppose $v_3 \not\sim c_2'$ for some $c_2' \in C \setminus \{c_1, c_2\}$. By (5), $c_2' \circledast Y$. By Lem. 13, $c_2' \not\sim v_1$ ($c_2 := c_2', u := v_3, v := v_4, x := v_1, y := v_2$) and $c_2' \not\sim x$ ($c_2 := c_2', u := v_3, v := v_4, x := x, y := y$). Moreover, $y \not\sim v_1$ otherwise $v_1, v_2, v_3, v_4, c_1, c_2, c_2'$ and y would induce a $\mathfrak{A}_{46}$. Now $v_1, v_2, v_3, v_4, c_1, c_2, c_2', x$ and y induce a $\mathfrak{A}_{47}$ (if $c_2 \not\sim x$) or a $\mathfrak{A}_{48}$ (if $c_2 \sim x$). Claim 2 follows. □

As $v_3 \in B$, there exist adjacent vertices $x', y' \in \overline{N}(v_3)$. As $v_3 \circledast C$, we may assume that $y' \in Y \setminus \{y, v_2, v_4\}$ and $x' \in X \setminus \{v_3\}$. Now, by symmetry, we obtain from (6), (7), and (8):

$$y' \not\sim v_1, x' \sim v_2, \quad \text{and} \quad x' \sim c_2.$$

In particular, $x' \in X \setminus \{x, v_1, v_3\}$. Moreover,

$$x \not\sim y',$$

otherwise $c_1, c_2, v_2, v_3, x, y'$ would induce a $\mathfrak{A}_{23}$ (if $x \not\sim c_2$ and $y' \not\sim c_1$), or $c_1, c_2, v_1, v_2, v_3, v_4, x, y'$ would induce a $\mathfrak{A}_{31}$ (if $x \sim c_2$ and $y' \sim c_1$), or a $\mathfrak{A}_{30}$ (otherwise).

Suppose $y \sim x'$. Then $y \not\sim v_1$ (otherwise $v_1, v_2, v_3, v_4, c_1, c_2, y, x'$ would induce a $\mathfrak{A}_{46}$ (if $x' \sim v_4$), or $v_1, v_2, v_3, v_4, c_1, c_2, y, x', y'$ would induce a $\mathfrak{A}_{47}$ (if $x' \not\sim v_4, y' \not\sim c_1$) or a $\mathfrak{A}_{48}$ (if $x' \not\sim v_4, y' \sim c_1$)), and, by symmetry, $x' \not\sim v_4$. Now $v_1, v_2, v_3, v_4, c_1, c_2, x, y, x', y'$ induce a $\mathfrak{A}_{49}$ or a $\mathfrak{A}_{50}$ (if $x \not\sim c_2$ or $y' \not\sim c_1$) or a $\mathfrak{A}_{51}$ (if $x \sim c_2$ and $y' \sim c_1$). Thus,

$$y \not\sim x'.$$

Furthermore,

$$y \not\sim v_1,$$

otherwise $v_1, v_3, v_4, c_1, c_2, y, x'$ would induce a $\mathfrak{A}_{28}$ (if $x' \sim v_4$), or $v_1, v_3, v_4, c_1, c_2, y, x', y'$ would induce a $\mathfrak{A}_{30}$ (if $x' \not\sim v_4, y' \not\sim c_1$) or a $\mathfrak{A}_{31}$ (if $x' \not\sim v_4, y' \sim c_1$). By symmetry,

$$x' \not\sim v_4.$$

Suppose $x \not\sim c_2$. Then $v_3, v_4, c_1, c_2, x, y, x', y'$ induce a $\mathfrak{A}_{30}$ (if $y' \not\sim c_1$) or a $\mathfrak{A}_{31}$ (if $y' \sim c_1$). Thus,

$$x \sim c_2,$$

and by symmetry,

$$y' \sim c_1.$$

Now $v_1, v_2, v_3, v_4, c_1, c_2, x, y, x', y'$ induce a $\mathfrak{A}_{52}$. □

Lemma 15 Let $v_1 v_2 \cdots v_{2k-1} v_{2k}$, $k \geq 2$, be an induced path in B connecting $v_1 \in X$ and $v_{2k} \in Y$. If $c_1 \not\sim v_{2k}$ and $c_2 \not\sim v_1$ then $c_1 \circledcirc \{v_{2i} \mid i = 1, \ldots, k\}$ or $c_2 \circledcirc \{v_{2j-1} \mid j = 1, \ldots, k\}$.

Proof: The proof is by induction on k. In case $k = 2$, Lem. 15 follows from Lem. 14.

Let $k > 2$. First, suppose $c_1 \not\sim v_{2k-2}$. Then, by induction (applied to $v_1 v_2 \cdots v_{2k-2}$),

$$c_1 \circledcirc \{v_{2i} \mid i = 1, \ldots, k-1\}, \tag{9}$$

or

$$c_2 \circledcirc \{v_{2j-1} \mid j = 1, \ldots, k-1\}. \tag{10}$$

As $c_1 \not\sim v_{2k}$, we are done in case (9). Also, if $c_2 \not\sim v_{2k-1}$ then we are done in case (10). So, let us assume that (9) does not hold, hence (10) does hold, and $c_2 \sim v_{2k-1}$. Let $c_1 \sim v_{2i}$ for some $i \in \{1, \ldots, k-2\}$. Now, if $i = k-2$ then $v_{2k-4}, v_{2k-3}, v_{2k-2}, v_{2k-1}, c_1, c_2$ induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$). If $i < k-2$, $v_{2i-1}, v_{2i}, v_{2k-3}, v_{2k-2}, v_{2k-1}, c_1, c_2$ induce a $\mathfrak{A}_{27}$. Thus, the case $c_1 \not\sim v_{2k-2}$ is settled.

Now suppose $c_1 \sim v_{2k-2}$, and by symmetry, $c_2 \sim v_3$. Then $c_1 \sim v_2$ because otherwise $v_1, v_2, v_{2k-2}, v_{2k-1}, c_1, c_2$ would induce a $\mathfrak{A}_{23}$ (if $c_2 \sim v_{2k-1}$), or $v_1, v_2, v_3, v_{2k-2}, v_{2k-1}, c_1, c_2$ would induce a $\mathfrak{A}_{27}$ (if $c_2 \not\sim v_{2k-1}$).

Now, if $c_1 \not\sim v_4$ then $v_1, v_2, v_3, v_4, v_5, c_1, c_2$ induce a $\mathfrak{A}_{25}$ (if $c_2 \not\sim v_5$) or a $\mathfrak{A}_{28}$ (if $c_2 \sim v_5$). If $c_1 \not\sim v_{2i}$ for some $3 \le i \le k-1$, then by induction (applied to $v_1 v_2 \cdots v_{2i}$), $c_1 \not\sim v_2$ or $c_2 \not\sim v_3$, a contradiction. Hence,

$$c_1 \circledast \{v_{2i} \mid i = 1, \ldots, k-1\}.$$

By symmetry we have

$$c_2 \circledast \{v_{2j-1} \mid j = 2, \ldots, k\}.$$

But then c_1, c_2 and $v_1, \ldots, v_{2k}$, induce a $\mathfrak{A}_3$. $\square$

Lemma 16 (Prop. 7) For every non-trivial connected component $B' = (X \cup Y, E')$ of $G[B]$, $X \circledast C$ or $Y \circledast C$.

Proof:

CLAIM 1 For every induced path $v_1 v_2 \cdots v_{2k-1} v_{2k}$, $k \ge 3$, in B and for every vertex $v \in C$: $v \sim v_1$ or $v \sim v_{2k}$

PROOF: The proof is by induction on k. First, let $k = 3$. If $v \not\sim v_1$ and $v \not\sim v_6$ then, as $v \in C$ and because of Prop. 6.a, $v \sim v_2, v \sim v_5$. Discussing the adjacencies between v and v_3, v_4 we conclude that there is an induced C_5, or a $\mathfrak{A}_{11}$ (antenna), or a $\mathfrak{A}_2$. Thus, $v \sim v_1$ or $v \sim v_6$, and Claim 1 is proved for $k = 3$.

Next, let $k \ge 4$, and assume that $v \not\sim v_1$ and $v \not\sim v_{2k}$. By Prop. 6.a, $v \sim v_2$, $v \sim v_{2k-1}$. Let i, $3 \le i \le 2k-2$, be the smallest index such that $v \not\sim v_i$; such an i exists otherwise $v, v_1, \ldots, v_{2k}$ would induce a $\mathfrak{A}_2$. Note that, by the choice of i, $v \sim v_j$, $j = 2, 3, \ldots i-1$, and by Prop. 6.a, $v \sim v_{i+1}$. Moreover, i must be odd otherwise v, $v_1, v_2, \ldots$, v_i would induce a $\mathfrak{A}_2$ (if $i \ge 6$) or $v, v_1, v_2, v_3, v_4, v_5$ would induce a $\mathfrak{A}_{11}$ (antenna) (if $i = 4$).

Now, if $i \le 2k-5$ then, by applying the induction hypothesis for the path $v_i, v_{i+1} \cdots v_{2k}$, $v \sim v_i$ or $v \sim v_{2k}$, a contradiction. Thus, $i \ge 2k-3$, hence $i = 2k-3$. Then v, v_{2k-4}, v_{2k-3}, v_{2k-2}, v_{2k-1}, v_{2k} induce a $\mathfrak{A}_{11}$ (antenna). $\square$

Let $B' = (X \cup Y, E(B'))$ be a nontrivial connected component of B, and assume to the contrary that $X \not\circledast C$ and $Y \not\circledast C$. We distinguish two cases.

Case 1: *There exists a vertex $c \in C$ non-adjacent to a vertex $a \in X$ and non-adjacent to a vertex $b \in Y$.* As $c \in C$ and because of Prop. 6.a, $a \not\sim b$. Hence, by Claim 1, every shortest path in B' connecting a and b must consist of exactly four vertices (including a, b); let $v_1v_2v_3v_4$ with $v_1 = a$ and $v_4 = b$ be a shortest path in B'.

As $c \in C$ and by Prop. 6.a, $c \sim v_2, c \sim v_3$. We will show that $v_2 \circledast C$ and $v_3 \circledast C$. As a first step, we prove $v_2 \circledast C$ or $v_3 \circledast C$:

CLAIM 2 $v_2 \circledast C$ or $v_3 \circledast C$.

PROOF: Suppose $v_2 \not\sim c'$ and $v_3 \not\sim c''$ for some $c', c'' \in C \setminus \{c\}$. Then c' is adjacent to v_1, v_3 and c'' is adjacent to v_2, v_4 because of Prop. 6.a. In particular, $c' \neq c''$. Moreover, $c' \sim v_4$ or $c'' \sim v_1$ (otherwise $c, c', c'', v_1, v_2, v_3, v_4$ would induce a $\mathfrak{A}_{20}$), but not both (otherwise $v_1, v_2, v_3, v_4, c', c''$ would induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$)). Now, $c, c', c'', v_1, v_2, v_3, v_4$ induce a $\mathfrak{A}_{10}$, regardless whether $c' \not\sim v_4$ or $c'' \not\sim v_1$, and Claim 2 follows. □

By symmetry, we may assume that

$$v_2 \circledast C.$$

As $v_2 \in B$, there exist adjacent vertices $x, y \in \overline{N}(v_2)$. As $v_2 \circledast C$, we may assume that $x \in X \setminus \{v_1, v_3\}$, $y \in Y \setminus \{v_2\}$.

Suppose that $x \sim v_4$. Then $c \sim x$ (as $c \in C$), and c, v_1, v_2, v_3, v_4, x induce a $\mathfrak{A}_{11}$ (antenna). Thus,

$$x \not\sim v_4. \tag{11}$$

In particular, $y \neq v_4$. Moreover,

$$y \sim c, \tag{12}$$

otherwise $c \sim x$ (by Prop. 6.a) and therefore $y \not\sim v_1$ (else y, v_1, v_2, c, x would induce a C_5), implying that either v_1, v_2, v_3, c, x, y induce a $\mathfrak{A}_{11}$ (antenna) (if $y \sim v_3$), or $v_1, v_2, v_3, v_4, c, x, y$ induce a $\mathfrak{A}_6$ (eiffeltower). Furthermore,

$$y \sim v_3, \tag{13}$$

otherwise v_1, v_2, v_3, v_4, c, y would induce a $\mathfrak{A}_{11}$ (antenna) (if $y \sim v_1$), or $v_1, v_2, v_3, v_4, c, x, y$ would induce a $\mathfrak{A}_6$ (eiffeltower) (if $y \not\sim v_1$ and $x \not\sim c$), or a $\mathfrak{A}_7$ (if $y \not\sim v_1$ and $x \sim c$).

We next show:

CLAIM 3 $v_3 \circledast C$

PROOF: Suppose $v_3 \not\sim c'$ for some $c' \in C \setminus \{c\}$. Then by Prop. 6.a, $c' \sim v_4$ and $c' \sim y$. Moreover, $y \not\sim v_1$ otherwise $v_1, v_2, v_3, v_4, c, c', y$ would induce a $\mathfrak{A}_{20}$ (if $c' \not\sim v_1$) or a $\mathfrak{A}_{10}$ (if $c' \sim v_1$). Now, $v_1, v_2, v_3, v_4, c, c', x$ and y induce a forbidden subgraph H as follows:

- $c' \not\sim v_1, c' \not\sim x, c \not\sim x$: H is a $\mathfrak{A}_{36}$
- $c' \sim v_1, c' \not\sim x, c \not\sim x$: H is a $\mathfrak{A}_{38}$.
- $c' \not\sim v_1, c' \sim x, c \not\sim x$: H is a $\mathfrak{A}_{38}$
- $c' \sim v_1, c' \sim x, c \not\sim x$: H is a $\mathfrak{A}_{39}$.
- $c' \not\sim v_1, c' \not\sim x, c \sim x$: H is a $\mathfrak{A}_{37}$
- $c' \sim v_1, c' \not\sim x, c \sim x$: H is a $\mathfrak{A}_{42}$.
- $c' \not\sim v_1, c' \sim x, c \sim x$: H is a $\mathfrak{A}_{41}$
- $c' \sim v_1, c' \sim x, c \sim x$: H is a $\mathfrak{A}_{43}$.

Claim 3 follows. $\square$

As $v_3 \in B$, there exist adjacent vertices $x', y' \in \overline{N}(v_3)$. As $v_3 \circledast C$, we may assume that $x' \in X \setminus \{v_3\}$, $y' \in Y \setminus \{v_2, v_4\}$. By symmetry, we obtain from (11), (12), and (13):

$$y' \not\sim v_1, x' \sim c, \text{ and } x' \sim v_2.$$

In particular, $x' \neq v_1$, and x, y, x', y' are pairwise distinct.

Suppose $x \sim y'$. Then c is adjacent to precisely one of x, y': otherwise either $v_1, v_2, v_3, v_4, c, x, y'$ would induce a $\mathfrak{A}_7$ (if $c \sim x, y'$), or v_2, v_3, c, x, y' would induce a $\mathfrak{A}_4$ ($K_2 \cup K_3$) (if $c \not\sim x, y'$). By symmetry we may assume that $c \not\sim x$ and $c \sim y'$. Then v_3, v_4, c, x, y, y' induce a $\mathfrak{A}_{11}$ (antenna). Thus,

$$x \not\sim y'.$$

Suppose $y \sim x'$. Then $x' \sim v_4$ or $y \sim v_1$ (otherwise $v_1, v_2, v_3, v_4, c, x, y, x', y'$ would induce a $\mathfrak{A}_{32}$, or a $\mathfrak{A}_{33}$, or a $\mathfrak{A}_{34}$), but not both (otherwise $v_1, v_2, v_3, v_4, c, y, x'$ would induce a $\mathfrak{A}_{20}$). By symmetry we may assume that $x' \not\sim v_4$ and $y \sim v_1$. Then $v_1, v_2, v_3, v_4, c, y, x', y'$ induce a $\mathfrak{A}_{36}$ (if $c \not\sim y'$), or a $\mathfrak{A}_{37}$ (if $c \sim y'$). Thus,

$$y \not\sim x'.$$

Then $c \sim x$ or $c \sim y'$ (otherwise v_3, v_4, c, x, y, x' would induce a $\mathfrak{A}_{11}$ (antenna) (if $x' \sim v_4$), or $v_3, v_4, c, x, y, x', y'$ would induce a $\mathfrak{A}_6$ (eiffeltower) (if $x' \not\sim v_4$)), but not both (otherwise $v_1, v_2, v_3, v_4, c, x, y, x', y'$ would induce a $\mathfrak{A}_{35}$ (if $y \not\sim v_1$, $x' \not\sim v_4$), or v_1, v_2, v_4, c, y, x' would induce a $\mathfrak{A}_{11}$ (antenna) (if $y \sim v_1$, $x' \sim v_4$), or $v_1, v_3, v_4, c, y, x', y'$ would induce a $\mathfrak{A}_7$ (if $y \sim v_1$, $x' \not\sim v_4$); the case $y \not\sim v_1$, $x' \sim v_4$ is symmetrical to the last one). By symmetry we may assume that $c \sim x$ and $c \not\sim y'$. But then v_1, v_2, c, y, x', y' induce a $\mathfrak{A}_{11}$ (antenna) (if $y \sim v_1$), or $v_1, v_2, c, x, y, x', y'$ induce a $\mathfrak{A}_7$ (if $y \not\sim v_1$).

Case 1 is settled.

Case 2: *For every vertex $c \in C$, $c \circledast X$ or $c \circledast Y$.*

In this case, since it is assumed that $C \not\circledast X$ and $C \not\circledast Y$, there exist vertices $c_1 \neq c_2 \in C$ such that

$$c_1 \circledast X,\ c_1 \not\circledast Y \quad \text{and} \quad c_2 \not\circledast X,\ c_2 \circledast Y.$$

Observe that the prerequisites for Lem. 13, 14, 15 hold. We distinguish two subcases.

Case 2.1: *For all $a \in X, b \in Y$, if $c_2 \not\sim a$ and $c_1 \not\sim b$ then $a \sim b$.*
Since $c_1, c_2 \in C$, there exist a neighbour $x \in X$ of c_2 and a neighbour $y \in Y$ of c_1. By Lem. 13, $x \not\sim y$. Consider an induced path $(x = v_1, v_2, \cdots, v_{2k-1}, v_{2k} = y)$ in B connecting x and y, $k \geq 2$. By Lem. 13 again, $c_1 \not\sim v_2$ and $c_2 \not\sim v_{2k-1}$. As we are in Case 2.1, $v_2 \sim v_{2k-1}$, hence $k = 2$. Now $v_1, v_2, v_3, v_4, c_1, c_2$ induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$).

Case 2.1 is settled.

Case 2.2: *There exist $a \in X, b \in Y$ such that $c_2 \not\sim a, c_1 \not\sim b$, and $a \not\sim b$.*
Let $a \in X$ and $b \in Y$ be two such vertices, and consider an induced path $(a = v_1, v_2, \cdots, v_{2k-1}, v_{2k} = b)$ in B connecting a and b. As $a \not\sim b$, $k \geq 2$. By Lem. 15, we may assume without loss of generality,

$$c_1 \circledcirc \{v_{2i} \mid 1 \leq i \leq k\}.$$

As $c_1 \in C$, c_1 has a neighbour $y \in Y \setminus \{v_2, v_4, \ldots, v_{2k}\}$.

Case 2.2.1: $c_2 \circledcirc \{v_{2j-1} \mid 1 \le j \le k\}$.
As $c_2 \in C$, c_2 has a neighbour $x \in X \setminus \{v_1, v_3, \ldots, v_{2k-1}\}$. By Lem. 13,

$$x \not\sim y.$$

CLAIM 4 $x \circledcirc \{v_{2i} \mid 1 \le i \le k\}$ or $y \circledcirc \{v_{2j-1} \mid 1 \le j \le k\}$
PROOF: Suppose $x \sim v_{2i}$ for some $1 \le i \le k$. Then $y \not\sim v_{2i-1}$, and if $i < k$, $y \not\sim v_{2i+1}$, otherwise c_1, c_2, x, y, v_{2i}, and v_{2i-1} (and if $i < k$, v_{2i+1}) would induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$), hence if $v \in \{v_{2j-1} \mid 1 \le j \le k\}$ were a neighbour of y, then $c_1, c_2, x, y, v, v_{2i}, v_{2i-1}$ would induce a $\mathfrak{A}_{27}$. Thus, $y \circledcirc \{v_{2j-1} \mid 1 \le j \le k\}$. By symmetry, if $y \not\circledcirc \{v_{2j-1} \mid 1 \le j \le k\}$, then $x \circledcirc \{v_{2i} \mid 1 \le i \le k\}$. □

By symmetry we may assume that

$$x \circledcirc \{v_{2i} \mid 1 \le i \le k\}.$$

As B is connected, x therefore has a neighbour $y' \in Y \setminus \{y, v_2, v_4, \ldots, v_{2k}\}$. By Lem. 13 (applied to $u = v_1, v = v_2, x$ and y'),

$$c_1 \not\sim y'.$$

Moreover,

$$y \circledcirc \{v_{2j-1} \mid 1 \le j \le k\} :$$

For, if $y \sim v_{2j-1}$ for some $1 \le j \le k$, then $y' \not\sim v_{2j-1}$ otherwise $c_1, c_2, x, y, v_{2j-1}, y'$ would induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$). Now $c_1, c_2, x, y, v_{2j-1}, y'$ and v_{2j} induce a $\mathfrak{A}_{27}$.

As B is connected, y must have a neighbour $x' \in X \setminus \{x, v_1, v_3, \ldots, v_{2k-1}\}$. By Lem. 13 again,

$$c_2 \not\sim x'.$$

Hence

$$x' \not\sim y',$$

otherwise c_1, c_2, x, y, x', y' would induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$). Now c_1, c_2, x, y, x', y', and v_1 (if $y' \sim v_1$) or v_2 (if $x' \sim v_2$) induce a $\mathfrak{A}_{27}$, or $c_1, c_2, x, y, x', y', v_1, v_2$ induce a $\mathfrak{A}_{19}$ (if $y' \not\sim v_1$ and $x' \not\sim v_2$).

Case 2.2.1 is settled.

Case 2.2.2: $c_2 \sim v_{2j-1}$ *for some* $1 \le j \le k$.
Fix such an index j. Since $c_2 \not\sim v_1$, $j > 1$. By Lem. 13 (applied to $u = v_1, v = v_2, x = v_{2j-1}$ and y),

$$y \not\sim v_{2j-1}.$$

Suppose $y \sim v_{2j'-1}$ for some $1 \le j' \le k$, $j' \ne j$. Then, by Lem. 13 again,

$$c_2 \not\sim v_{2j'-1}.$$

If $j' = j - 1$ then $c_1, c_2, y, v_{2j-1}, v_{2j-2}, v_{2j-3}$ induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$). If $j' = j + 1$ (implying $j < k$) then $c_1, c_2, y, v_{2j-1}, v_{2j}, v_{2j+1}$ induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$). If $j' \ne j \pm 1$, $c_1, c_2, y, v_{2j-1}, v_{2j}, v_{2j'-1}, v_{2j'}$ induce a $\mathfrak{A}_{27}$. Thus,

$$y \circledcirc \{v_1, v_3, \ldots, v_{2k-1}\}.$$

As B is connected, y has a neighbour $x \in X \setminus \{v_1, v_3, \ldots, v_{2k-1}\}$. By Lem. 13 again,

$$c_2 \not\sim x.$$

Now, if $x \sim v_{2j-2}$ then $c_1, c_2, x, y, v_{2j-2}, v_{2j-3}$ induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$), and if $x \not\sim v_{2j-2}$ then $c_1, c_2, x, y, v_{2j-1}, v_{2j-2}, v_{2j-3}$ induce a $\mathfrak{A}_{27}$.

Case 2.2.2, hence Case 2, is settled, and the proof of Prop. 7 is complete. □

2.3 The proof of Proposition 8

Let $v \in V \setminus (B \cup C)$, that is, $N(v)$ induces a split graph and $\overline{N}(v)$ is independent. Assume that $G - v$ is probe split and let (C, S, I) be a probe split partition of $G - v$. Create subpartitions of S and I as:

$$\begin{aligned} S_1 &= S \cap N(v), \\ S_2 &= S \setminus S_1, \\ I_1 &= I \cap N(v), \\ I_2 &= I \setminus I_1. \end{aligned}$$

As $\overline{N}(v)$ is an independent set, $S_2 \circledcirc I_2$ and $|C \cap \overline{N}(v)| \leq 1$. Moreover, $C \circledast S_2$ and $S_2 \subseteq \overline{N}(v)$ implies that if $S_2 \neq \emptyset$, then $|C \cap \overline{N}(v)| = 0$, that is $v \circledast C$.

Lemma 17 If $S_1 \circledcirc I_1$ then G is probe split.

Proof: We split the proof into two cases.

1. $v \circledast C$. We may assume $S_1 \not\circledcirc I_2$ (otherwise $(C, S_2 \cup \{v\}, I \cup S_1)$ is a valid probe split partition of G), $I_1 \not\circledcirc S_2$ (otherwise $(C \cup \{v\}, S_1, I \cup S_2)$ is a valid probe split partition), and $C \neq \emptyset$ (otherwise $(\{v\}, S_1 \cup I_1, S_2 \cup I_2)$ is a valid probe split partition). In particular I_1, I_2, S_1, S_2 all are non-empty. Define
$$\begin{aligned} I_{21} &= I_2 \cap N(S_1), \\ I_{22} &= I_2 \setminus I_{21}, \\ I_{11} &= I_1 \cap N(S_2), \\ I_{12} &= I_1 \setminus I_{11}. \end{aligned}$$
Note that I_{21} and I_{11} are non-empty. We show that $I_{21} \circledast C$ or $I_{11} \circledast C$. Let $i_{11} \in I_{11}$ with a neighbour $s_2 \in S_2$ and a non-neighbour $c \in C$ and let $i_{21} \in I_{21}$ with a neighbour $s_1 \in S_1$ and a non-neighbour $c' \in C$. If $c' = c$ then $vci_{11}i_{21}s_1s_2$ induce a $\mathfrak{A}_{11}$ (antenna), and if $c \neq c'$ then $vcc'i_{11}i_{21}s_1s_2$ induce a $\mathfrak{A}_{28}$. Thus either $I_{21} \circledast C$ or $I_{11} \circledast C$. If $I_{21} \circledast C$, then $(C, \{v\} \cup S_2 \cup I_{21}, I_1 \cup I_{22} \cup S_1)$ is a valid probe split partition and if $I_{11} \circledast C$ then $(C \cup \{v\}, S_1 \cup I_{11}, I_{12} \cup S_2 \cup I_2)$ is a valid probe split partition.

2. $v \not\circledast C$. Because $\overline{N}(v)$ is independent, v is non-adjacent to precisely one vertex c of C. Then, for the same reason, $S_2 = \emptyset$ and $c \circledcirc I_2$. Define
$$\begin{aligned} I_{11} &= I_1 \cap N(c), \\ I_{12} &= I_1 \setminus I_{11}. \end{aligned}$$

We may assume I_{11} to be non-empty, because otherwise $(C - c + v, S_1, I + c)$ would be a valid probe split partition. We may also assume that $I_{11} \not\circledast (C - c)$ because otherwise $(C - c + v, I_{11} \cup S_1, I_2 \cup I_{12} \cup \{c\})$ would be a valid probe split partition. Let $i_{11} \in I_{11}$ and $c_{11} \in C - c$ such that $i_{11}c_{11} \notin E$ (note that $i_{11}c \in E$). Partition I_2 as:

$$\begin{aligned} I_{21} &= \{x \in I_2 \mid |N(x) \cap S_1| \geq 2\}, \\ A &= \{x \in I_2 \mid |N(x) \cap S_1| = 1 \wedge x \circledast (C - c)\}, \\ B &= \{x \in I_2 \mid |N(x) \cap S_1| = 1 \wedge x \not\circledast (C - c)\}, \\ R &= I_2 \setminus (I_{21} \cup A \cup B). \end{aligned}$$

Note that by this definition $R \circledcirc S_1$ and $A \circledast (C - c)$. We first show that

$$I_{21} \circledast (C - c). \tag{14}$$

Let $x \in I_{21}$ with two neighbours $u, u' \in S_1$ and a non-neighbour $c_x \in C - c$. If $i_{11}c_x \notin E$ then $vuu'cxi_{11}c_x$ induces a $\mathfrak{A}_{20}$, if $xc_{11} \notin E$ then $vuu'cxi_{11}c_{11}$ induces a $\mathfrak{A}_{20}$, and if both are edges then $vuu'cxc_xi_{11}c_{11}$ induces a $\mathfrak{A}_{46}$. This proves (14).

We now may assume that $B \neq \emptyset$ because otherwise $(C - c, I_{21} \cup A \cup \{c, v\}, I_1 \cup R \cup S_1)$ is a valid probe split partition. We next show that precisely one vertex $s_B \in S_1$ exists such that s_B is the unique neighbour in S_1 of every vertex in B, that is

$$N(B) \cap S_1 = \{s_B\}. \tag{15}$$

Assume there are $b_1, b_2 \in B$ with unique neighbour $s_1, s_2 \in S_1$, respectively ($b_1 \neq b_2, s_1 \neq s_2$). Then $c_{11}b_1$ and $c_{11}b_2$ are both edges, because otherwise $b_1s_1b_2s_2vci_{11}c_{11}$ would induce a $\mathfrak{A}_{37}$ or a $\mathfrak{A}_{36}$. According to the definition of B, a vertex $c_1 \in C - c$ exists that is a non-neighbour of b_1. $c_1i_{11} \in E$, because otherwise we again have a $\mathfrak{A}_{37}$ or a $\mathfrak{A}_{36}$ ($b_1s_1b_2s_2vci_{11}c_1$). But now $b_1s_1b_2s_2vci_{11}c_{11}c_1$ induces a $\mathfrak{A}_{47}$ (if $c_1b_2 \notin E$) or a $\mathfrak{A}_{48}$ (if $c_1b_2 \in E$). This proves $s_1 = s_2$ and thus (15).

Finally we show that

$$s_B \circledast (I_{21} \cup A). \tag{16}$$

Let $x \in I_{21} \cup A$ be a non-neighbour of s_B and let $s_1 \in S_1$ be a neighbour of x. Let $b \in B$ with non-neighbour $c_b \in C - c$. By (14) and the definition of A, $x \circledast (C - c)$. Both $i_{11}c_b$ and $c_{11}b$ are edges because otherwise $i_{11}cvs_1s_Bbxc_b$, and $i_{11}cvs_1s_Bbxc_{11}$, respectively, induce a $\mathfrak{A}_{36}$. But then $i_{11}cvs_1s_Bbxc_{11}c_b$ induce a $\mathfrak{A}_{48}$. (16) follows.

We claim that (Q, U, J) with

$$\begin{aligned} Q &= C - c + s_B, \\ U &= I_{21} \cup A \cup \{c, v\}, \\ J &= I_1 \cup R \cup B \cup (S_1 - s_B) \end{aligned}$$

is a valid probe split partition: Independence of U and J is clear by definition of A, B, R and s_B and $Q \circledast U$ follows from (14) and (16).

□

We turn to the case when $S_1 \not\oslash I_1$. Let $x \in S_1, y \in I_1$ with $xy \in E$ such that $|N(x) \cap I_1| + |N(y) \cap S_1|$ is maximum. Partition $I_1 - y$ and $S_1 - x$ as:

$$\begin{aligned} I_{11} &= (I_1 - y) \cap N(x), \\ I_{12} &= (I_1 - y) \setminus I_{11}, \\ S_{11} &= (S_1 - x) \cap N(y), \\ S_{12} &= (S_1 - x) \setminus S_{11}. \end{aligned}$$

For some of the following lemmas it is important to recall that $G[N(v)]$ is assumed to be a split graph and therefore $(2K_2, C_4, C_5)$-free.

Lemma 18 $(S_{11} \cup S_{12}) \circledcirc (I_{11} \cup I_{12})$

Proof: First, there cannot be an edge between S_{11} and I_{11} or between S_{12} and I_{12}, because together with x, y this edge would induce a C_4 or $2K_2$, respectively, in $N(v)$. Secondly, suppose an edge $x'y'$ exists, with $x' \in S_{11}, y' \in I_{12}$. Then $I_{11} = \emptyset$, because otherwise x, x', y' and a vertex from I_{11} would induce a $2K_2$ in $N(v)$, which implies that $|N(x') \cap I_1| + |N(y) \cap S_1| > |N(x) \cap I_1| + |N(y) \cap S_1|$, contradicting the choice of x, y. Similarly, there is no edge between I_{11} and S_{12}. □

Lemma 19 If $S_{11} \neq \emptyset$ then $y \circledast (C \cap N(v))$.

Proof: If y is non-adjacent to some $c \in (C \cap N(v))$ then x, y, c and a vertex from S_{11} induce a C_4 in $N(v)$. □

Lemma 20 Either $S_2 \circledcirc I_{11}$ or $I_2 \circledcirc S_{11}$.

Proof: Assume there are edges $s_2 i_{11}$ and $i_2 s_{11}$ with $s_2 \in S_2, i_{11} \in I_{11}, i_2 \in I_2, s_{11} \in S_{11}$. If $xi_2 \in E$ or $ys_2 \in E$ then $vs_2 i_2 s_{11} i_{11}$ together with x or y, respectively, induce a $\mathfrak{A}_{11}$ (antenna); if neither are edges then $vs_2 i_2 s_{11} i_{11} xy$ induce a $\mathfrak{A}_2$. □

Lemma 21 **a)** Either $S_2 \circledcirc I_{12}$ or $S_2 \circledcirc (I_{11} + y)$ or $I_2 \circledcirc (S_{11} + x)$.

b) Either $I_2 \circledcirc S_{12}$ or $I_2 \circledcirc (S_{11} + x)$ or $S_2 \circledcirc (I_{11} + y)$.

Proof: We prove the first statement; the second is symmetrical. Assume there are edges $s_2 i_{12}$, $s_2' z$ and $i_2 t$ with $s_2, s_2' \in S_2,\ i_{12} \in I_{12},\ i_2 \in I_2,\ z \in I_{11} + y,\ t \in S_{11} + x$. By Lem. 20, $z = y$ or $t = x$, implying that z and t are adjacent. If $zs_2 \in E$ or $i_{12} s_2' \in E$ then $vzti_2$ together with s_2 or s_2', respectively, induces a $\mathfrak{A}_{11}$ (antenna); if neither are edges then $vzti_{12} i_2 s_2 s_2'$ induces a $\mathfrak{A}_6$ (eiffeltower). □

Lemma 22 If $I_2 \not\oslash S_1$ then $(I_{12} \cap N(S_2)) \circledast C$.

Proof: Seeking a contradiction, assume $i_{12} \in I_{12} \cap N(S_2)$ is non-adjacent to $c \in C$. Let $s_2 \in S_2$ be a neighbour of i_{12}. We distinguish two cases matching the partition of S_1.

1. $I_2 \not\oslash (S_{11} + x)$
   Consider an edge $i_2 z$ with $i_2 \in I_2, z \in S_{11} + x$. Then z is adjacent to c, v, y and non-adjacent to s_2, i_{12}. Thus v must be adjacent to c because otherwise $v i_{12} s_2 c z$ would induce a C_5. Furthermore, y is non-adjacent to s_2 (otherwise $vyzs_2i_2i_{12}$ would induce a $\mathfrak{A}_{11}$ (antenna)) and hence non-adjacent to c (otherwise $vyzs_2i_{12}c$ would induce a $\mathfrak{A}_{23}$). But now either $vczs_2i_2i_{12}$ induces a $\mathfrak{A}_{11}$ (antenna) (if $i_2c \notin E$) or $vczs_2i_2i_{12}y$ induces a $\mathfrak{A}_{17}$ (if $i_2c \in E$).

2. $I_2 \not\oslash S_{12}$
   Consider an edge i_2s_{12} with $i_2 \in I_2, s_{12} \in S_{12}$. We may assume that $xi_2 \notin E$, because otherwise the previous case applies. $vc \in E$ because otherwise $vi_{12}s_2cs_{12}$ would induce a C_5 and hence $i_2c \in E$ because otherwise $vi_{12}s_2cs_{12}i_2$ would induce a $\mathfrak{A}_{11}$ (antenna). Now, if $yc \notin E$ then either $vxycs_2i_2s_{12}i_{12}$ induces a $\mathfrak{A}_{19}$ (if $ys_2 \notin E$) or $vxycs_2i_2s_{12}$ induces a $\mathfrak{A}_{27}$. If $yc \in E$ then either $vxycs_2i_{12}$ induces a $\mathfrak{A}_{23}$ (if $ys_2 \notin E$) or $vxycs_2i_2s_{12}i_{12}$ induces a $\mathfrak{A}_{40}$ (if $ys_2 \in E$).

$\square$

Lemma 23 If $S_2 \not\oslash I_{12}$ and $I_2 \not\oslash (S_{11} + x)$ then G is probe split.

Proof: We show that (Q, U, J) with

$$\begin{aligned} Q &= C + v, \\ U &= S_1 \cup \{x\} \cup (I_{12} \cap N(S_2)), \\ J &= S_2 \cup I \setminus (I_{12} \cap N(S_2)) \end{aligned}$$

is a valid probe split partition. By assumption $S_2 \neq \emptyset$, therefore $v \circledast C$ and Q is a clique. By Lem. 18, U is independent and by Lem. 22, $U \circledast Q$. Finally, by Lem. 21.a, J is independent. $\square$

Lemma 24 If $I_2 \not\oslash S_{12}$ and $S_2 \not\oslash (I_{11} + y)$ then G is probe split.

Proof: We show that (Q, U, J) with

$$\begin{aligned} Q &= C + v, \\ U &= (I_{11} + y) \cup (I_{12} \cap N(S_2)) \cup S_{12}, \\ J &= (S \setminus S_{12}) \cup (I_{12} \setminus N(S_2)) \cup I_2 \end{aligned}$$

is a valid probe split partition. Again is by assumption $S_2 \neq \emptyset$ and thus Q a clique. By Lem. 18, U is independent and by Lem. 21.b, J is independent. First we show that

$$((I_{11} + y) \cap N(S_2)) \circledast C. \tag{17}$$

Let $z \in (I_{11} + y) \cap N(S_2)$ be non-adjacent to $c \in C$ and adjacent to $s_2 \in S_2$. Let $i_2 \in I_2$ and $s_{12} \in S_{12}$ be neighbours, and note that by Lem. 21.b, $i_2 x \notin E$. But now either $vczs_2s_{12}i_2$ induce a $\mathfrak{A}_{11}$ (antenna), or $vczs_2s_{12}i_2x$ induce a $\mathfrak{A}_{27}$. This proves (17).

Next we show that

$$((I_{11} + y) \setminus N(S_2)) \circledast C. \tag{18}$$

Let $z \in (I_{11} + y) \setminus N(S_2)$ be non-adjacent to $c \in C$. Let $i_2 \in I_2$ and $s_{12} \in S_{12}$ be neighbours, and let $s_2 \in S_2$ and $t \in I_{11} + y$ be neighbours as well. Note that $t \neq z$ and that by (17) $tc \in E$. But now $vcxzts_2s_{12}i_2$ induce either a $\mathfrak{A}_{29}$ (if $ci_2 \notin E$) or a $\mathfrak{A}_{30}$ (if $ci_2 \in E$). $\square$

Lemma 25 If $I_2 \not\oslash S_{12}$ and $S_2 \circledcirc (I_{11} + y)$ then G is probe split.

Proof: We distinguish two cases:

1. $v \circledast C$

   We show that (Q, U, J) with

   $$\begin{aligned} Q &= C + v, \\ U &= S_1 \cup (I_{12} \cap N(S_2)), \\ J &= (I \setminus N(S_2)) \cup S_2 \end{aligned}$$

   is a valid probe split partition. By assumption $v \circledast C$ and thus Q is a clique. By Lem. 18, U is independent and by Lem. 22, $U \circledast Q$. Finally, by assumption J is independent.

2. $v \not\circledast C$

   By the discussion after the definition of S_2, we have that $S_2 = \emptyset$ and that v is non-adjacent to precisely one vertex $c \in C$. Furthermore, $c \circledcirc I_2$, because $\overline{N}(v)$ is assumed to be independent. We start by showing that

   $$\text{if } c \not\oslash (I_{11} + y) \text{ then } \; x \circledast (I_2 \cap N(S_{12})) \;\wedge\; (C - c) \circledast (I_2 \cap N(S_{12})) \;\wedge\; S_{11} = \emptyset. \tag{19}$$

   Let $z \in N(c) \cap (I_{11} + y)$, and $i_2 \in I_2$ with a neighbour $s_{12} \in S_{12}$. Then $xi_2 \in E$, and thus $x \circledast (I_2 \cap N(S_{12}))$, because otherwise $cvxzi_2s_{12}$ induces a $\mathfrak{A}_9$ (co-fish). Next, assume $c' \in C - c$ is non-adjacent to i_2. But then $cvxzi_2s_{12}c'$ induce either a $\mathfrak{A}_{10}$ (if $zc' \notin E$) or a $\mathfrak{A}_{18}$ (if $zc' \in E$). Thus $(C - c) \circledast (I_2 \cap N(S_{12}))$. Finally, assume a vertex $s_{11} \in S_{11}$ exists. If $yc \notin E$ then $y \neq z$ and $xyvczs_{11}$ induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$). If $yc \in E$ then either $xyvcs_{11}i_2s_{12}$ induce a $\mathfrak{A}_{21}$ (if $s_{11}i_2 \in E$) or $yvcs_{11}i_2s_{12}$ induce a $\mathfrak{A}_9$ (co-fish) (if $s_{11}i_2 \notin E$). Thus $S_{11} = \emptyset$, finishing the proof of (19).

   Next we show that

   $$\text{if } (C - c) \not\circledast (I_{12} \cap N(c)) \text{ then } \; S_{11} = \emptyset \;\wedge\; x \circledast (I_2 \cap N(S_{12})) \;\wedge\; (C - c) \circledast (I_2 \cap N(S_{12})). \tag{20}$$

   By (19), we may assume that $c \circledcirc I_{11} + y$. Let $i_{12} \in (I_{12} \cap N(c))$ be non-adjacent to $c' \in C - c$. First, assume a vertex $s_{11} \in S_{11}$ exists. Then $cc'vxyi_{12}s_{11}$ induce either a $\mathfrak{A}_{18}$ (if $yc' \in E$) or a $\mathfrak{A}_{10}$ (if $yc' \notin E$). Thus $S_{11} = \emptyset$. Consider now neighbours $i_2 \in I_2$ and $s_{12} \in S_{12}$. If $xi_2 \notin E$ then $vcc'xyi_{12}i_2s_{12}$ induce a $\mathfrak{A}_{38}$ (if $c'i_2, c'y \notin E$), a $\mathfrak{A}_{42}$ (if $c'i_2 \in E, c'y \notin E$) a

$\mathfrak{A}_{41}$ (if $c'i_2 \notin E, c'y \in E$) or a $\mathfrak{A}_{45}$ (if $c'i_2, c'y \in E$). Thus $x \circledast (I_2 \cap N(S_{12}))$. It follows that $c' \circledast I_2 \cap N(S_{12})$ because otherwise $cc'vxs_{12}i_{12}i_2$ induce a $\mathfrak{A}_{20}$. Finally, assume that a vertex $c'' \in C \setminus \{c, c'\}$ is non-adjacent to a vertex $i'_2 \in I_2 \cap N(S_{12})$. Let $s'_{12} \in S_{12}$ be a neighbour of i'_2 (note that $i'_2 = i_2$ or $s'_{12} = s_{12}$ is possible). $c'i'_2 \in E$ because $c' \circledast I_2 \cap N(S_{12})$ and $c''i_{12} \in E$ because otherwise $cc''vxs'_{12}i_{12}i'_2$ would induce a $\mathfrak{A}_{20}$. But then $cc'c''vxs'_{12}i_{12}i'_2$ induce a $\mathfrak{A}_{46}$. This finishes the proof of (20).

We are now ready to prove the lemma. If $c \not\circledcirc (I_{11} + y)$ or $(C - c) \not\circledast (I_{12} \cap N(c))$ then by (19), (20), (Q, U, J) with

$$Q = C - c + x,$$
$$U = \{c, v\} \cup (I_2 \cap N(S_{12})),$$
$$J = I \setminus (I_2 \cap N(S_{12})) \cup S_{12}$$

is a valid probe split partition. Otherwise (Q, U, J) with

$$Q = C - c + v,$$
$$U = S \cup (I_{12} \cap N(c)),$$
$$J = I \setminus (I_{12} \cap N(c)) \cup \{c\}$$

is a valid probe split partition.

□

Lemma 26 If $I_2 \circledcirc S_{12}$ and $S_2 \not\circledcirc (I_{11} + y)$ then G is probe split.

Proof: Let

$$A = (I_{11} + y) \cap N(S_2),$$
$$B = (I_{11} + y) \setminus A.$$

By assumption, $A \neq \emptyset$ and $S_2 \neq \emptyset$ and thus $v \circledast C$. We first show that

$$\text{Either } A \circledast C \text{ or } \exists! c_A \in C : c_A \circledcirc A \wedge (C - c_A) \circledast A. \tag{21}$$

Let $z \in (I_{11} + y) \cap N(S_2)$ be non-adjacent to $c_1, c_2 \in C$ ($c_1 \neq c_2$), then $zxvc_1c_2$ and a neighbour of z in S_2 induce a $\mathfrak{A}_{24}$ ($\overline{2P_3}$). This proves that every vertex in $(I_{11} + y) \cap N(S_2)$ has at most one non-neighbour in C. Consider two vertices $z, z' \in (I_{11} + y) \cap N(S_2)$ and $c \in C$ such that $zc \in E$ but $z'c \notin E$. If z and z' have a common neighbour $s \in S_2$ then $vzz'cxs$ induce a $\mathfrak{A}_{24}$ ($\overline{2P_3}$) and if they have private neighbours s, s', respectively, then $vzz'cxss'$ induce a $\mathfrak{A}_{25}$. This proves that every two vertices in A have the same neighbours in C, thereby implying the validity of (21).

We now distinguish two cases:

1. $S_2 \circledcirc I_{12}$
   If $S_{11} = \emptyset$ then (Q, U, J) with

$$
\begin{aligned}
Q &= (C \cap N(A)) \cup \{v, x\},\\
U &= A \cup (C \setminus N(A)),\\
J &= (I \setminus A) \cup (S - x)
\end{aligned}
$$

   is a valid probe split partition: It is clear that Q is a clique. By (21) U is an independent set and $Q \circledast U$. Since it is assumed that $I_{12} \circledcirc S_2$ and $S_2 \circledcirc I_{12}$, J is an independent set.

   So, assume that $S_{11} \neq \emptyset$. Since $v \circledast C$, by Lem. 19, $y \circledast C$. If $A = \{y\}$, then $S_2 \circledcirc I_{11}$ and thus (Q, U, J) with

$$
\begin{aligned}
Q &= C \cup \{v, y\},\\
U &= S_{11} + x,\\
J &= (I - y) \cup S_{12} \cup S_2
\end{aligned}
$$

   is a valid probe split partition.

   So, assume that $A - y \neq \emptyset$, then, by Lem. 20, $I_2 \circledcirc S_{11}$. Note that $A \circledast C$ both when $y \in A$ and when $y \notin A$: If $y \in A$ then this follows from $y \circledast C$ and (21); if $y \notin A$ and $a \in A$ is non-adjacent to $c \in C$ and adjacent to $s \in S_2$ then $avcys$ together with a vertex from S_{11} induce a $\mathfrak{A}_{23}$. Now we can show that (Q, U, J) with

$$
\begin{aligned}
Q &= C \cup \{v, x\},\\
U &= A + y,\\
J &= (S - x) \cup (I \setminus (A + y))
\end{aligned}
$$

   is a valid probe split partition (note that y may or may not be in A): We have demonstrated that $A \circledast C$ and $y \circledast C$ and thus $Q \circledast U$. By assumption $I_2 \circledcirc S_{12}$ and $S_2 \circledcirc I_{12}$, we have demonstrated that $I_2 \circledcirc S_{11}$, and thus J is an independent set.

2. $S_2 \not\circledcirc I_{12}$
   By Lem. 23 we may assume that $I_2 \circledcirc (S_{11} + x)$. We first show

$$
\forall i_{12} \in I_{12} \cap N(S_2) \ \ \forall z \in I_{11} + y : N(i_{12}) \cap C \subseteq N(z) \cap C. \tag{22}
$$

   Let $i_{12} \in I_{12}$, $z \in I_{11} + y$, $c \in C$ and assume $i_{12}c \in E$ but $zc \notin E$. Let $s_2 \in S_2$ be a neighbour of i_{12}. First consider the case $z \in A$ and let $s_2' \in S_2$ be a neighbour of z. Then either $vcxzi_{12}$ and s_2 or s_2' induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$) (if $zs_2 \in E$ or $i_{12}s_2' \in E$) or $vcxzi_{12}s_2s_2'$ induce a $\mathfrak{A}_{27}$ (otherwise). Next consider the case $z \notin A$. Let $a \in A$ with a neighbour $s_2' \in S_2$ (by assumption $A \neq \emptyset$), then by the previous case $ac \in E$. But then either $vcxazi_{12}$ and s_2 or s_2' induce a $\mathfrak{A}_{28}$ (if $as_2 \in E$ or $i_{12}s_2' \in E$), or $vcxazi_{12}s_2s_2'$ induce a $\mathfrak{A}_{30}$ (otherwise). Thus (22) holds.

   Next we show

$$
\begin{aligned}
B &\circledcirc C \setminus N(A),\\
B &\circledast C \cap N(A),\\
I_{11} \neq \emptyset &\longrightarrow (I_{12} \cap N(S_2)) \circledast (C \cap N(A)).
\end{aligned} \tag{23}
$$

Suppose $z \in B$ is adjacent to c_A (remember that $C \setminus N(A) = \{c_A\}$). Let $a \in A$ and let $s_2 \in S$, $i_{12} \in I_{12}$ be neighbours. By (22), $c_A i_{12} \notin E$ and thus $vcxzi_{12}s_2$ induce a $\mathfrak{A}_{23}$. This proves the first statement.

Let $a \in A$, $b \in B$, $c \in C - c_A$ and assume $cb \notin E$. Let $s_2' \in S_2$ and $i_{12} \in I_{12}$ be neighbours. If $i_{12}c \notin E$ then either $vxcai_{12}s_2'$ induce a $\mathfrak{A}_{23}$ (if $as_2' \notin E$) or $vxcabi_{12}s_2'$ induce a $\mathfrak{A}_{25}$ (if $as_2' \in E$). If $i_{12}c \in E$ then $vxcabi_{12}s_2'$ induce a $\mathfrak{A}_{28}$ (if $as_2' \in E$) or $vxcabi_{12}s_2's_2$ induce a $\mathfrak{A}_{30}$ (if $as_2' \notin E$ and with $s_2 \in S_2$ a neighbour of a). This proves the second statement.

Finally assume $i_{12} \in I_{12} \cap N(S_2)$ with neighbour $s_2 \in S_2$ is non-adjacent to $c \in C \cap N(A)$. We will show that $I_{11} = \emptyset$. Let $i_{11} \in I_{11}$, then, by the second statement, $yc \in E$ and $i_{11}c \in E$. If $ys_2 \notin E$ or $i_{11} \notin E$ then $vxcs_2i_{12}y$ or $vxcs_2i_{12}i_{11}$, respectively, induce a $\mathfrak{A}_{23}$; otherwise $vxcs_2i_{12}yi_{11}$ induce a $\mathfrak{A}_{18}$. This finishes the proof of (23).

Note that (23) implies that all vertices in $I_{11} + y$ are adjacent to $C - c_A$ and non-adjacent to c_A. The last intermediate result we need is

$$\text{either } (I_{12} \cap N(S_2)) \circledast (C \cap N(y)) \quad \text{or } (S_2 \cap N(I_{12})) \circledast y \wedge C \circledast y. \tag{24}$$

Assume $i_{12} \in I_{12} \cap N(S_2)$ is non-adjacent to $c \in C \cap N(y)$, and let $s_2 \in S_2$ be adjacent to i_{12}. First we show that $S_2 \cap N(I_{12}) \circledast y$. Assume $s_2' \in S_2 \cap N(I_{12})$ is non-adjacent to y and let $i_{12}' \in I_{12}$ be a neighbour of s_2'. Now $ys_2 \in E$, $i_{12}'c \in E$ and $i_{12}s_2' \notin E$, because otherwise $vcxy$ and, respectively, $i_{12}s_2$, $i_{12}'s_2'$, $i_{12}s_2'$ would induce a $\mathfrak{A}_{23}$. But this implies $s_2 \neq s_2', i_{12} \neq i_{12}'$ and thus $vcxys_2s_2'i_{12}i_{12}'$ either induce a $\mathfrak{A}_{40}$ (if $i_{12}'s_2 \notin E$) or a $\mathfrak{A}_{45}$ (if $i_{12}'s_2 \in E$). Next we show that $C \circledast y$. Assume not, then $yc' \notin E$ for some $c' \in C$ ($c' \neq c$). If $i_{12}c' \notin E$ then $vxycc'i_{12}s_2$ induce a $\mathfrak{A}_{18}$; if $i_{12}c' \in E$ then $vxyc'i_{12}s_2$ induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$). This finishes the proof of (24).

For the remainder of the proof of the lemma we consider two cases:

(a) $I_{12} \cap N(S_2) \circledast C - c_A$
We show that (Q, U, J) with

$$\begin{aligned} Q &= C - c_A + v, \\ U &= (I_{11} + y) \cup (I_{12} \cap N(S_2)) \cup \{c_A\}, \\ J &= S \cup (I_{12} \setminus N(S_2)) \cup I_2 \end{aligned}$$

is a valid probe split partition (note that c_A might not exist). It is clear that Q is a clique and that J is independent. By (21), (22) and (23), U is independent and by (23) and the case distinction $U \circledast Q$.

(b) $I_{12} \cap N(S_2) \not\circledast (C - c_A)$
By (23), $I_{11} = \emptyset$, and by (24), $(S_2 \cap N(I_{12})) \circledast y$ and $C \circledast y$. Thus (Q, U, J) with

$$\begin{aligned} Q &= C + y, \\ U &= \{v\} \cup (S_2 \cap N(I_{12})), \\ J &= I_2 \cup (S \setminus (S_2 \cap N(I_{12}))) \end{aligned}$$

is a valid probe split partition.

□

Lemma 27 If $I_2 \circledcirc S_{12}$ and $S_2 \circledcirc (I_{11}+y)$, then G is probe split.

Proof:

1. $I_{12} \circledcirc S_2$
   We may assume $v \not\circledcirc C$ because otherwise $(C+v, S \setminus S_2, I \cup S_2)$ is a valid probe split partition. So let $c \in C$ be the single non-neighbour of v in C. Then $S_2 = \emptyset$ and $c \circledcirc I_2$. We may also assume that $S_{11} \neq \emptyset$ because otherwise $(C-c+x, \{c,v\}, I \cup S_{12})$ is a valid probe split partition. We start by proving
   $$\text{if } yc \notin E \text{ then } c \not\circledast I_{11} \wedge (C-c) \circledast (I_{12} \cap N(c)). \tag{25}$$
   First, let $s_{11} \in S_{11}$. If c is non-adjacent to $i_{11} \in I_{11}$, then $vxyci_{11}s_{11}$ induce a $\mathfrak{A}_{26}$ ($\overline{P_6}$). Next, if there are neighbours $s_{11} \in S_{11}$, $i_{12} \in I_{12}$ such that $c'i_{12} \notin E$, for some $c' \in C-c$, then $vxycc'i_{12}s_{11}$ induce a $\mathfrak{A}_{18}$. This proves (25).

   By (25), (Q,U,J) with
   $$\begin{aligned} Q &= C-c+v, \\ U &= S \cup (I_{12} \cap N(c)), \\ J &= I \setminus (I_{12} \cap N(c)) + c \end{aligned}$$
   is a valid probe split partition if $yc \notin E$. So assume $yc \in E$. We may then also assume $I_2 \not\circledcirc S_{11}$ because otherwise $(C-c+x+y, \{c,v\}, I \cup S)$ is a valid probe split partition. We prove that
   $$\text{if } I_{11} \neq \emptyset \text{ then } c \circledcirc (I_{11} \cup I_{12}) \wedge S_{12} = \emptyset. \tag{26}$$
   Let $i_2 \in I_2$ and $s_{11} \in S_{11}$ be neighbours. First, consider a vertex $i_{11} \in I_{11}$ that is adjacent to c. Then either $vxcs_{11}i_2i_{11}$ induce a $\mathfrak{A}_9$ (co-fish) (if $xi_2 \notin E$) or $vxycs_{11}i_2i_{11}$ induce a $\mathfrak{A}_{10}$ (otherwise). Next, assume $i_{12} \in I_{12}$ is adjacent to c. Let $i_{11} \in I_{11}$, then $ci_{11} \notin E$. Now either $vxyci_{11}s_{11}i_2i_{12}$ induce a $\mathfrak{A}_{38}$ (if $xi_2 \notin E$) or $vxycs_{11}i_2s_{12}$ induce a $\mathfrak{A}_{20}$ (if $xi_2 \in E$). Finally assume $s_{12} \in S_{12}$ exists. Let $i_{11} \in I_{11}$, then $ci_{11} \notin E$. Now either $vxyci_{11}s_{11}i_2s_{12}$ induce a $\mathfrak{A}_{38}$ (if $xi_2 \notin E$) or $vxycs_{11}i_2s_{12}$ induce a $\mathfrak{A}_{20}$ (if $xi_2 \in E$). This finishes the proof of (26).

   By (26), $(C-c+v+y, S, I-y+c)$ is a valid probe split partition if $I_{11} \neq \emptyset$. Now we assume $I_{11} = \emptyset$.
   $$\text{If } S_{12} \neq \emptyset \text{ then } x \circledcirc I_2. \tag{27}$$
   Let $s_{12} \in S_{12}$. First, if $i_2 \in I_2 \cap N(S_{11})$ with neighbour $s_{11} \in S_{11}$ is adjacent to c, then $vxycs_{12}i_2s_{11}$ induce either a $\mathfrak{A}_{20}$ (if $ci_2 \notin E$) or a $\mathfrak{A}_{10}$ (if $ci_2 \in E$). Next, let $i_2 \in I_2$ be adjacent to $s_{11} \in S_{11}$ and assume $i_2' \in I_2$ is non-adjacent to x. But then $vxycs_{12}i_2s_{11}i_2'$ induce either a $\mathfrak{A}_{36}$ (if $ci_2, ci_2' \notin E$) or a $\mathfrak{A}_{39}$ (if $ci_2, ci_2' \in E$) or a $\mathfrak{A}_{38}$ (otherwise). This finishes the proof of (27).

   $$\text{If } S_{12} \cup (I_{12} \cap N(c)) \neq \emptyset \text{ then } \exists! s_i \in S_{11} : S_{11} \cap N(I_2) = \{s_i\}. \tag{28}$$
   We already know we may assume $|S_{11} \cap N(I_2)| \geq 1$. Let $t \in S_{12} \cup (I_{12} \cap N(c))$ and $s, s' \in S_{11} \cap N(I_2)$. If s, s' have a common neighbour $i_2 \in I_2$ then $vyctss'i_2$ induce a $\mathfrak{A}_{20}$, if they have private neighbours $i_2, i_2' \in I_2$, respectively, then $vyctss'i_2i_2'$ induce a $\mathfrak{A}_{36}$. This finishes the proof of (28).

   We are now ready to prove the first case of the lemma, distinguishing two cases:

(a) $S_{12} \neq \emptyset$
We claim that (Q, U, J) with

$$\begin{aligned} Q &= C - c + y + s_i, \\ U &= \{v, c\}, \\ J &= (I - y) \cup (S_{11} - s_i + x) \cup S_{12} \end{aligned}$$

is a valid probe split partition. By Lem. 19, Q is a clique and by assumption U is independent. By (27), x is non-adjacent to $J - x$ and by (28), $S_{11} - s_i$ is non-adjacent to I, thus J is independent.

(b) $S_{12} = \emptyset$
If $I_{12} \cap N(c) = \emptyset$, then is $(C - c + v + y, S_{11} - x, I - y + c)$ a valid probe split partition. Otherwise (Q, U, J) with

$$\begin{aligned} Q &= C - c + y + s_i, \\ U &= \{v, c\}, \\ J &= (I - y) \cup (S_{11} - s_i + x) \end{aligned}$$

is a valid probe split partition.

2. $I_{12} \not\oslash S_2$
Since $S_2 \neq \emptyset$, $v \circledast C$. We may assume $I_2 \circledcirc (S_{11} + x)$, because otherwise by Lem. 22, (Q, U, J) with

$$\begin{aligned} Q &= C + v, \\ U &= (S \setminus S_2) \cup (I_{12} \cap N(S_2)), \\ J &= I \setminus (I_{12} \cap N(S_2)) \cup S_2 \end{aligned} \tag{29}$$

is a valid probe split partition. Furthermore, if $S_{11} \neq \emptyset$, then by Lem. 19, $y \circledast C$. Then $C \circledast (I_{12} \cap N(S_2))$, because otherwise $vxyci_{12}s_2$, with neighbours $i_{12} \in I_{12}$ and $s_2 \in S_2$ such that $ci_{12} \notin E$, would induce a $\mathfrak{A}_{23}$, and again (Q, U, J) from (29) is a valid probe split partition. Thus, we may also assume $S_{11} = \emptyset$ and $C \not\circledast (I_{12} \cap N(S_2))$.

First, observe that

$$\forall c \in C : c \not\circledast (I_{12} \cap N(S_2)) \rightarrow cy \notin E : \tag{30}$$

If $i_{12} \in I_{12} \cap N(S_2)$ with neighbour $s_2 \in S_2$ and non-neighbour $c \in C$, then $vxyzi_{12}s_2$ induce a $\mathfrak{A}_{23}$. We now prove

$$\exists! c \in C : (C - c) \circledast (I_{12} \cap N(S_2)) \ \wedge \ c \circledcirc (I_{11} + y) \ \wedge \ c \circledcirc (I_{12} \cap N(S_2)) \ \wedge \ C - c \circledast (I_{11} + y) \tag{31}$$

Let $i_{12} \in I_{12} \cap N(S_2)$ with neighbour $s_2 \in S_2$ and non-neighbour $c \in C$. By (30), $yc \notin E$. If $c' \in C - c$ exists such that $c' \not\circledast (I_{12} \cap N(S_2))$, then $c'y \notin E$ and $vxycc'i_{12}s_2$ either induce a $\mathfrak{A}_{25}$ (if $c'i_{12} \notin E$), or a $\mathfrak{A}_{28}$ (if $c'i_{12} \in E$). Thus, $(C - c) \circledast (I_{12} \cap N(S_2))$. Next, assume c is adjacent to $i_{11} \in I_{11}$. Let $i_{12} \in I_{12}$ be non-adjacent to c with neighbour $s_2 \in S_2$. But then $vxci_{11}i_{12}s_2$ induce a $\mathfrak{A}_{23}$. So $c \circledcirc I_{11}$, and by (30), $c \circledcirc y$. Now assume $i'_{12} \in I_{12} \cap N(S_2)$ exists such that $ci'_{12} \in E$. Let $s'_2 \in S_2$ be adjacent to i'_{12} (possibly $s_2 = s'_2$). If $s_2 = s'_2$ then $xyvci_{12}i'_{12}s_2$ induce a $\mathfrak{A}_{27}$, otherwise $xyvci_{12}i'_{12}s_2s'_2$ induce a $\mathfrak{A}_{19}$. Thus $c \circledcirc I_{12} \cap N(S_2)$. Finally, assume $z \in I_{11}+y$ is non-adjacent to $c' \in C-c$. Then $vxzcc'i_{12}s_2$ induce a $\mathfrak{A}_{28}$. Thus $(C-c) \circledast (I_{11}+y)$, finishing the proof of (31).

We claim that (Q, U, J) with

$$Q = C - c + v,$$
$$U = (I_{11} + y + c) \cup (I_{12} \cap N(S_2)),$$
$$J = S \cup I_2 \cup (I_{12} \setminus N(S_2))$$

is a valid probe split partition. It is clear that Q is a clique. By (31), U is independent and $U \circledast Q$ and by assumption J is independent.

□

Lemma 28 (Prop. 8) If $G - v$ is a probe split graph, then so is G. Moreover, a probe split partition of G can be obtained from a probe split partition of $G - v$ in linear time.

Proof: Lem. 24 to 27 cover all possible relations between I_2 and S_{12} and between S_2 and $I_{11}+y$ and therefore prove that if $G - v$ is probe split, then so is G. The proofs of those lemmas also show how a probe split partition of G can be calculated from a probe split partition of $G - v$. It is straightforward that this can be done in linear time using suitable data structures. □

3 Partitioned probe split graphs

In this section we investigate graphs $G = (N, P, E)$ with a given partition of its vertex set into probes P and non-probes N, with $G[N]$ an independent set.

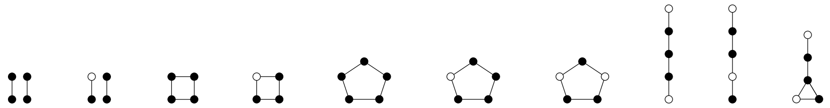
Definition 29 A graph $G = (N, P, E)$, with N an independet set, is a partitioned probe split graph (with respect to the partition $V(G) = N \cup P$) if there exist $E' \subseteq N \times N$ such that $G' = (N \cup P, E \cup E')$ is a split graph.

First we prove a characterisation by 10 forbidden induced subgraphs for partitioned probe split graphs, and then we give a polynomial recognition algorithm.

In this section we shall write X_N, X_P for the intersection of a vertex set X with N, P, respectively in a partitioned probe split graph (N, P, E).

Lemma 30 If $G = (N, P, E)$ is a partitioned probe split graph and (C, S, I) is a valid probe split partition for G, then for all $x \in I$ either $|S \cap N(x)| \leq 1$, or $x \circledast C_P$, or $x \notin N \wedge S_P \cap N(x) = \emptyset$.

Proof: Let $x \in I$ be adjacent to $s_1, s_2 \in S$ and non-adjacent to $c \in C_P$. Then $x \notin N$ because otherwise xcs_1s_2 would induce a $\mathfrak{B}_4$, and s_1 and s_2 are both in N, because otherwise xcs_1s_2 would induce a $\mathfrak{B}_3$ or a $\mathfrak{B}_4$. □



$\mathfrak{B}_1$ $\mathfrak{B}_2$ $\mathfrak{B}_3$ $\mathfrak{B}_4$ $\mathfrak{B}_5$ $\mathfrak{B}_6$ $\mathfrak{B}_7$ $\mathfrak{B}_8$ $\mathfrak{B}_9$ $\mathfrak{B}_{10}$

Fig. 2: Forbidden subgraphs for partitioned probe split graphs (black vertices probes, white vertices non-probes).

Theorem 31 $G = (N, P, E)$ with N an independent set is a partitioned probe split graph iff G has none of the graphs of Fig. 2 as an induced subgraph.

Proof: It is clear that if G is probe split, then G does not contain any of the graphs of Fig. 2 as an induced subgraph. For the other direction it is easy to verify that if G does not contain any of the graphs in Fig. 2, then G does not contain any of the graphs in Fig. 1 either, and thus G is probe split. Let (C, S, I) be a probe split partition of G. We distinguish two cases.

1. $S_P \circledcirc I_P$
   Let
$$A = \{v \in I_N \cap N(S_P) \mid v \circledast C_P\},$$
$$B = I_N \cap N(S_P) \setminus A.$$
   Suppose $B = \emptyset$. If we add all edges between vertices in $N' = S_N \cup A \cup C_N$ then we get a split graph with clique $Q = N' \cup C_P$ and independent set $V(G) \setminus Q$. As $N' \subseteq N$, it follows that G is probe split with respect to the partition (N, P).
   So assume $B \neq \emptyset$. Then
$$S_N \circledcirc I_P : \tag{32}$$
   Let $n \in S_N$ be adjacent to $i \in I_P$ and let $b \in B$ with neighbour $s \in S_P$ and non-neighbour $c \in C_P$. Then $bscni$ induce either a $\mathfrak{B}_9$ (if $ic \notin E$) or a $\mathfrak{B}_{10}$ (if $ic \in E$).
   Moreover
$$|S_P \cap N(B)| \leq 1. \tag{33}$$
   To see this, note first that by Lem. 30 and definition of B, every vertex in B has exactly one neighbour in S_P. Suppose now that vertices $s, s' \in S_P \cap N(B)$ exist ($s \neq s'$), with neighbour $b, b' \in B$ ($b \neq b'$), respectively. Let $c \in C_P$ be non-adjacent to b. Then $b's'csb$ induce either a $\mathfrak{B}_8$ (if $b'c \notin E$) or a $\mathfrak{B}_{10}$ (if $b'c \in E$). Contradiction.
   Let $\{s_B\} = S_P \cap N(B)$. Then
$$s_B \circledast A : \tag{34}$$
   Suppose s_B is non-adjacent to $a \in A$. Let $b \in B$ be non-adjacent to $c \in C_P$ and let $s \in S_P$ be a neighbour of a. Then bs_Bcsa induce a $\mathfrak{B}_{10}$.
   Now, if we add all edges between vertices in $N' = A \cup C_N$ then we get a split graph with clique $Q = N' \cup C_P + s_B$ and independent set $V(G) \setminus Q$. As $N' \subseteq N$, it follows that G is probe split with respect to the partition (N, P). This settles the first case.

2. $S_P \not\circledcirc I_P$
   Let $x \in S_P$, $y \in I_P$ with $xy \in E$ such that $|N(x) \cap I_P| + |N(y) \cap S_P|$ is maximum. Partition $S_P - x$ and $I_P - y$ as:

$$\begin{aligned} S_1 &= (S_P - x) \cap N(y), \\ S_0 &= (S_P - x) \setminus S_1, \\ I_1 &= (I_P - y) \cap N(x), \\ I_0 &= (I_P - y) \setminus I_1. \end{aligned}$$

   Then $S_1 \circledcirc I_1$ (otherwise there would be a $\mathfrak{B}_3$) and $S_0 \circledcirc I_0$ (otherwise there would be a $\mathfrak{B}_1$). Moreover, $S_1 \circledcirc I_0$: If $x_1 \in S_1$ is adjacent to $y_0 \in I_0$, then $I_1 = \emptyset$ (otherwise there would be a $\mathfrak{B}_1$), which implies that $|N(x_1) \cap I_P| + |N(y) \cap S_P| > |N(x) \cap I_P| + |N(y) \cap S_P|$, contradicting the choice of x, y. Similarly, $I_1 \circledcirc S_0$. Let

$$\begin{aligned} A &= \{v \in I_N \cap N(x) \mid v \circledast C_P\}, \\ B &= I_N \setminus A. \end{aligned}$$

   Then

$$A \circledcirc S_1 \quad \wedge \quad B \circledcirc S_0 \quad \wedge \quad S_N \circledcirc I_0 \tag{35}$$

   because if $a \in A$ were adjacent to $s_1 \in S_1$ then $xyas_1$ induce a $\mathfrak{B}_4$; if $b \in B$ were adjacent to $s_0 \in S_0$, then by Lem. 30 and definition of B, $bx \notin E$ and thus $xybs_0$ induce a $\mathfrak{B}_2$; if $s \in S_N$ were adjacent to $i \in I_0$ then $xysi$ induce a $\mathfrak{B}_2$. Furthermore,

$$|(S_1 + x) \cap N(B)| \leq 1 \quad \wedge \quad |(I_1 + y) \cap N(S_N)| \leq 1 : \tag{36}$$

   If $b \in B$ has two neighbours $s, s' \in S_1 + x$, then $ybss'$ induce a $\mathfrak{B}_4$. And if $s, s' \in S_1 + x$ ($s \neq s'$) have different neighbours $b, b' \in B$, respectively, then $bsys'b'$ induce a $\mathfrak{B}_8$. The proof of the second statement is similar.

   We now consider three cases.

   (a) $B \not\circledcirc S_1$
   By (36), we let $S_1 \cap N(B) = \{s_B\}$, then $B \circledcirc S_1 + x - s_B$. Let $b \in B$ be adjacent to s_B. Then

$$A = \emptyset, \quad I_1 = \emptyset, \quad y \circledast C,$$

   because otherwise $axys_Bb$, with $a \in A$, would induce a $\mathfrak{B}_8$; xi_1s_Bb, with $i_1 \in I_1$, would induce a $\mathfrak{B}_2$; and $xycs_B$, with $c \in C$ non-adjacent to y, would induce a $\mathfrak{B}_3$ or a $\mathfrak{B}_4$.

   Then G is a split graph with clique $C + s_B + y$ and independent set $(S - s_B) \cup (I - y)$, and thus trivially probe split.

   (b) $B \circledcirc S_1 \wedge S_N \not\circledcirc (I_1 + y)$
   By (36), we let $(I_1 + y) \cap N(S_N) = \{i_S\}$, then $S_N \circledcirc I_1 + y - i_S$. Then

$$A \circledcirc S_0, \quad i_S \circledast C,$$

because otherwise si_Sxas_0, with $a \in A$ adjacent to $s_0 \in S_0$ and $s \in S_N$ adjacent to i_S, would induce a $\mathfrak{B}_9$; and $xcsi_S$, with $s \in S_N$ adjacent to and $c \in C$ non-adjacent to i_S, would induce a $\mathfrak{B}_4$.

Then G is a split graph with clique $C + i_S + x$ and independent set $(S - x) \cup (I - i_S)$ and thus trivially probe split.

(c) $B \circledcirc S_1 \wedge S_N \circledcirc (I_1 + y)$
First assume that $S_1 \neq \emptyset$. Let $s_1 \in S_1$, then

$$A \circledcirc S_0, \quad y \circledast C,$$

because otherwise as_0s_1y, with $a \in A$ and $s_0 \in S_0$ neighbours, would induce a $\mathfrak{B}_2$; and $xycs_1$, with $c \in C$ non-adjacent to y, would induce a $\mathfrak{B}_3$ or a $\mathfrak{B}_4$.

Then G is a split graph with clique $C + x + y$ and independent set $(S - x) \cup (I - y)$ and thus trivially probe split.

Next assume $S_1 = \emptyset$. If we add all edges between vertices in A then we get a split graph with clique $C + x \cup A$ and independent set $(S - x) \cup (I \setminus A)$. As $A \subseteq N$, if follows that Q is probe split with respect to the partition (N, P).

This settles the second case, and Th. 31 is proved.

□

Together, Lem. 30 and Th. 31 prove that partitioned probe split graphs can be recognized in polynomial time:

Theorem 32 Given $G = (N, P, E)$ with N an independent set, it can be decided in $O(V^2 + VE)$ time whether G is probe split.

Proof: First check with Alg. 1 that G is probe split and, if so, calculate a valid partition (C, S, I). Next check that Lem. 30 holds and work through Th. 31. This can be done in the desired timebound. □

References

[1] G. J. Chang, A. J. J. Kloks, J. Liu, and S. L. Peng. The PIGs full monty - A floor show of minimal separators. In *Proceedings of 22nd Symposium on Theoretical Aspects of Computer Science (STACS) February 24-26, Stuttgart, Germany. Lecture Notes in Computer Science*, volume 3404, pages 521–532, 2005.

[2] G. J. Chang, T. Kloks, and S.-L. Peng. Probe interval bigraphs. In *2nd Brazilian Symposium on Graphs, Algorithms and Combinatorics*, 2005.

[3] M.-S. Chang, T. Kloks, D. Kratsch, J. Liu, and S. L. Peng. On the recognition of probe graphs of some self-complementary classes of perfect graphs. In *Proceedings of the 11th International Computing and Combinatorics Conference (COCOON) August 16-19 2005, Kunming, Yunnan, China, Lecture Notes in Computer Science*, volume 3595, pages 808–817, 2005.

[4] S. Foldes and P. L. Hammer. Split graphs. In *Proc. 8th southeast. Conf. on Combinatorics, graph theory, and computing; Baton Rouge*, pages 311–315. 1977.

[5] M. C. Golumbic and M. Lipshteyn. Chordal probe graphs. *Discrete Appl. Math.*, 143(1-3):221–237, 2004.

[6] P. L. Hammer and B. Simeone. The splittance of a graph. *Combinatorica*, 1:275–284, 1981.

[7] A. Hertz. Slim graphs. *Graphs Comb.*, 5(2):149–157, 1989.

[8] C. T. Hoàng and F. Maffray. On slim graphs, even pairs, and star-cutsets. *Discrete Math.*, 105(1-3):93–102, 1992.

[9] Information System on Graph Classes and their Inclusions (ISGCI). `http://wwwteo.informatik.uni-rostock.de/isgci`.

[10] J. L. Johnson and J. P. Spinrad. A polynomial time recognition algorithm for probe interval graphs. In *Proceedings of the 12th annual ACM-SIAM symposium on discrete algorithms. Washington, DC, USA, January 7-9, 2001. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM)*, pages 477–486. 2001.

[11] R. M. McConnell and J. P. Spinrad. Construction of probe interval models. In *Proceedings of the thirteenth annual ACM-SIAM symposium on discrete algorithms, San Francisco, CA, USA, January 6-8, 2002. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM)*, pages 866–875. 2002.

[12] F. R. McMorris, C. Wang, and P. Zhang. On probe interval graphs. *Discrete Appl. Math.*, 88(1-3):315–324, 1998.

[13] N. Przulj and D. G. Corneil. 2-tree probe interval graphs have a large obstruction set. *Discrete Appl. Math.*, 150(1-3):216–231, 2005.

[14] L. Sheng. Cycle free probe interval graphs. *Congr. Numerantium*, 140:33–42, 1999.

[15] P. Zhang. Probe interval graphs and its application to physical mapping of DNA. Unpublished, 1994.