

Clique-transversal sets and weak 2-colorings in graphs of small maximum degree[†]

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A clique-transversal set in a graph is a subset of the vertices that meets all maximal complete subgraphs on at least two vertices. We prove that every connected graph of order n and maximum degree three has a clique-transversal set of size $\lfloor 19n/30 + 2/15 \rfloor$. This bound is tight, since $19n/30 - 1/15$ is a lower bound for infinitely many values of n . We also prove that the vertex set of any connected claw-free graph of maximum degree at most four, other than an odd cycle longer than three, can be partitioned into two clique-transversal sets. The proofs of both results yield polynomial-time algorithms that find corresponding solutions.

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1 Introduction

We consider finite, simple undirected graphs $G = (V, E)$, with vertex set V and edge set E . The number of vertices will be denoted by n .

There are slight differences in the usage of the term ‘clique’ in graph theory. Throughout this paper, we use *clique* with the following restricted meaning: *inclusion-wise maximal complete subgraph with at least two vertices*. Hence, isolated vertices will *not* be called cliques, and maximality under inclusion will be required.

A *clique-transversal set* is a set $S \subseteq V$ that meets all cliques of G . The smallest cardinality of a clique-transversal set in G , called *clique-transversal number*, is denoted by $\tau_C(G)$. A *weak 2-coloring* of G is a mapping $\phi : V \rightarrow \{r, g\}$ (say red, green) such that both $\phi^{-1}(r)$ and $\phi^{-1}(g)$ are clique-transversal sets. If such ϕ exists, we say that G is *weakly 2-colorable*.

This notion can be extended to *weak k -coloring*, also called *k -clique-coloring* in the literature, which assigns one of k colors to each vertex in such a way that no clique is monochromatic. Equivalently, graph G is *weakly k -colorable* if there exists a partition $V_1 \cup \dots \cup V_k = V$ such that no V_i contains any cliques of G . The smallest nonnegative integer k admitting a weak k -coloring will be denoted by $\chi_C(G)$.

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1.1 Some standard terminology

For $d = 3, 4$ we denote by $\mathcal{G}_d$ the class of *connected* graphs of *maximum degree* at most d . The members of $\mathcal{G}_3$ are the connected *subcubic* graphs, and those 3-regular ones are called *cubic*. The degree of vertex v will be denoted by $d(v)$. A vertex is *simplicial* if its neighbors are mutually adjacent.

Given a “forbidden” graph F , graph G is called *F-free* if no *induced* subgraph of G is isomorphic to F . In the cases of $F = K_3$, $F = K_{1,3}$, and $F = K_4 - e$ (one edge removed from K_4), we use the standard terms *triangle-free*, *claw-free*, and *diamond-free*, respectively. A *hole* is a chordless cycle of length at least four.

1.2 Results and history

The graph invariant τ_C was introduced by Gallai [13] and then studied by various authors. The earliest published results deal with chordal graphs [22], relating τ_C with the number of vertices under the assumption that each edge is contained in some clique of given order. The case of $\tau_C \leq n/2$ has been analyzed for line graphs and their complements [4], and some general bounds on τ_C appeared in [11].

It should be noted that if G is triangle-free, then a set is a clique-transversal set of G if and only if it meets all edges—i.e., it is a vertex cover—therefore $\tau_C(G)$ is equal to the number of vertices minus the independence number of G in this case. This also means that the determination of τ_C is algorithmically hard, on many restricted classes of graphs. Structured hard classes with respect to τ_C can be found in [9, 14, 8], whereas polynomial-time algorithms for other classes are given in [7, 9, 16].

Liang *et al.* [17] proved the following estimates on the clique-transversal number.

1. Every connected cubic graph G of order $n > 4$ has $5n/14 \leq \tau_C(G) \leq 2n/3$.
2. There exist infinitely many cubic connected graphs with $\tau_C(G) = 3n/5$.

The estimates of 1 were proved also in [21], and the graphs attaining the lower bound were characterized in both papers. On the other hand, it remained an open problem to determine a tight upper bound on $\tau_C(G)$ as a function of n . It has been explicitly raised in [17, page 114] whether $\lceil 3n/5 \rceil$ is a valid upper bound. Here we disprove this guess and prove the following estimates.

Theorem 1 *Consider the class $\mathcal{G}_3$ of connected subcubic graphs.*

1. *If $G \in \mathcal{G}_3$ of order n is not cubic, or contains a triangle, then $\tau_C(G) \leq 19n/30 + 1/30$.*
2. *If $G \in \mathcal{G}_3$ of order n is cubic and triangle-free, then $\tau_C(G) \leq 19n/30 + 2/15$.*

These bounds are tight in the sense that there exist infinitely many $G \in \mathcal{G}_3$, say of order n , such that

3. *G is cubic and $\tau_C(G) = 19n/30 - 1/15$.*
4. *G is not cubic and $\tau_C(G) = 19n/30 - 3/10$.*

Moreover, clique-transversal sets of sizes guaranteed in Parts 1 and 2 can be found in polynomial time for any $G \in \mathcal{G}_3$.

Determining $\chi_C(G)$ is hard: to decide $\chi_C = 2$ is NP-complete on 3-chromatic *perfect* graphs [15], and can be even harder: it is Σ_2^P -complete on unrestricted input graphs [19]. On the positive side, all planar graphs have $\chi_C \leq 3$ [20] and $\chi_C = 2$ can be tested in polynomial time if the input is restricted to planar instances [15], hence χ_C on planar graphs can be determined efficiently.

A necessary and sufficient condition for $\chi_C \leq k$ on line graphs was given in [4]. Moreover, claw-free perfect graphs are weakly 2-colorable [6]. It was erroneously stated in [17, page 114] that the upper bound $\tau_C \leq n/2$ implies $\chi_C \leq 2$ for claw-free cubic graphs; later, however, in an unpublished manuscript the authors of [17] gave a proof for weak 2-colorability. Here we extend this latter result by dropping the condition of regularity and also weakening the condition on vertex degrees.

Theorem 2 *Every connected claw-free graph of maximum degree at most four, other than an odd hole, is weakly 2-colorable. Moreover, a weak 2-coloring can be found in polynomial time.*

These results are proved in Sections 2 and 3, respectively. Some related problems are mentioned in the concluding section.

2 Transversal sets

In this section we prove Theorem 1. Let us begin with the proof of tightness, and then proceed with the upper bounds.

Proof of Parts 3 and 4. Locke [18] constructed an infinite family of connected cubic triangle-free graphs with $n := 30k + 22$ vertices and independence number $11k + 8$. Thus, in every such graph G we have

$$\tau_C(G) = 19k + 14 = 19(n - 22)/30 + 14 = 19n/30 - 1/15.$$

If a non-regular connected graph is needed, we omit just one non-cutting vertex. Denoting $n := 30k + 21$ we obtain

$$\tau_C(G) = 19k + 13 = 19(n - 21)/30 + 13 = 19n/30 - 3/10.$$

Proof of Parts 1 and 2. Let $G = (V, E)$ be a subcubic connected graph of order n . Suppose first that G is triangle-free. If G is not 3-regular, we first run the $O(n^4)$ algorithm of Fraughnaugh and Locke [12], which finds an independent set W of size at least $11n/30 - 1/30$ in G . Then the set

$$S := V \setminus W, \quad |S| \leq 19n/30 + 1/30$$

meets all edges of G and hence is a clique-transversal set of required size, found in polynomial time. If G is triangle-free and cubic, then the algorithm in [12] guarantees a slightly weaker lower bound $|W| \geq 11n/30 - 2/15$ on the size of independent set W , and we obtain $|S| \leq 19n/30 + 2/15$ in this case.

Suppose from now on that G contains a triangle, say T with vertex set $\{x_1, x_2, x_3\}$. Each $x_i \in T$ ($i = 1, 2, 3$) has at most one neighbor outside T . We assume $d(x_1) \geq d(x_2) \geq d(x_3)$, and if $d(x_i) = 3$ then denote the neighbor of x_i outside T by y_i .

If $d(x_1) = 2$, then $G \simeq K_3$; and if $d(x_3) = 3$ and $y_1 = y_2 = y_3$, then $G \simeq K_4$. In either case, $\tau_C(G) = 1 \leq n/3$ holds, and we have nothing to prove. Similarly, it is easy to check that $\tau_C(G) \leq n/2$ is valid if $n \leq 4$. Hence, we assume $d(x_1) = 3$ and $n > 4$.

We shall apply induction on n , assuming that the upper bound $\tau_C(G') \leq 19n'/30 + 1/30$ is valid for all non-cubic $G' \in \mathcal{G}_3$ of order $n' < n$. For disconnected subcubic graphs with K components, none of which is cubic, this equivalently means $\tau_C(G') \leq 19n'/30 + K/30$. Note that no proper subgraph of G can have cubic components, because G is connected. The following simple fact will also be useful.

Remark 1 Removing any set U of vertices, the number of components in the remaining graph cannot be larger than the edges connecting U with $V \setminus U$.

We now proceed with the inductive step for the upper bound on τ_C . If $d(x_2) = 2$, then $G - T$ is connected and it has a clique-transversal set S' of size at most $19(n-3)/30 + 1/30$ by the induction hypothesis. Since $S := S' \cup \{x_1\}$ is a clique-transversal set in G , the upper bound $\tau_C(G) \leq 19n/30 - 13/15$ follows.

Suppose $d(x_2) = 3$ and $y_1 \neq y_2$. If $d(x_3) = 2$, or $d(x_3) = 3$ but $y_3 = y_1$ (or $y_3 = y_2$), we consider the graph $G - T - y_1$ (or $G - T - y_2$). Since it has at most three connected components by Remark 1, it contains a clique-transversal set S' of size at most $19(n-4)/30 + 3/30$, and then $S := S' \cup \{y_1, x_2\}$ meets all cliques of G . Thus, $\tau_C(G) \leq 19n/30 + (3 - 76 + 60)/30 = 19n/30 - 13/30$.

Finally, suppose $d(x_3) = 3$ and $y_1 \neq y_2 \neq y_3 \neq y_1$. We now consider $G - T - y_1 - y_2$. By Remark 1 it has at most five connected components. Hence, by the induction hypothesis, it has a clique-transversal set S' of size at most $19(n-5)/30 + 5/30$, and $S := S' \cup \{y_1, y_2, x_3\}$ is a clique-transversal set in G . Thus, $\tau_C(G) \leq 19n/30 + (5 - 95 + 90)/30 = 19n/30$.

Time analysis. Let us choose a polynomial $P(x)$ satisfying the following properties: $P(x)$ is monotone increasing for $x > 0$, $P(n)$ is an upper bound for all n on the running time of the $O(n^4)$ algorithm in [12] for triangle-free subcubic graphs, moreover

$$P(x') + P(x'') \leq P(x' + x'') \quad \text{and} \quad P(x-3) + cx \leq P(x)$$

for all $x', x'' \geq 1$, all $x \geq 4$, and for some constant c to be fixed later. For instance, if $\sum_{i=0}^4 a_i x^i$ is a valid bound for [12], then $P(x) := \sum_{i=0}^4 |a_i| x^i + cx^2$ will do; and any faster algorithm for triangle-free graphs would yield a stronger estimate for the general case, too.

If G is triangle-free, then the algorithm terminates in at most $P(n)$ steps by assumption. Otherwise, triangle T can be found in $c_1 n$ steps for some constant c_1 , e.g. applying breadth-first search and checking at each vertex whether its two descendants (or possibly three for the root vertex) are adjacent or not.

The removal of 3, 4, or 5 vertices takes constant time. Assuming that the remaining graph has connected components of orders $n_1, \dots, n_k$, we need at most $c_2(n_1 + \dots + n_k)$ steps to determine its components and at most $P(n_1) + \dots + P(n_k) \leq P(n_1 + \dots + n_k) \leq P(n-3)$ steps to find the partial clique-transversal set S' . In this way, choosing $c = c_1 + c_2$ we obtain that $P(n)$ is an upper bound on the total running time. $\square$

3 Weak 2-coloring

In this section we prove Theorem 2. Since even cycles are trivial to 2-color, we assume that G is not a cycle. It will turn out that diamond-free graphs admit a more elegant approach than general ones, therefore we treat them first; and afterwards the idea will be to identify a diamond D , find a weak 2-coloring of $G - D$, and prove that it can be extended to a weak 2-coloring of G .

So, assume first that G is connected, claw-free and also diamond-free, has maximum degree at most four, and is not a chordless cycle of length greater than three. Under these conditions we say that G is a *safe graph*. Moreover, Let us call a vertex x *safe* if it satisfies the following requirements:

1. $G - x$ is connected,
2. $G - x$ is not a cycle longer than three,

3. x is either a pendant vertex or contained in a $K_3 \subseteq G$.

For a safe vertex x we define its *critical neighbor* y —whose choice is not always unique—as follows.

- If $d(x) = 1$, then y is the unique neighbor of x .
- If x is in some triangle T_x , let K_x be the (unique) clique containing T_x .
 - If x has neighbor(s) outside K_x , let $y \notin K_x$ be any such neighbor.
 - Otherwise, let $y \in K_x$ be any neighbor of x .

Note that K_x is well-defined because each edge (and hence also each triangle) of G lies in a unique clique, otherwise G would not be diamond-free. For the same reason, x cannot occur in two triangle cliques which share a further vertex. And x cannot be involved in two cliques of size two either, because they would induce a claw with a vertex of T_x . On the other hand, it can happen that x is incident with two edge-disjoint triangles, in this case $T_x = K_x$ can be chosen as any one of them.

We proceed with some properties concerning safe vertices in safe graphs.

Lemma 1 *If x is a safe vertex in a safe graph G , and x is contained in a triangle T_x , then also $K_x - x$ is a clique in $G - x$ for the unique clique K_x containing T_x in G .*

Proof: Otherwise, there is a vertex z adjacent to all vertices of K_x . In this case, xy must be a non-edge, by the maximality of K_x . But then $T_x \cup \{z\}$ induces a diamond, a contradiction. $\square$

Lemma 2 *Every safe graph of order greater than one has a safe vertex.*

Proof: Let G be a safe graph. Suppose first that G has a leaf x . The only safe-vertex-defining condition which could be violated is 2, but then we would find a claw in G . Thus, x is safe.

Assume next that G has no pendant vertices. Then G is not a tree, and it contains a chordless cycle. If this cycle can be chosen with length at least four, then we denote it by C . Since G is not a cycle, there exists some vertex u adjacent to C . Claw-freeness implies that there is an edge $e = xy$ in C such that xyu is a triangle. If $G - x$ is disconnected, then the two neighbors of x on C and a third neighbor in another component of $G - x$ form a claw with center x . Hence, $G - x$ has to be connected, and again it suffices to check whether Condition 2 is valid.

Suppose on the contrary that the graph $G - x$ is a chordless cycle. Let $z \neq y$ be the other neighbor of x on C . In this case, $G - x$ consists of two paths, namely $P := C - x$ from y to z and a $z-u$ path Q , completed to a chordless cycle with edge uy . The neighbors of x are y, u, z , and the neighbor of z on Q . This is the only situation where x violates Condition 2. But then both y and z are safe in G .

Finally, if G has no chordless cycles of length at least four, then G is chordal, by definition. It is a well-known fact that a chordal graph has a simplicial vertex x , which clearly is safe. $\square$

Lemma 3 *Let x be a safe vertex in a safe graph $G = (V, E)$, with critical neighbor y . If $\phi : V \setminus \{x\} \rightarrow \{r, g\}$ is a weak 2-coloring of $G - x$, then $\phi(x) := \{r, g\} \setminus \{\phi(y)\}$ extends it to a weak 2-coloring of G .*

Proof: Suppose on the contrary that some monochromatic clique R occurs in G , say completely red. Of course, $x \in R$ and $|R| \geq 2$. Let W be the complete subgraph $R - x$. This W is not maximal in $G - x$

since ϕ is a weak 2-coloring of $G - x$. Hence, By Lemma 1 we have $|W| = 1$, say $W = \{w\}$. Note that $w \neq y$ because $\phi(w) = \phi(x) \neq \phi(y)$.

Vertex x is not pendant, therefore its K_x is well-defined. Since the edge wx is a clique in G and so it cannot be contained in any triangle, we see that $wy \notin E$, moreover w is not in $K_x - x$.

By Lemma 1, $K_x - x$ is a clique in $G - x$, consequently both y and w have some non-neighbors in $K_x - x$; denote one non-neighbor by y' and w' , respectively. Then yy' is an edge, otherwise $\{x, y, w, w'\}$ would induce a claw. But now $yy' \notin E$ implies $y' \neq w'$ and that $\{x, y, y', w'\}$ induces a diamond, a contradiction. $\square$

Based on these lemmas, we design Algorithm 1 as a subroutine for the general algorithm to find a weak 2-coloring.

Algorithm 1 SAFECOL(G) — Weak 2-coloring of safe graphs

Require: Safe graph $G = (V, E)$.

Ensure: Weak 2-coloring $\phi : V \rightarrow \{r, g\}$.

- 1: **if** $|V| = 1$ **then** {assume $V = \{v\}$ }
 - 2: $\phi(v) := g$
 - 3: **else**
 - 4: Find safe vertex x and its critical neighbor y
 - 5: SAFECOL($G - x$)
 - 6: $\phi(x) := \{r, g\} \setminus \{\phi(y)\}$
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Time analysis for Algorithm 1. Apart from the recursive call in Step 5, the only time-consuming instruction is to identify a safe vertex in Step 4. Efficient implementation is ensured by the following claim.

Lemma 4 *A safe vertex in a safe graph can be found in linear time.*

Proof: The non-cutting vertices x of G can be enumerated in $O(n)$ steps, and since G has bounded maximum degree (and also because it is claw-free), for each x it can be tested in constant time whether or not x is incident with a triangle. Finally, $G - x$ can be a cycle for at most one choice of x . $\square$

Hence, storing the eliminated vertices in a stack, the recursive call of Step 5 (which yields iterated executions of Steps 4 and 6) can be implemented efficiently. As a consequence, Algorithm 1 requires not more than $O(n^2)$ steps.

The following side-product of our method appears to be of interest on its own right, too.

Remark 2 *Since every subgraph of any safe $G \not\cong K_1$ contains a safe vertex, a “safe elimination order” can be determined.*

From now on we suppose that G contains a diamond $D \simeq K_4 - e$. Some cliques of G have vertices in both D and $G - D$; we call them *crossing cliques*. If a crossing clique Q has just one vertex in D , we say that Q is a *strong crossing clique*; and otherwise we say that Q is *weak*.

As for notation, we assume that the diamond D found in G has vertex set $\{c_1, c_2, d_1, d_2\}$, where the only non-edge is $\{c_1, c_2\}$. By the degree assumption, there can occur at most one edge from d_i to $M := G - D$, and at most two edges from c_i to M ($i = 1, 2$). Due to these degree constraints and the assumption that G is claw-free, combinations of the following crossing cliques may occur:

- strong edge: $c_i a_i$ (at most one for each $i \in \{1, 2\}$)
- strong triangle: $c_i b'_i b''_i$ (at most one for each $i \in \{1, 2\}$)
- weak triangle: $c_i d_j w_{i,j}$ (at most one for each pair (i, j))
- weak 4-clique: $c_i d_1 d_2 z_i$ (at most one for each $i \in \{1, 2\}$)

Degree bounds on d_1, d_2 imply that if both $w_{1,j}, w_{2,j}$ exist, then $w_{1,j} = w_{2,j}$; and similarly, if both z_1, z_2 exist, then $z_1 = z_2$. Moreover, weak triangles of type $d_1 d_2 v$ would create a claw, hence are excluded.

The procedure can now be formalized as described in Algorithm 2. The heart of the proof is expressed in the following assertion.

Algorithm 2 CLQCOL(G) — Determination of weak 2-coloring

Require: Claw-free connected graph $G = (V, E)$ of maximum degree at most 4, not a hole.

Ensure: Weak 2-coloring $\phi : V \rightarrow \{r, g\}$.

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1: if  $G$  is diamond-free then  $\{G$  is safe $\}$ 
2:   SAFECOL( $G$ )
3: else
4:   Find diamond  $D$ , label its vertices  $c_1, c_2, d_1, d_2$  such that  $c_1 c_2 \notin E$ 
5:   for all components  $H$  of  $G - D$  do
6:     if  $H$  not a cycle longer than 3 then
7:       CLQCOL( $H$ )
8:     else  $\{$ assume  $H \simeq C_\ell, \ell \geq 4$ , vertices labeled  $x_1, \dots, x_\ell$  sequentially along  $H$  $\}$ 
9:       if  $\ell$  is even then
10:         $\phi(x_i) := g$  for  $i$  odd ( $i = 1, 3, \dots, \ell - 1$ ),  $\phi(x_i) := r$  for  $i$  even ( $i = 2, 4, \dots, \ell$ )
11:       if  $\ell$  is odd then
12:        Find edge  $e \in E(H)$  contained in a crossing clique  $Q$   $\{$ assume  $e = x_1 x_\ell$  $\}$ 
13:         $\phi(x_i) := g$  for  $i$  odd ( $i = 1, 3, \dots, \ell$ ),  $\phi(x_i) := r$  for  $i$  even ( $i = 2, 4, \dots, \ell - 1$ )
14:   Find  $\phi : \{c_1, c_2, d_1, d_2\} \rightarrow \{r, g\}$  with  $\phi(d_1) \neq \phi(d_2)$ , s.t. no monochromatic crossing clique occurs
        $\{$ such  $\phi$  exists; see text $\}$ 

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Lemma 5 *Let $G \in \mathcal{G}_4$ be claw-free, and D a diamond in G . If no component of $G - D$ is an odd hole, then every weak 2-coloring of $G - D$ can be extended to a weak 2-coloring of G in such a way that the two vertices of degree three inside D get distinct colors.*

Proof: Suppose that a weak 2-coloring ϕ of $G - D$ has been fixed. We wish to extend it to the entire G without changing any color in $G - D$; the extension will also be denoted by ϕ .

Once we decide that $\phi(d_1) \neq \phi(d_2)$ holds, all cliques of G with three vertices in D are 2-colored. This includes the triangles of D and the weak 4-cliques, too, if there are any. Therefore, we only have to show that the crossing cliques of orders two and three—strong edge, strong triangle, weak triangle—are 2-colorable under this condition.

A strong crossing clique may determine the color of c_i . Namely, $\phi(c_i) = \{r, g\} \setminus \{\phi(a_i)\}$ must hold in a strong edge, and likewise, $\phi(b'_i) = \phi(b''_i)$ in a strong triangle forces $\phi(c_i) = \{r, g\} \setminus \{\phi(b'_i)\}$. Since each c_i is incident with at most one strong clique, two contradictory conditions of this kind cannot occur at c_i . Moreover, apart from these situations, we have no *a priori* restriction on the colors of c_1 and c_2 .

Suppose first that c_1a_1 is a strong edge. Then c_1 cannot be incident with any crossing triangles: a strong one is impossible by the degree condition, and a weak triangle $c_1d_1w_{1,1}$ would create a claw on $\{c_1, d_2, a_1, w_{1,1}\}$ because c_1a_1 is a clique and hence a_1 cannot be adjacent to any neighbor of c_1 . Consequently, $\phi(c_1) := \{r, g\} \setminus \{\phi(a_1)\}$ yields a 2-coloring for all crossing cliques incident with c_1 . The same argument applies if there is a strong edge c_2a_2 .

The situation is similar and only slightly more complicated if there is a strong triangle, say $c_1b'_1b''_1$. In this case further edges b'_1d_1 and/or b''_1d_2 may be present, creating one or two weak triangles (or weak 4-cliques). If $\phi(b'_1) = \phi(b''_1)$, the choice $\phi(c_i) := \{r, g\} \setminus \{\phi(b'_1)\}$ 2-colors those weak triangles as well, and the proof is done. On the other hand, if $\phi(b'_1) \neq \phi(b''_1)$, then we may disregard the strong triangle because it is already 2-colored, independently of the actual color of c_2 .

From now on we may assume that c_1 and c_2 are contained in weak triangles only. We select one $c_1d_iw_{1,i}$ and one $c_2d_jw_{2,j}$, and define $\phi(c_1) := \{r, g\} \setminus \{\phi(w_{1,i})\}$, $\phi(c_2) := \{r, g\} \setminus \{\phi(w_{2,j})\}$. This leaves at most one monochromatic weak triangle on each of c_1 and c_2 . If such a triangle remains on one of c_1 and c_2 only, then some of $(\phi(d_1), \phi(d_2)) := (g, r)$ and $(\phi(d_1), \phi(d_2)) := (r, g)$ surely makes it 2-colored. In the other case both d_1 and d_2 occur in two weak triangles; but each d_i has only one neighbor in $G - D$, therefore we must have $w_{1,1} = w_{2,1} \neq w_{1,2} = w_{2,2}$. Here $w_{2,1} \neq w_{1,2}$ holds because otherwise two weak 4-cliques would occur instead of four weak triangles.

If $\phi(w_{2,1}) \neq \phi(w_{1,2})$, a simple completion of the coloring is to put $\phi(d_1) := \phi(w_{1,2})$ and $\phi(d_2) := \phi(w_{2,1})$; and if $\phi(w_{2,1}) = \phi(w_{1,2})$, then all the four weak triangles have a vertex of opposite color at c_1 or c_2 , and we obtain a weak 2-coloring by assigning $(\phi(d_1), \phi(d_2)) := (g, r)$. $\square$

Based on Lemma 5, the soundness of Algorithm 2 can be verified easily, although it needs a little case distinction because odd hole components in $G - D$ are not weakly 2-colorable. If a component $H \not\cong K_3$ of $G - D$ is an odd cycle longer than three, however, then any edge connecting H with D has to be extendable to a triangle with two vertices in H , for otherwise a claw would occur. Hence, edge e in Step 10 is well-defined, and it induces a strong triangle with c_1 or c_2 . That is, the situation is the same as if the strong triangle occurred from a non-cycle component, and the argument given in the proof of Lemma 5 verifies that all crossing cliques are 2-colored.

Time analysis for Algorithm 2. As it has been shown, Algorithm 1 called in Step 2 runs within cn_i^2 time on any graph of order n_i , for some absolute constant c . Observe further that, no matter how many times it is performed during the recursive calls of Step 7, the safe subgraphs occurring in the procedure are mutually vertex-disjoint. Consequently, the overall running time of this part of Algorithm 2 does not exceed cn^2 .

Even better, cycles in Steps 8–13 need time proportional to ℓ , and also those cycles are mutually vertex-disjoint. Hence, they require $O(n)$ time altogether. Also, Step 14 requires constant time for D , because only few crossing cliques can occur and they can be enumerated in constant time. These constants sum up to $O(n)$ through all iterations.

Since the vertex degrees are bounded, we need at most $c'n$ time to determine diamond D in Step 4. Also, we can enumerate the components of $G - D$ in Step 5 and check the condition in Step 6 in linear time. Hence, reduction to a smaller problem instance takes linear time. Thus, the overall running time of the algorithm is $O(n^2)$. $\square$

4 Concluding remarks

Here we put a couple of simple observations and mention some problems, which would be of interest for future research.

NP-completeness. From the well-known fact that the independence number is NP-complete to determine on cubic graphs, in connection with Theorem 1 we can derive that the complexity of finding τ_C is NP-complete on triangle-free cubic graphs. The proof can be done in two steps:

- Given a cubic graph $G = (V, E)$, replace each edge $e = xy \in E$ by a path $xv_e w_e y$ of length three. This operation yields a subcubic triangle-free graph H , and increases the independence number by exactly $|E|$.
- Take two copies H', H'' of H and insert the edges $v'_e w''_e$ and $v''_e w'_e$ for all $e \in E$. This results in a cubic triangle-free graph whose independence number is the double of that in H .

Optimum running time. Although our algorithms run in polynomial time, we expect that the orders of those polynomials are not optimal. For this reason, it is natural to ask:

Problem 1 *Determine the best asymptotic running time of an algorithm for*

1. *finding clique-transversal sets of size at most $19n/30 + O(1)$ in connected subcubic graphs,*
2. *finding weak 2-colorings in claw-free graphs of maximum degree four.*

Clique-transversal number vs. clique size. The flavor of results in [22] is that if every edge of a ‘nicely structured’ graph lies in a ‘large’ clique, then τ_C is ‘small’. This direction has been pursued in [3] and recently in [5]. We think that there are many further classes of graphs for which such kind of results would be of interest to study.

Line graphs. The line graph of K_6 is 8-regular and is not weakly 2-colorable. This fact, together with our Theorem 2, leads to the following problem.

Problem 2 *Find the largest integer d such that every claw-free graph of maximum degree d is weakly 2-colorable.*

Perfect graphs. A long-standing open problem of Duffus *et al.* [10] asks whether χ_C is bounded above by a constant on the class of *perfect* graphs. In fact, no examples of perfect graphs G with $\chi_C(G) > 3$ are known. The upper bound $\chi_C \leq 3$ has been proved for some classes of perfect graphs in [6]. Moreover, it is immediate by definition that every *strongly perfect* graph is weakly 2-colorable.

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