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Covering codes in Sierpiński graphs

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For a graph G and integers a and b, an (a, b)-code of G is a set C of vertices such that any vertex from C has exactly a neighbors in C and any vertex not in C has exactly b neighbors in C. In this paper we classify integers a and b for which there exist (a, b)-codes in Sierpiński graphs.

Keywords: codes in graphs, perfect codes, Sierpiński graphs

1 Introduction

In coding theory, a binary code is defined as a subset of $\{0, 1\}^n$. Since a code should at least correct one error, the Hamming distance between any pair of code vertices must be at least 3. In terms of graph theory, we seek for a vertex subset C of the n-cube Q_n, such that $d(u, v) \geq 3$ for any $u, v \in V(Q_n)$, $u \neq v$. (As usual, $d(u, v)$, is the shortest-path distance between u and v.)

The above concepts can be extended from hypercubes to arbitrary graphs. Let G be an arbitrary (connected) graph. Then a subset C of vertices of G is called a 1-*code* (or simply a *code*) if $d(u, v) \geq 3$ holds for any $u, v \in C$, $u \neq v$. Moreover, C is a *perfect code* provided that the closed neighborhoods of elements of C form a partition of V. It was Biggs [2] who initiated the study of perfect codes in distance regular graphs. Kratochvíl with his co-workers follows with the study of codes in general graphs, see the monograph [16], references therein, and [9] for result on the related complexity issues.

Tower of Hanoi graphs model the classical Tower of Hanoi puzzle with 3 pegs and n discs. Their nice fractal structure enables to observe many nice properties. In particular, Cull and Nelson [5] proved that they contain (essentially) unique perfect codes, see also [17]. The Tower of Hanoi graphs extend naturally to graphs $S(n, k)$, $n, k \geq 1$, where $S(n, 3)$ are isomorphic to the graphs of the puzzle with 3 pegs and n discs. The theorem of Cull and Nelson was extended to $S(n, k)$ in [14], see also [8] where in particular shorter arguments are provided.

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In [4], Cohen, Honkala, Lytsin and Mattson introduced a generalization of covering codes using weights and named them *weighted codes*. For a study of small radius weighted coverings see [3]. Independently, Axenovich [1] studied some special cases of perfect weighted codes calling them (t,i,j)-*coverings*.

In order to have a more convenient definition of perfect weighted coverings of radius one, (a,b)-codes were introduced in the following way [6]. Let G be a graph and a,b nonnegative integers. Then a set C of vertices of G is an (a,b)-*code* of G if any vertex from C has exactly a neighbors in C and any vertex from $G \setminus C$ has exactly b neighbors in C. Defined in this way, an (a,b)-code is exactly a perfect $(\frac{b-a}{b}, \frac{1}{b})$-covering as defined in [4]. Moreover, $(1,i,j)$-coverings from [1] are exactly $(i-1,j)$-codes. Finally, Telle defines $[i,j]$-dominating sets in [21] which are exactly (i,j)-codes, see also [7]. We will simply speak about (a,b)-codes when referring to such sets.

Graphs $S(n,k)$, $n,k \geq 1$, form a two-parametric family of graphs of fractal type and have been well-studied by now. (See the next section for their definition and basic properties.) In this paper we give a characterization of the parameters n and k for which $S(n,k)$ admits an (a,b)-code. Since $S(n,1)$, $n \geq 1$, and $S(1,k)$, $k \geq 1$, are of no special interest, let us assume in the rest that $n \geq 2$ and $k \geq 2$. Then our main result is the following.

Theorem 1.1 *Let $n,k \geq 2$. Then $S(n,k)$ contains an (a,b)-code if and only if $a < k$ and one of the following cases holds:*

(i) $a \geq 1$, $b = a$, k *even;*

(ii) $a \geq 2$ *even,* $b = a$, k *odd;*

(iii) $a = 0$, $b = 1$*;*

(iv) $a \geq 1$, $b = a+1$, n *odd;*

(v) $a \geq 1$, $b = a+2$, $n = 2$, $k = 2a+1$.

Note that $(0,1)$-codes coincide with perfect codes. Indeed, if two vertices from a $(0,1)$-code would be at distance 2, then their common neighbor would have two neighbors in the code. Hence Theorem 1.1 (iii) covers the before mentioned result on perfect codes in graphs $S(n,k)$.

We proceed as follow. In the next section we introduce and describe graphs $S(n,k)$. Then, in Section 3, we give necessary conditions on a and b for the existence of (a,b)-codes in graphs $S(n,k)$. In the subsequent sections we construct the claimed (a,a), $(a,a+2)$, and $(a,a+1)$-codes, therefore completing the proof of the Theorem 1.1. We conclude with some ideas for further research.

2 Graphs $S(n,k)$

Graphs $S(n,k)$ were introduced in [13] and later named after Sierpiński in [14]. The motivation for their introduction were topological studies from [19, 20]. For this aspect of the graphs $S(n,k)$ see the recent Lipscomb's book [18], where these graphs are addressed as Klavžar-Milutinović graphs.

Graphs $S(n,k)$ were studied from many different points of view, we have already mentioned perfect codes. Other aspects include $L(2,1)$-labelings [8], crossing numbers [15], and different colorings [12, 11].

The graph $S(n,k)$ $(n,k \geq 1)$ is defined on the vertex set $\{0,1,2,\ldots,k-1\}^n$, two different vertices $u = (i_1,i_2,\ldots,i_n)$ and $v = (j_1,j_2,\ldots,j_n)$ being adjacent if and only if there exists an index h in $\{1,2,\ldots,n\}$ such that

(i) $i_t = j_t$, for $t = 1, \dots, h-1$;
(ii) $i_h \neq j_h$; and
(iii) $i_t = j_h$ and $j_t = i_h$ for $t = h+1, \dots, n$.

In the rest of the paper we will write $\langle i_1 i_2 \dots i_n \rangle$ as short for $(i_1, i_2, \dots, i_n)$. The graphs $S(2,3)$, $S(3,3)$, and $S(2,4)$ are shown in Fig. 1.

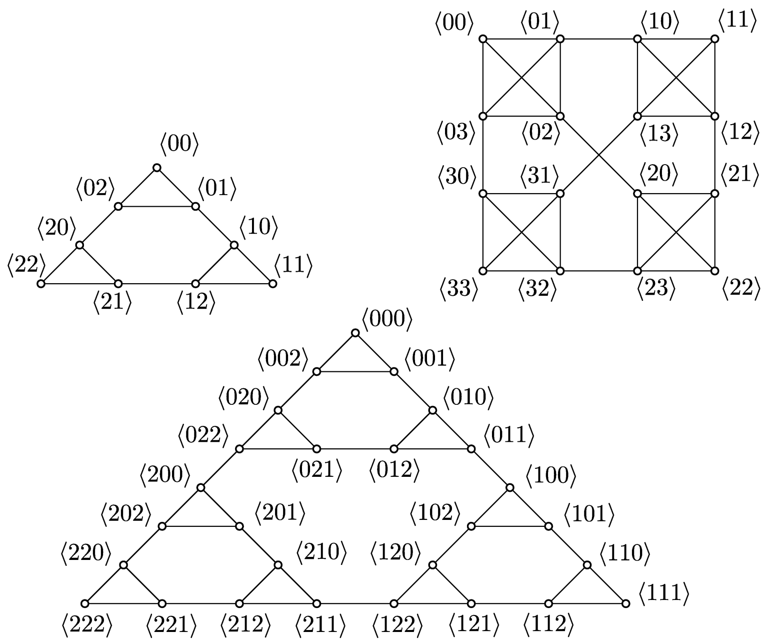

Fig. 1: $S(2,3)$, $S(2,4)$, and $S(3,3)$

Note that the graph $S(n,1)$, $n \geq 1$, has only one vertex, that is, $S(n,1) = K_1$. On the other hand, if $n = 1$, then $S(1,k) = K_k$.

A vertex of the form $\langle ii \dots i \rangle$ of $S(n,k)$ is called *extreme*, other vertices are *inner*. The set of extreme vertices of $S(n,k)$ will be denoted $X(S(n,k))$. Note that the extreme vertices of $S(n,k)$ are of degree $k-1$ and the degree of the inner vertices is k. Note also that $|X(S(n,k))| = k$ and that $|S(n,k)| = k^n$.

Finally, for a vertex $u = \langle i_1 i_2 \dots i_n \rangle$ of $S(n,k)$ let $K(u)$ denote the k-clique induced by vertices $\langle i_1 i_2 \dots i_{n-1} j \rangle$, $1 \leq j \leq k$.

3 Necessary conditions

In this section we restrict the possible couples (a,b) for which (a,b)-codes in graphs $S(n,k)$ are possible.

Note that $S(n,2)$ is isomorphic to the path on 2^n vertices. The only possible (a,b) codes for them are $(0,1)$, $(1,1)$ and $(1,2)$. More precisely, P_{2^n} has a $(0,1)$-code and a $(1,1)$-code for any $n \geq 1$, while it contains a $(1,2)$-code if and only if n is odd. In the latter case, such a code is $1\,101\,101 \cdots 101\,1$, where

1 denotes code vertices. Hence Theorem 1.1 holds for $k = 2$, in particular, item (iv) is covered with the $(1, 2)$-codes for odd n. We will therefore assume in the rest that $k \geq 3$.

Lemma 3.1 *Let $C \neq \emptyset$ be an (a, b)-code in $S(n, k)$. Then $a < k$ and $b > 0$.*

Proof: Suppose $b = 0$. Then it follows that $C = V(S(n, k))$, which is neither a $(k, 0)$-code nor a $(k - 1, 0)$-code. Therefore $b > 0$.

Clearly $a \leq k$. Suppose $a = k$ and let $x \in X(S(n, k))$. Since x has degree $k - 1$, it is not in the code. Consider y a neighbor of x. Since $n \geq 2$, y has degree k, and at least one of them is not in the code (namely x). Therefore, y cannot have a neighbors in the code and y is not in the code. Thus, none of the neighbors of x is in the code and $b = 0$, which is not possible, as shown at the beginning of the proof. □

Lemma 3.2 *Let C be an (a, b)-code in $S(n, k)$ and K_k any of its k-cliques. Then*

$$b - 1 \leq |C \cap K_k| \leq a + 1\,.$$

Proof: If there is a clique K_k with $|C \cap K_k| > a+1$, then any vertex $v \in C \cap K_k$ satisfies $|N(v) \cap C| > a$, which yields a contradiction.

Since $n \geq 2$, $S(n, k)$ is not regular and thus, the code cannot be the whole graph. There exists a vertex u which is not in C. Consider the clique $K(u)$. Then u has at most one neighbor not in $K(u)$ and $|C \cap K(u)| \geq b - 1$ so that $b \leq k$.

Consider a clique K_k. Either there is a vertex v in $K_k \setminus C$ and the preceding study yields $|C \cap K_k| \geq b - 1$, or $K_k \subset C$ and $|C \cap K_k| = k > b - 1$. □

Lemma 3.3 *Let C be an (a, b)-code of $S(n, k)$ with d extreme vertices in C. Then*

$$|C| \cdot (k - a + b) = bk^n + d\,.$$

Proof: Consider the bipartite subgraph B of $S(n, k)$ with bipartition $V_1 = C$ and $V_2 = V(S(n, k)) \setminus C$, keeping only the edges between V_1 and V_2. Every vertex in V_2 has degree b in B. Let v be a vertex in $X(S(n, k)) \cap V_1$. Then its degree in B is $k - a - 1$. Other vertices of V_1 have degree $k - a$. Then counting the number of edges in B in two ways we have

$$|C| \cdot (k - a) - d = (k^n - |C|) \cdot b\,,$$

from which the lemma follows. □

Corollary 3.4 *Let C be an (a, b)-code of $S(n, k)$ without extreme vertices. Then $(k - a + b) | bk^n$.*

Lemma 3.5 *Let C be an (a, b)-code of $S(n, k)$. Then $a \leq b$.*

Proof: If $a = 0$, the statement holds trivially. We now assume $a > 0$.

Suppose first that there exists $u \in X(S(n, k)) \setminus C$. Then u is adjacent to b vertices in $K(u)$. An arbitrary code vertex from $K(u)$ has $b - 1$ neighbors in $K(u) \cap C$ and, maybe, one more in C. Thus $a \leq b$.

Now, suppose $X(S(n,k)) \subset C$. Let u be an extreme vertex of $S(n,k)$. Let $w \neq u$ be a vertex in $C \cap K(u)$, such a vertex exists since $a > 0$. Then the neighbor x of w, $x \notin K(u)$, does not belong to C, for otherwise w would be adjacent to more code vertices than u. Hence x is adjacent precisely to $b-1$ code vertices in $K(x)$.

If $b > 1$, each code vertex of $K(x)$ has at most one other neighbor in $C \setminus K(x)$. Thus $a \leq (b-2)+1 = b-1$.

Else, $b = 1$. Therefore any clique K_k contains either 0, 1 or k vertices of the code. Since $K(u)$ contains u and w, it means that $K(u) \subset C$. Let v be another neighbor of u (we recall that $k \geq 3$), it is in the code and its neighbor y not in $K(u)$ is not for the same reason as x. Since y and x have already one neighbor in the code, we may state that $K(x)$ and $K(y)$ contain no vertex of the code. Consider vertices x' and y' linking these two cliques. They have no neighbor in the code which is impossible. Thus we may conclude that $b > 1$. □

Note that in the proof of Lemma 3.5 we need $k > 2$ when we study the case $b = 1$. However, we have assumed in the beginning of the section that this is indeed the case.

From Lemmas 3.2 and 3.5 we deduce that the only possible (a,b)-codes are (a,a), $(a,a+1)$, and $(a,a+2)$. Next we will obtain some additional necessary conditions for the existence of such codes.

Lemma 3.6 *If a and k are odd then there is no (a,a)-code in $S(n,k)$.*

Proof: Let C be an (a,a)-code in $S(n,k)$ with a odd, and k odd.

Suppose first that there is some $x \in X(S(n,k)) \cap C$. Since a is odd, $a \geq 1$, there exists a vertex $y \in K(x) \cap C$, $y \neq x$. Since $k \geq 3$, there exists another vertex v in $K(x)$. Either it is in C and $a > 1$ or it is not in C and has at least two neighbors in the code so that $a \geq 2$. Let z be the neighbor of y that is not in $K(x)$. Then $z \notin C$, for otherwise y would be adjacent to more code vertices than x. Now, in $K(z)$, z has $a-1$ neighbors from C. Since $a-1 > 0$ we can consider such a vertex. It can have at most $a-1$ adjacent code vertices. It follows that there is no extreme vertex in C.

We next claim that for any vertex u, $|K(u) \cap C| = a$. By way of contradiction, suppose that there is a vertex u such that $|K(u) \cap C| < a$. Then, by Lemma 3.2, we get $|K(u) \cap C| = a-1$. Then $x \in K(u) \cap C$ is adjacent to at most $a-1$ code vertices, a contradiction.

We have thus shown that for every vertex u, $|K(u) \cap C| = a$. Since we have assumed that $k \geq 3$, the number of k-cliques in $S(n,k)$ is k^{n-1}. Thus the above implies that $|C| = a \cdot k^{n-1}$. Because the subgraph induced by C is a-regular with a odd, it means that $|C|$ must be even. But $a \cdot k^{n-1}$ is odd. □

Note that it follows from the above proof that no extreme vertex can belong to an (a,a)-code for any $a \geq 1$.

Lemma 3.7 *If an $(a,a+2)$-code exists in $S(n,k)$, then $n = 2$ and $k = 2a+1$.*

Proof: By Lemma 3.2, $|C \cap K_k| = a+1$ for any k-clique K_k. Thus

$$|C| = k^{n-1} \cdot (a+1)\,.$$

On the other hand, Lemma 3.3 implies that

$$|C| \cdot (k - a + (a+2)) = k^{n-1} \cdot (a+1) \cdot (k+2) = k^n \cdot (a+2) + d\,,$$

where $0 \leq d \leq k$. Dividing both sides of the equality by k^{n-1} we arrive at

$$2a + 2 - k = \frac{d}{k} \cdot \frac{1}{k^{n-2}}\,.$$

By the right-hand side of the last equality we infer that this number is between 0 and 1 (recall that $n \geq 2$), and by the left side of the equality it is an integer, thus it can only be 1 or 0.

The first case implies $n = 2$, $d = k$ and $k = 2a + 1$. The second case is impossible, because it implies $d = 0$, thus extreme vertices would have in this case exactly $a + 1$ neighbors in the code. □

4 Existence results for (a, a) and $(a, a + 2)$

In this section we construct (a, a)- and $(a, a + 2)$-codes. We first observe the following useful fact.

Lemma 4.1 *Suppose there exists an (a, b)-code in $S(2, k)$ that does not include any of the extreme vertices. Then there exists (a, b)-codes in $S(n, k)$ for all $n \geq 3$.*

Proof: Suppose $S(2, k)$ contains an (a, b)-code that includes none of the extreme vertices. We extend this code inductively to $S(n, k)$ for all $n \geq 3$ as follows. Let $n \geq 3$ and suppose that $S(n - 1, k)$ contains an (a, b)-code C that does not include any of its extreme vertices. We extend C to C' by setting $\langle i_1 i_2 \dots i_n \rangle \in C'$ if and only if $\langle i_2 \dots i_n \rangle \in C$. It is now straightforward to see that C' is an (a, b)-code in $S(n, k)$ that does not include any of the extreme vertices. □

Lemma 4.2 *Let $n \geq 2$ and $a < k$. Then an (a, a)-code of $S(n, k)$ exists if and only if*

(i) a is even or

(ii) a is odd and k is even.

Proof: By Lemma 3.6, a and k cannot both be odd if an (a, a)-code exists. Moreover, we have observed that no extreme vertex belongs to such a code. Hence by Lemma 4.1 it suffices to prove the existence of codes for the graphs $S(2, k)$ when at least one of a and k is even. We distinguish two cases.

Case 1. $a = 2p$.
We claim that

$$C = \{ij \mid j = i \pm \varepsilon \ (\mathrm{mod}\ k),\ 0 \leq i \leq k - 1,\ 1 \leq \varepsilon \leq p\}$$

is an (a, a)-code in $S(2, k)$.

Let $\langle ij \rangle \in C$. Then there exists ε, $1 \leq \varepsilon \leq p$, such that $j = i \pm \varepsilon \ (\mathrm{mod}\ k)$. Without loss of generality let $j = i + \varepsilon \ (\mathrm{mod}\ k)$ (the proof follows similar lines if $j = i - \varepsilon \ (\mathrm{mod}\ k)$). Neighbors of ij in $K(\langle ij \rangle) \cap C$ are of the form $\langle il \rangle$ with $l \equiv i \pm \varepsilon' \ (\mathrm{mod}\ k)$, where $\varepsilon' \neq \varepsilon$, $1 \leq \varepsilon' \leq p$, and one more additional neighbor $\langle ii - \varepsilon \rangle$. Since by Lemma 3.1, $a < k$, it follows that $p < k$ and vertex ij has precisely $2p - 1$ neighbors in $K(\langle ij \rangle) \cap C$. Moreover $\langle ij \rangle$ is also adjacent to $\langle ji \rangle \in C$. Therefore $\langle ij \rangle$ has exactly $2p = a$ neighbors in C.

Let $\langle ij \rangle \notin C$. First we observe that the vertex $\langle ii \rangle$ has $2p$ neighbors in C. Second, if $i \neq j$ it follows that $(j - i) \ (\mathrm{mod}\ k) > p$, since $\langle ij \rangle \notin C$. Then $\langle ij \rangle$ has $2p$ neighbors in $K(\langle ij \rangle) \cap C$ and vertex $\langle ji \rangle$

which is the neighbor of $\langle ij\rangle$ not belonging to the clique $K(\langle ij\rangle)$ also does not belong to C. Therefore C is an (a,a)-code in $S(2,k)$.

Case 2. $a = 2p + 1$ and $k = 2q$.
We claim that $C = \{\langle ij\rangle \mid j = i \pm \varepsilon \pmod{k},\ 0 \le i \le k-1,\ 1 \le \varepsilon \le p\} \cup \{\langle il\rangle \mid 0 \le i \le k-1,\ l = i + q \pmod{k}\}$ is an (a,a)-code in $S(n,k)$. By Lemma 3.1, $a < k$ and it follows $2p+1 < 2q$ and therefore $p < q$. Let $ij \in C$. Then $\langle ij\rangle$ has $2p$ neighbors in $K(\langle ij\rangle) \cap C$. Moreover for vertex $\langle ji\rangle$ it follows that $\langle ji\rangle \in C$, since $i + q \equiv i - q \pmod{k}$. Hence $\langle ij\rangle$ has precisely $2p+1 = a$ neighbors in C. Let $\langle ij\rangle \notin C$. Then $\langle ij\rangle$ has precisely $2p+1 = a$ neighbors in $K(\langle ij\rangle) \cap C$ and its neighbor $\langle ji\rangle$ does not belong to C. Therefore C is an (a,a)-code in $S(2,k)$.

Note finally that in both cases C contains no extreme vertex, hence Lemma 4.1 applies. □

Combining Lemma 3.3 with the fact that an (a,a)-code contains no extreme vertices we infer:

Corollary 4.3 *Let C be an (a,a)-code in $S(n,k)$. Then $|C| = a \cdot k^{n-1}$.*

Lemma 4.4 *$S(2, 2a+1)$ contains an $(a, a+2)$-code.*

Proof: Let Q be the complete graph on $k = 2a+1$ vertices and let $1, 2, \ldots, 2a+1$ be its vertices. Then Q is an Eulerian graph and let $v_1, \ldots, v_{a(2a+1)}$ be an Eulerian tour T in Q. Using T we describe an $(a, a+2)$-code C in $S(2,k)$ as follows.

First set $\langle ii\rangle \in C$ for all $i = 0, \ldots, k-1$. In addition, for $j \ne i$ set $\langle ij\rangle \in C$ if and only if there is some t such that $v_t = i$ and $v_{t+1} = j$ (where subscript are modulo $a(2a+1)$). Note that $\langle ji\rangle \notin C$. (The construction is illustrated in Fig. 2 where a $(2,4)$-code is constructed in $S(2,5)$ using an Eulerian tour in K_5.)

Second, let $\langle ij\rangle \notin C$. By definition of an Eulerian tour, the edge ij of Q must belong in one way in T. Since $\langle ij\rangle \notin C$, there must be some t such that $v_t = j$ and $v_{t+1} = i$. Then $\langle ij\rangle$ has $a+1$ neighbors in $C \cap K(\langle ij\rangle)$ plus a neighbor $\langle ji\rangle$.

Therefore C is an $(a, a+2)$-code of $S(2,k)$. □

One may remark that if there is an $(a, a+2)$-code C in $S(2, 2a+1)$ then all extreme vertices must belong to C moreover from C one may exhibit an Eulerian tour of K_{2a+1} (contract all vertices of C belonging to the same $(2a+1)$-clique and consider the orientation of the edges between different cliques of $S(2, 2a+1)$, where an edge is always oriented from a vertex in the code to the vertex not in the code). So all $(a, a+2)$-codes in $S(2, 2a+1)$ are obtained by the construction given in the proof of Lemma 4.4.

Corollary 4.5 *Let C be an $(a, a+2)$-code in $S(2, 2a+1)$. Then $|C| = 2a^2 + 3a + 1$.*

5 Case of $(a, a+1)$-code

To settle the case of $(a, a+1)$-codes in Theorem 1.1, we prove a stronger result. Before we need some additional definitions. Given a set C of vertices of a graph G and a vertex x, let

$$w(x) = \begin{cases} a - |N(x) \cap C|; & x \in C, \\ a + 1 - |N(x) \cap C|; & \text{otherwise}. \end{cases}$$

be the *weight* of x. (The weight of x is also a function of C and a (and of G), but since these will be clear from the context we write simply $w(x)$.) Then we say that a subset C of vertices of $S(n,k)$ is a *near code*

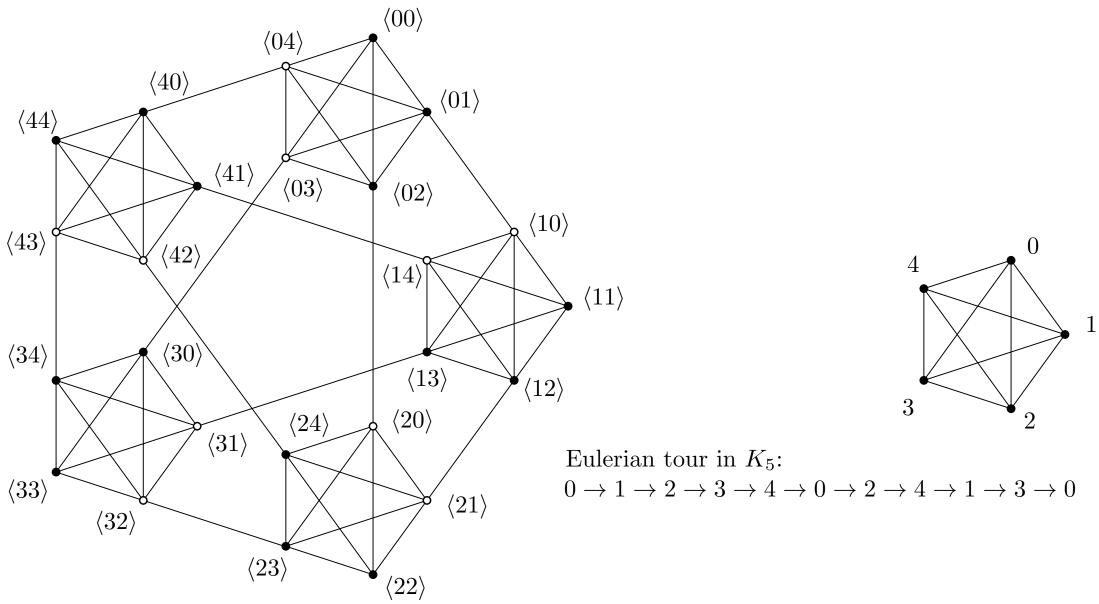

Fig. 2: A $(2, 4)$ code (black vertices) in $S(2, 5)$ obtained via an Eulerian tour in K_5

if $w(x) = 0$ for all $x \in V(S(n, k)) \setminus X(S(n, k))$, and $w(x) \leq 1$ for vertices x in $X(S(n, k))$. Given n and k, we denote $S_i(n, k)$ (for short S_i) the subgraph of $S(n, k)$ induced by $\{\langle ii_2 \dots i_n \rangle$ for all $i_2, \dots, i_n \in \{0, \dots k-1\}\}$. The graph S_i is isomorphic to $S(n-1, k)$.

From the definition of the near code and since the only edges between S_i and $G - S_i$ are incident to extreme vertices of S_i, we have that $C \cap V(S_i)$ is a near code in S_i if C is a near code in $S(n, k)$.

We denote by $\bullet$ extreme vertices that lie in the near code and by $\circ$ the other extreme vertices. Furthermore, we add the subscript $*$ for a vertex of weight 0 and the subscript $+$ for weight 1. For example, $\circ_+$ is an extreme vertex which is not in the code and which has a neighbors in the code. Let us now define the following special near codes C:

- SO^n is a near code where n is odd, $a + 1$ many extreme vertices are $\bullet_*$, the other $k - a - 1$ extreme vertices are $\circ_*$.
- WO^n is a near code where n is odd, a many extreme vertices are $\bullet_+$, the other $k - a$ extreme vertices are $\circ_+$
- SE^n is a near code where n is even, a many extreme vertices are $\bullet_+$, the other $k - a$ extreme vertices are $\bullet_*$.
- WE^n is a near code where n is even, $a + 1$ many extreme vertices are $\circ_+$, the other $k - a - 1$ extreme vertices are $\circ_*$

Note that by definition SO^n is an $(a, a + 1)$-code. Now we are ready to state:

Theorem 5.1 *Let $n \geq 1$, $a \geq 0$ and $k > a$ be integers. The near codes of $S(n,k)$ are precisely SO^n and WO^n if n is odd and SE^n and WE^n if n is even.*

Proof: Let C be a near code of $S(n,k)$ with $n \geq 2$. The main argument of the proof is that the only possible matchings between extreme vertices of different subgraphs S_i, $0 \leq i \leq k-1$, are : $\bullet_+ — \bullet_+$, $\circ_* — \circ_*$ and $\bullet_* — \circ_+$. The proof works by induction on n.

For $n = 1$, it is clear that any near code in the clique K_k is a set of a or $a+1$ vertices which give respectively a WO^1 and a SO^1 code.

Further we divide induction step into two cases, according to the parity of n. Assume first that n is odd. Suppose that there is an extreme vertex x of kind $\bullet_+$. Without loss of generality one may assume that x belongs to S_1. By induction hypothesis, $C \cap S_1$ is isomorphic to SE^{n-1}. Now for each extreme vertex $y \neq x$ of S_1 of kind $\bullet_+$ we need an S_i containing a $\bullet_+$ as extreme vertex. Therefore, there are $a-1$ graphs S_i such that $S_i \cap C$ is isomorphic to SE^{n-1}. Similarly, for the $\bullet_*$ extreme vertices of S_1, we need $k-a$ graphs S_j having extreme vertex of kind $\circ_+$ which implies that $S_j \cap C$ is isomorphic to WE^{n-1}.

Now to obtain a near code, each remaining $\bullet_+$ not in an extreme vertex of $S(n,k)$ must be matched together in an arbitrary way. Since there are a graphs S_i containing a extreme vertices $\bullet_+$ and only one edge between two distinct S_i, it implies that each extreme vertex of $S(n,k)$ belonging to some S_i must be $\bullet_+$. Now, the $k-a$ other S_j contain $a+1$ extreme vertices $\circ_+$. If such a vertex is not extreme in $S(n,k)$ it must be matched to a $\bullet_*$ vertex. Since we have only $(k-a)\cdot a$ vertices of kind $\bullet_*$, it follows that each S_j must contain an extreme vertex $\circ_+$ of $S(n,k)$. Hence in this case we get $C = WO^n$.

If C contains a $\circ_+$ the proof goes along similar lines as above.

Suppose next that there is an extreme vertex x of kind $\bullet_*$. Without loss of generality one may assume that x belongs to S_1. By induction hypothesis, $C \cap S_1$ is isomorphic to SE^{n-1}. Now for each extreme vertex of S_1 of kind $\bullet_+$ we need an S_i containing a $\bullet_+$ as extreme vertex. Therefore, there are a additional graphs S_i such that $S_i \cap C$ is isomorphic to SE^{n-1} (altogether there $a+1$ such graphs). Similarly, for the $\bullet_*$ extreme vertex $y \neq x$ of S_1, we need $k-a-1$ graphs S_j having extreme vertex of kind $\circ_+$ which by induction hypothesis implies that $S_j \cap C$ is isomorphic to WE^{n-1}.

To obtain a near code, each remaining $\bullet_+$ not an extreme vertex of $S(n,k)$ must be matched together. Since there are $a+1$ graphs S_i containing a extreme vertices $\bullet^+$ and only one edge between two distinct S_i, it implies that each extreme vertex of $S(n,k)$ belonging to some S_i must be $\bullet_*$. The $k-a-1$ other S_j graphs contain $a+1$ extreme vertices $\circ_+$. If such a vertex is not extreme in $S(n,k)$ it must be matched to a $\bullet_*$ vertex. We have exactly $(k-a-1)\cdot(a+1)$ vertices of kind $\bullet_*$ that are not in $X(S(n,k))$ (and are from $(a+1)$ SE^{n-1} graphs) and the same number of $\circ_+$ vertices - $a+1$ of them in each of $k-a-1$ copies of WE^{n-1}, hence they can be matched together. Also the remaining vertices $\circ_*$ can be matched together where $k-a-1$ of such vertices belong to $X(S(n,k))$. Therefore $C = SE^n$, in particular C is an $(a, a+1)$-code in $S(n,k)$.

If C contains a $\circ_*$ the arguments are similar.

Next let n be even and $n \geq 2$. Suppose that there is an extreme vertex x of the kind $\bullet_*$. Without loss of generality one may assume that x belongs to S_1. By induction hypothesis, $C \cap S_1$ is isomorphic to SO^{n-1}. Now for each extreme vertex $y \neq x$ of S_1 of kind $\bullet_*$ we need an S_i containing a $\circ_+$ as extreme vertex. Therefore, there are a graphs S_i such that $S_i \cap C$ is isomorphic to WO^{n-1}. Similarly, for the $\circ_*$ extreme vertices of S_1, we need $k-a-1$ graphs S_j having extreme vertex of kind $\circ_*$ which implies that $S_j \cap C$ is isomorphic to SO^{n-1}.

To obtain a near code, each of the remaining $\bullet_+$ vertices that are not extreme vertices of $S(n,k)$ must be matched together. Since there are a graphs S_i containing a extreme vertices $\bullet_+$ and only one edge between two distinct S_i. It follows that each extreme vertex of $S(n,k)$ belonging to some S_i, that is of type WO^{n-1} must be $\bullet_+$. Altogether there are a extreme vertices $\bullet_+$ of $S(n,k)$. The remaining unmatched vertices from SO^{n-1} are all $\bullet_*$ and there is precisely one from each component, hence there are $k-a$ such extreme vertices in $S(n,k)$. It follows $C = SE^n$.

When C contains a $\bullet_+$ the proof works similarly.

Suppose finally that there is an extreme vertex x of kind $\circ_*$. Without loss of generality one may assume that x belongs to S_1. By induction hypothesis, $C \cap S_1$ is isomorphic to SO^{n-1}. Now for each extreme vertex y of S_1 of kind $\circ_+$, $y \neq x$, we need an S_i containing a $\circ_+$ as extreme vertex. Therefore, there are $k-a-2$ additional graphs S_i such that $S_i \cap C$ is isomorphic to SO^{n-1} (altogether there $k-a-1$ such graphs). Similarly, for the $\bullet_*$ extreme vertex of S_1, we need $a+1$ graphs S_j having extreme vertex of kind $\bullet_*$ which by induction hypothesis implies that $S_j \cap C$ is isomorphic to WO^{n-1}. Now to obtain a near code, each remaining $\circ_+$ not an extreme vertex of $S(n,k)$ must be matched together. Since we have $k-a-1$ graphs of type SO^{n-1} and in each of those graphs the same number of vertices $\circ_+$, it follows that among extreme vertices of $S(n,k)$ there are $k-a-1$ vertices of type $\circ_+$. Similarly we can conclude that the remaining $a+1$ vertices of $X(S(n,k))$ are $\circ_*$. It follows $C = WE^n$.

Again, if C contains a $\circ_+$ the proof works similarly. Hence we have completed the induction step and hence the proof. □

The recursive construction of an $(1,2)$-code in $S(3,3)$ is illustrated in Fig. 3.

The next consequences are immediate.

Corollary 5.2 *Graphs $S(n,k)$ admits an $(a,a+1)$-code if and only if n is odd and $0 \leq a \leq k-1$.*

Corollary 5.3 *Let C be an $(a,a+1)$-code in $S(n,k)$, where n is an odd number. Then $|C| = (a+1) \cdot \frac{k^n+1}{k+1}$.*

6 Concluding remarks

We have thus characterized integers a,b for which there exist (a,b)-codes in graphs $S(n,k)$. For the case of $(0,1)$-codes, that is, perfect codes, the uniqueness (modulo obvious symmetries) has been proved in [14]. As exhibited in our paper all existing (a,b)-codes of graphs $S(n,k)$, $n \geq 2$, are constructed in a unique way from (a',b')-codes of graphs $S(n-1,k)$ or their weak codes, where the unique means that there are always the same k building blocks of graphs $S(n-1,k)$ with corresponding (weak) codes. (Graphs $S(n-1,k)$ might have, comparing to each other, different types of weak codes, as in the case of $(a,a+1)$-codes, but altogether the set of all k graphs $S(n-1,k)$ with the corresponding (weak) codes is as a set always the same.) Different types of extreme vertices of graphs $S(n-1,k)$ are matched together in $S(n,k)$ in a unique way. As shown in [15], the automorphism group of a graph $S(n,k)$ is isomorphic to the symmetric group $Sym(k)$, where any automorphism of $S(n,k)$ permutes extreme vertices and the corresponding subgraphs $S(n-1,k)$. It follows that also all existing (a,b)-codes in graphs $S(n,k)$ are unique up to symmetries. A more precise description (using labels) of the constructed $(a,a+1)$-codes would be of interest.

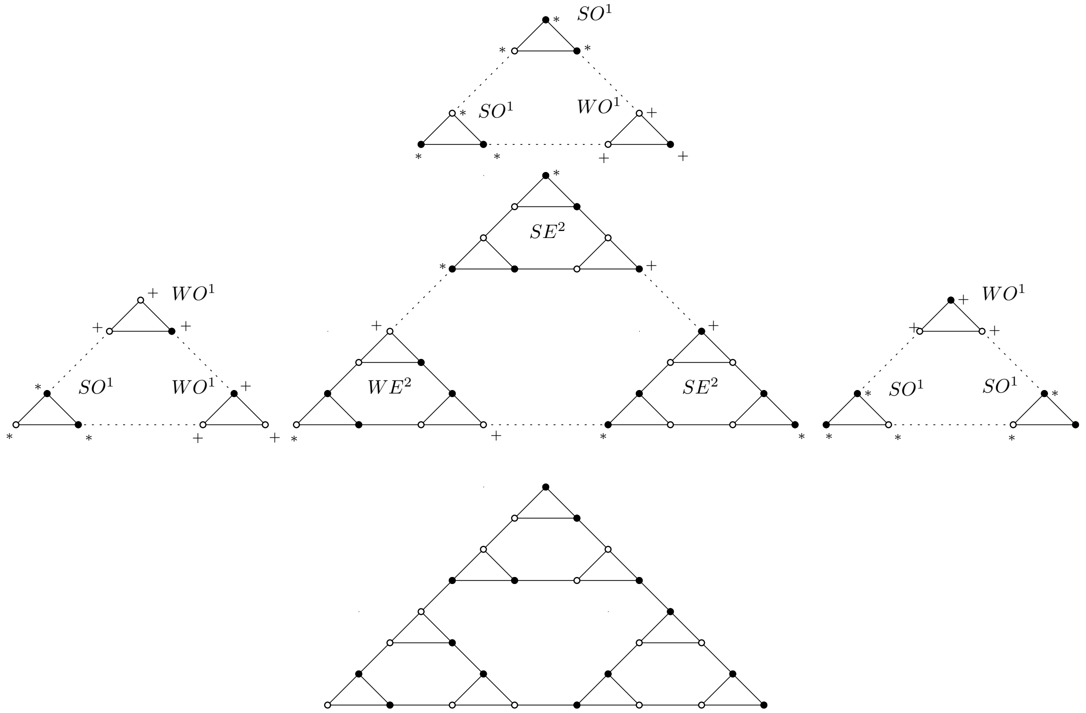

Fig. 3: Recursive construction of an $(1, 2)$-code in Sierpiński graph $S(3, 3)$.

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