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Structure of spanning trees on the two-dimensional Sierpinski gasket

Shu-Chiuan Chang[1] and Lung-Chi Chen[2]

[1]*Department of Physics, National Cheng Kung University, Tainan 70101, Taiwan*
[2]*Department of Mathematics, Fu Jen Catholic University, Taipei 24205, Taiwan*

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Consider spanning trees on the two-dimensional Sierpinski gasket $SG(n)$ where stage n is a non-negative integer. For any given vertex x of $SG(n)$, we derive rigorously the probability distribution of the degree $j \in \{1, 2, 3, 4\}$ at the vertex and its value in the infinite n limit. Adding up such probabilities of all the vertices divided by the number of vertices, we obtain the average probability distribution of the degree j. The corresponding limiting distribution ϕ_j gives the average probability that a vertex is connected by 1, 2, 3 or 4 bond(s) among all the spanning tree configurations. They are rational numbers given as $\phi_1 = 10957/40464$, $\phi_2 = 6626035/13636368$, $\phi_3 = 2943139/13636368$, $\phi_4 = 124895/4545456$.

Keywords: Spanning trees; Sierpinski gasket; exact solutions; limiting distribution; fractal geometry

1 Introduction

The enumeration of the number of spanning trees $N_{ST}(G)$ on a graph G was first considered by Kirchhoff in the analysis of electric circuits [Kirchhoff(1847)]. It is a problem of fundamental interest in mathematics [Biggs(1993), Burton and Pemantle(1993), Lyons(2005), Welsh(1993)] and physics [Temperley(1972), Wu(1977)]. The number of spanning trees corresponds to a special $q \to 0$ limit of the partition function of the q-state Potts model in statistical mechanics [Fortuin and Kasteleyn(1972), Wu(1982)], which in turn is related to the sandpile model [Cori and Borgne(2003), Dhar(2006)]. Just like other limits of the q-state Potts model, the spanning tree problem has been investigated intensely for decades, and has various applications in many areas. See, for example, [Wu and Chao(2004)] and references therein. It is also well known that there is a bijection between close-packed dimer coverings with spanning tree configurations on two related lattices [Temperley(1974)]. Some studies on the enumeration of spanning trees and the calculation of their asymptotic growth constants on regular lattices were carried out in Refs. [Chang and Shrock(2006)], [Chang and Wang(2006)], [Chang(2009)], [Shrock and Wu(2000)], [Tzeng and Wu(2000)]. Once the total number of spanning trees and its asymptotic growth constant is obtained, the next step is to understand the geometric structure of spanning trees. One interesting question is the probability distribution of the degree of a certain vertex among all the spanning trees [Aldous(1991)]. The geometric properties of spanning trees on $\mathbb{Z}^d$ lattices, especially the square lattice, had been considered in [Burton and Pemantle(1993), Manna et al.(1992)].

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Fractals are geometric structures of (generally non-integer) Hausdorff dimension realized by repeated construction of an elementary shape on progressively smaller length scales, see [Falconer(2003)] and [Mandelbrot(1982)]. A well-known example of a fractal is the Sierpinski gasket that has been extensively studied in several contexts: [Alexander(1983)], [Daerden and Vanderzande(1998)], [Dhar and Dhar(1997)], [Dhar(2005)], [Domany et al.(1983)], [Gefen et al.(1980)], [Gefen and Aharony(1981)], [Gefen et al.(1983,1984)], [Guyer(1984)], [Hattori et al.(1990,1992)], [Kozak and Balakrishnan(2002)(a)], [Kozak and Balakrishnan(2002)(b)], [Rammal and Toulouse(1982)], [Chang and Chen (2008)(a)], [Chang and Chen (2008)(b)], [Chang and Chen (2008)(c)], [Chang and Chen (2009)]. Recently, the authors derived rigorously the number of spanning trees on the Sierpinski gasket and conjectured the result for arbitrary dimension [Chang et al.(2007)]. It is of interest to consider the geometric structure of spanning trees on self-similar fractal lattices which have scaling invariance rather than translational invariance. Different from the lattices that have translational invariance, e.g. the square lattice with periodic boundary conditions, the probability distribution of the degree on Sierpinski gasket depends on the vertex location. Thereby, it is natural to investigate the average of the probability distribution of the degree over all the vertices on $SG(n)$ as n tends to infinite, and compare the values with the corresponding results on the infinite square lattice which is also 4-regular. In this paper, we shall present the probability distribution of the degree at any given vertex x on the two-dimensional Sierpinski gasket and the average over all the vertices on $SG(n)$, and the limiting distribution when the number of vertices goes to infinity.

2 Preliminaries

We first recall some relevant definitions for spanning trees and the Sierpinski gasket in this section. A connected graph (without loops) $G = (V, E)$ is defined by its vertex (site) and edge (bond) sets V and E [Biggs(1993), Harary(1969)]. Let $v(G) = |V|$ be the number of vertices and $e(G) = |E|$ the number of edges in G. A spanning subgraph G' is a subgraph of G with the same vertex set V and an edge set $E' \subseteq E$. As a tree is a connected graph with no circuits, a spanning tree on G is a spanning subgraph of G that is a tree and hence $e(G') = v(G) - 1$. The degree or coordination number k_i of a vertex $v_i \in V$ is the number of edges attached to it. A k-regular graph is a graph with the property that each of its vertices has the same degree k. In general, one can associate an edge weight to each edge connecting adjacent vertices v_i and v_j (see, for example [Tzeng and Wu(2000)]). For simplicity, all edge weights are set to one throughout this paper, so that the weight of each spanning tree is the same.

The construction of the two-dimensional Sierpinski gasket $SG(n)$ at stage n is shown in Fig. 1. At stage $n = 0$, it is an equilateral triangle; while stage $n+1$ is obtained by the juxtaposition of three n-stage structures. For the two-dimensional Sierpinski gasket $SG(n)$, the numbers of edges and vertices are given by

$$e(SG(n)) = 3^{n+1} , \qquad v(SG(n)) = \frac{3}{2}(3^n + 1) .$$

Except the three outmost vertices which have degree two, all other vertices of $SG(n)$ have degree four. In the large n limit, SG is 4-regular.

Let us define the notation for the vertices of $SG(n)$ to be used. An illustration for $SG(4)$ is shown in Fig. 2. The denotation of the vertices is given progressively with increasing number of digits in the subscript as follows. First of all, fix o as the leftmost vertex. Consider the $SG(m)$ with $0 \leq m \leq n$ which has o as its leftmost vertex, and denote a_m and b_m as its rightmost and topmost vertices, respectively. c_m is defined such that the vertices a_m, b_m and c_m demarcate the largest lacunary triangle of $SG(m+1)$. We

then define the vertex in the middle of the line connecting a_m and a_{m+1} with $m \geq 1$ as $a_{m,1}$. Similarly, $a_{m,1}$ and the associated $b_{m,1}$ and $c_{m,1}$ demarcate a lacunary triangle with $b_{m,1}$ on the left and $c_{m,1}$ on the right. Next for $m \geq 2$, we append the subscript $m, 1, 0$ for the vertices of the largest lacunary inside the triangle with outmost vertices a_m, $a_{m,1}$, $b_{m,1}$; the subscript $m, 1, 1$ for the vertices of the largest lacunary inside the triangle with outmost vertices $a_{m,1}$, a_{m+1}, $c_{m,1}$; the subscript $m, 1, 2$ for the vertices of the largest lacunary inside the triangle with outmost vertices $b_{m,1}$, $c_{m,1}$, c_m, etc. In general for the vertices of $SG(n)$, we use the notation $x_{\vec{\gamma}}$ where $x = a, b, c$ and the subscript $\vec{\gamma} = (\gamma_1, ..., \gamma_s)$ has s components with $1 \leq s \leq n$, $1 \leq \gamma_1 < n$ and $\gamma_k \in \{0, 1, 2\}$ for $k \in \{2, 3, ..., s\}$. For the vertices above the extended line connecting o and c_0, we will also use the notation $\tilde{x}_{\vec{\gamma}}$ such that it is the reflection of the vertex $x_{\vec{\gamma}}$ with respect to this line. For example, $a_{22} = \tilde{b}_{21}$, $b_{221} = \tilde{a}_{212}$, $c_{222} = \tilde{c}_{211}$, etc. The advantage of such a vertex notation is that the quantities to be studied for the vertices $x_{\gamma_1,...,\gamma_s}$ with $s \geq 2$ components in the subscript can be expressed in terms of the quantities for the vertices with $s - 1$ components in the subscript as shown in Section 5.

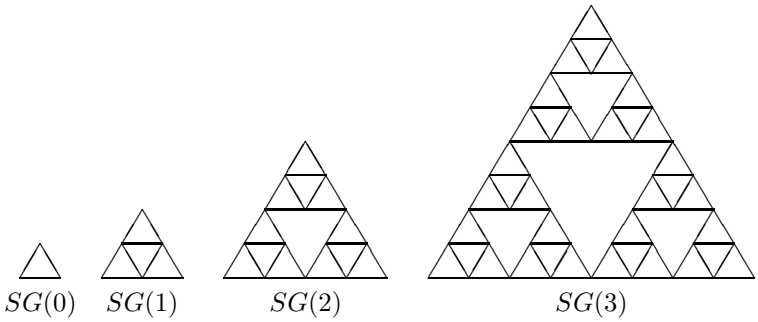

Fig. 1: The first four stages $n = 0, 1, 2, 3$ of the two-dimensional Sierpinski gasket $SG(n)$.

Let us define the following quantities as in [Chang et al.(2007)].

Definition 2.1 *Consider the two-dimensional Sierpinski gasket $SG(n)$ at stage n. (i) Define $f(n) \equiv N_{ST}(SG(n))$ as the number of spanning trees. (ii) Define $g(n)$ as the number of spanning subgraphs with two trees such that the vertex b_n belongs to one tree and the set of vertices $\{o, a_n\}$ belongs to the other tree. (iii) Define $h(n)$ as the number of spanning subgraphs with three trees such that each of the outmost vertices o, a_n and b_n belongs to a different tree.*

Notice that for the spanning subgraph configurations counted by $g(n)$, it is possible that the vertex b_n is an isolated vertex with no bonds of trees connecting to it. A similar statement applies to the outmost vertices o, a_n, b_n for the spanning subgraph configurations counted by $h(n)$. For a given vertex, we would like to investigate the number of bonds of spanning trees connecting to it among all the spanning tree configurations. We have the following definitions.

Definition 2.2 *Consider the two-dimensional Sierpinski gasket $SG(n)$ at stage n. For a certain vertex $x \in V(SG(n))$, the number of bond(s) connecting to it in a spanning tree configuration is denoted as $j \in \{1, 2, 3, 4\}$ or $i \in \{0, 1, 2, 3, 4\}$. (i) Define $f_j(n, x)$ as the number of spanning trees such that there is (are) j bond(s) connecting the vertex x. Define the probability $F_j(n, x) = f_j(n, x)/f(n)$. (ii) Define $g_i(n, x)$ as the number of spanning subgraphs with two trees such that the vertex b_n belongs to one tree*

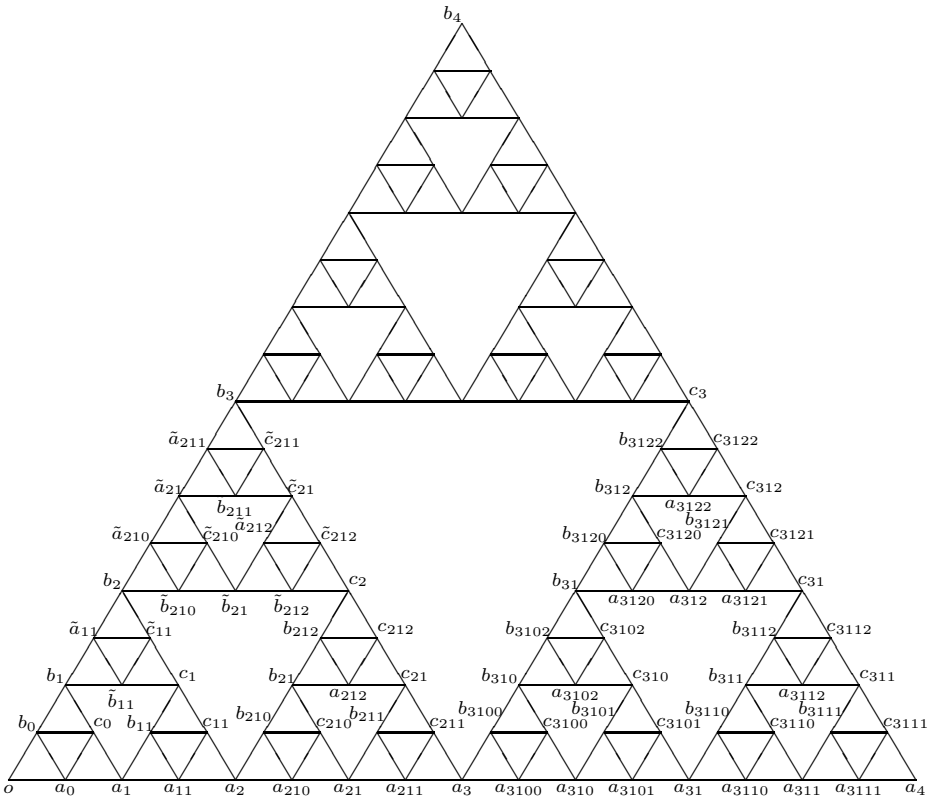

Fig. 2: The notation for the vertices of the Sierpinski gasket $SG(4)$. The vertices $\tilde{x}_{\vec{\gamma}}$ inside the triangle (b_3, c_3, b_4) are reflections of the vertices $x_{\vec{\gamma}}$ inside the triangle (a_3, c_3, a_4) with respect to the line connecting o and c_3, and are not shown.

and the set of vertices $\{o, a_n\}$ belongs to the other tree, and there is (are) i bond(s) connecting the vertex x. Define the probability $G_i(n,x) = g_i(n,x)/g(n)$. (iii) Define $h_i(n,x)$ as the number of spanning subgraphs with three trees such that each of the outmost vertices o, a_n, b_n belongs to a different tree, and there is (are) i bond(s) connecting the vertex x. Define the probability $H_i(n,x) = h_i(n,x)/h(n)$.

For any vertex x of $SG(n)$, the following relations for the probabilities should be satisfied,

$$\sum_{j=1}^{4} F_j(n,x) = \sum_{j=0}^{4} G_j(n,x) = \sum_{j=0}^{4} H_j(n,x) = 1 \ ,$$

which serves as a check for the results obtained.

In this paper, we derive rigorously $F_j(n,x)$ or $f_j(n,x)$ for an arbitrary vertex $x \in V(SG(n))$ with $j = 1,2,3,4$. Such probability on translational invariance lattices in the infinite-vertex limit is independent of the vertex location. In contrast, as the Sierpinski gasket is a self-similar fractal lattice which has scaling invariance rather than translational invariance, our results depend on the location of x. We shall consider the simplest vertex $x = o$ to obtain $F_j(n,o)$ as Theorem 3.1 and its infinite n limit as Corollary 3.1 in Section 3, then move on to the vertices $x \in \{a_m, b_m, c_m\}$ with $0 \le m < n$ to have Theorem 4.1 and Corollary 4.1 in Section 4. $F_j(n,x)$ for the remaining vertices will be treated in Section 5 as Propositions 5.1 and 5.2. The summation and average of all $F_j(n,x)$ for a given stage n will be studied in Section 6, and the average in the infinite n limit will be obtained as Theorem 6.1.

3 $F_j(n,o)$ with $j \in \{1,2\}$

Consider the Sierpinski gasket $SG(n)$ at stage n. We will derive $F_j(n,x)$ for the vertex $x = o$ in this section. Since the leftmost vertex o has degree two, $f_j(n,o) = 0$ for $j = 3,4$ and any $n \ge 0$. Similarly, we only need $i \in \{0,1,2\}$ for $g_i(n,x)$ and $h_i(n,x)$ with $x \in \{o, a_n, b_n\}$. Due to the symmetry of $SG(n)$, we have $f_j(n,o) = f_j(n,a_n) = f_j(n,b_n)$, $g_j(n,o) = g_j(n,a_n)$ with $j = 1,2$, and $h_i(n,o) = h_i(n,a_n) = h_i(n,b_n)$ with $i = 0,1,2$. According to the definition, $g_0(n,o) = g_0(n,a_n) = 0$, but $g_0(n,b_n) \neq 0$ for any $n \ge 0$. In fact, $g_0(n,b_n)$ is the only $g_0(n,x)$ with non-zero value, and $h_0(n,x)$ is non-zero only when $x \in \{o, a_n, b_n\}$. The initial values for $x = o$ at stage $n = 0$ are $f(0) = 3$ with decompositions $f_1(0,o) = 2$ and $f_2(0,o) = 1$, $g(0) = 1$ with decompositions $g_1(0,o) = 1$ and $g_2(0,o) = 0$, $h(0) = 1$ with decompositions $h_0(0,o) = 1$ and $h_j(0,o) = 0$ for $j = 1,2$. We also have $g_0(0,b_0) = 1$ and $g_j(0,b_0) = 0$ for $j = 1,2$.

The following recursion relations were derived in [Chang et al.(2007)] for $n \ge 0$,

$$\left\{ \begin{array}{rcl} f(n+1) & = & 6f(n)^2 g(n) \ , \\ g(n+1) & = & f(n)^2 h(n) + 7f(n)g(n)^2 \ , \\ h(n+1) & = & 12f(n)g(n)h(n) + 14g(n)^3 \ , \end{array} \right. \tag{1}$$

as illustrated in Figs. 3-5. $f(n)$, $g(n)$, $h(n)$ were solved exactly in [Chang et al.(2007)] such that they satisfy the relation $3g(n)^2 = f(n)h(n)$. It follows that the second and third lines of (1) can be simplified to

$$\left\{ \begin{array}{rcl} g(n+1) & = & 10f(n)g(n)^2 = \frac{10}{3} f(n)^2 h(n) \ , \\ h(n+1) & = & 50g(n)^3 = \frac{50}{3} f(n)g(n)h(n) \ . \end{array} \right. \tag{2}$$

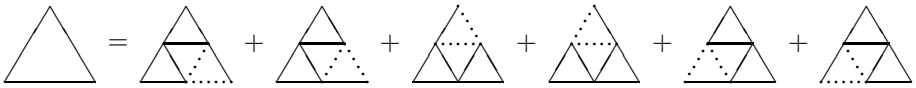

Fig. 3: Illustration for the expression of $f(n+1)$.

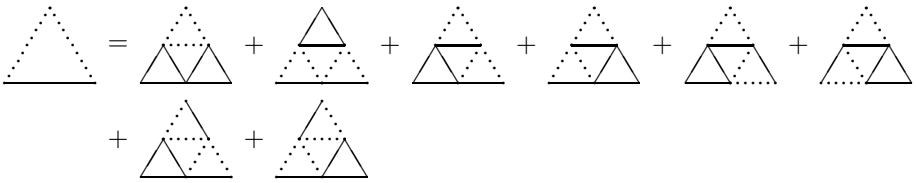

Fig. 4: Illustration for the expression of $g(n+1)$.

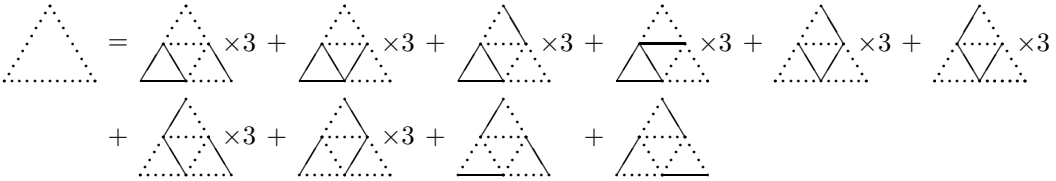

Fig. 5: Illustration for the expression of $h(n+1)$. The multiplication for the eight configurations on the right-hand-side corresponds to three possible orientations.

Using Figs. 3-5 for vertex o, we obtain the following recursion relations for $j = 1, 2$:

$$\begin{cases} f_j(n+1,o) &= 4f_j(n,o)f(n)g(n) + 2g_j(n,o)f(n)^2 \,, \\ g_j(n+1,o) &= f_j(n,o)f(n)h(n) + 3f_j(n,o)g(n)^2 + 4g_j(n,o)f(n)g(n) \,, \end{cases} \tag{3}$$

and

$$\begin{cases} g_j(n+1,b_{n+1}) &= h_j(n,o)f(n)^2 + f_j(n,o)g(n)^2 + 2[g_j(n,o) + 2g_j(n,b_n)]f(n)g(n) \,, \\ h_j(n+1,o) &= 4h_j(n,o)f(n)g(n) + 6g_j(n,b_n)g(n)^2 + 4f_j(n,o)g(n)h(n) \\ & \quad +8g_j(n,o)g(n)^2 + 2[g_j(n,o) + g_j(n,b_n)]f(n)h(n) \,. \end{cases} \tag{4}$$

Setting $f_0(n,o) = 0$ and $g_0(n,o) = 0$, (4) reduces to

$$\begin{cases} g_0(n+1,b_{n+1}) &= h_0(n,o)f(n)^2 + 4g_0(n,b_n)f(n)g(n) \,, \\ h_0(n+1,o) &= 4h_0(n,o)f(n)g(n) + 6g_0(n,b_n)g(n)^2 + 2g_0(n,b_n)f(n)h(n) \,. \end{cases} \tag{5}$$

The initial values for the probabilities are

$$\begin{cases} F_1(0,o) = 2/3 \,,\ F_2(0,o) = 1/3 \,,\ G_1(0,o) = 1 \,,\ G_2(0,o) = 0, \\ H_0(0,o) = 1 \,,\ G_0(0,b_0) = 1 \,,\ G_j(0,b_0) = H_j(0,o) = 0 \text{ with } j = 1,2 \,. \end{cases} \tag{6}$$

Divide the quantities in (3)-(5) by $f(n+1)$, $g(n+1)$ or $h(n+1)$ given in (1) or (2), we get

$$\begin{cases} F_j(n+1,o) &= \frac{2}{3}F_j(n,o) + \frac{1}{3}G_j(n,o) \,, \\ G_j(n+1,o) &= \frac{3}{5}F_j(n,o) + \frac{2}{5}G_j(n,o) \,, \end{cases} \tag{7}$$

$$\begin{cases} G_j(n+1,b_{n+1}) &= \frac{2}{5}G_j(n,b_n) + \frac{3}{10}H_j(n,o) + \frac{1}{10}F_j(n,o) + \frac{1}{5}G_j(n,o) \,, \\ H_j(n+1,o) &= \frac{6}{25}G_j(n,b_n) + \frac{6}{25}H_j(n,o) + \frac{6}{25}F_j(n,o) + \frac{7}{25}G_j(n,o) \,, \end{cases} \tag{8}$$

for $j = 1, 2$, and

$$\begin{cases} G_0(n+1,b_{n+1}) &= \frac{2}{5}G_0(n,b_n) + \frac{3}{10}H_0(n,o) \,, \\ H_0(n+1,o) &= \frac{6}{25}G_0(n,b_n) + \frac{6}{25}H_0(n,o) \,. \end{cases} \tag{9}$$

The probabilities $F_j(n,o)$ and $G_j(n,o)$ with $j = 1, 2$ can be solved exactly by linear algebra with recursion relations (7) and the initial values given in (6).

Theorem 3.1 *For the Sierpinski gasket $SG(n)$ with non-negative integer n,*

$$\begin{aligned} F_1(n,o) &= \frac{11}{14} - \frac{5}{42}\Big(\frac{1}{15}\Big)^n \,, \quad G_1(n,o) = \frac{11}{14} + \frac{3}{14}\Big(\frac{1}{15}\Big)^n \,, \\ F_2(n,o) &= \frac{3}{14} + \frac{5}{42}\Big(\frac{1}{15}\Big)^n \,, \quad G_2(n,o) = \frac{3}{14} - \frac{3}{14}\Big(\frac{1}{15}\Big)^n \,. \end{aligned}$$

From Theorem 3.1 and the exact expressions of $f(n)$, $g(n)$, $h(n)$ in [Chang et al.(2007)], we have the following corollary.

Corollary 3.1 *For the Sierpinski gasket $SG(n)$ with non-negative integer n,*

$$\begin{aligned} f_1(n,o) &= \Big[\frac{11}{14} - \frac{5}{42}\big(\frac{1}{15}\big)^n\Big]\Big[2^{\alpha(n)}3^{\beta(n)}5^{\gamma(n)}\Big] , \\ f_2(n,o) &= \Big[\frac{3}{14} + \frac{5}{42}\big(\frac{1}{15}\big)^n\Big]\Big[2^{\alpha(n)}3^{\beta(n)}5^{\gamma(n)}\Big] , \end{aligned}$$

where $\alpha(n) = \frac{1}{2}(3^n-1)$, $\beta(n) = \frac{1}{4}(3^{n+1}+2n+1)$ and $\gamma(n) = \frac{1}{4}(3^n-2n-1)$. The limiting probabilities for the vertex o are

$$\lim_{n\to\infty} F_1(n,o) = \frac{11}{14} , \qquad \lim_{n\to\infty} F_2(n,o) = \frac{3}{14} .$$

In order to derive the probability $F_j(n,x)$ for an arbitrary vertex $x \neq o$, we need the following lemma:

Lemma 3.1 *For the Sierpinski gasket $SG(n)$ with non-negative integer n,*

$$\begin{aligned} G_0(n,b_n) &= \frac{33}{28}\big(\frac{3}{5}\big)^n - \frac{5}{28}\big(\frac{1}{25}\big)^n , \qquad H_0(n,o) = \frac{11}{14}\big(\frac{3}{5}\big)^n + \frac{3}{14}\big(\frac{1}{25}\big)^n , \\ G_1(n,b_n) &= \frac{11}{14} - \frac{2}{7}\big(\frac{1}{15}\big)^n - \frac{6}{7}\big(\frac{3}{5}\big)^n + \frac{5}{14}\big(\frac{1}{25}\big)^n , \\ G_2(n,b_n) &= \frac{3}{14} + \frac{2}{7}\big(\frac{1}{15}\big)^n - \frac{9}{28}\big(\frac{3}{5}\big)^n - \frac{5}{28}\big(\frac{1}{25}\big)^n , \\ H_1(n,o) &= \frac{11}{14} + \frac{3}{14}\big(\frac{1}{15}\big)^n - \frac{4}{7}\big(\frac{3}{5}\big)^n - \frac{3}{7}\big(\frac{1}{25}\big)^n , \\ H_2(n,o) &= \frac{3}{14} - \frac{3}{14}\big(\frac{1}{15}\big)^n - \frac{3}{14}\big(\frac{3}{5}\big)^n + \frac{3}{14}\big(\frac{1}{25}\big)^n . \end{aligned}$$

The proof of this lemma can be found in the online version of this paper [Chang and Chen].

4 $F_j(n+m+1, x_n)$ with $x \in \{a,b,c\}$ and $n \geq 0$, $m \geq 0$

Consider the Sierpinski gasket $SG(n+m+1)$ with $n \geq 0$, $m \geq 0$. We will derive $F_j(n+1,x_n)$ with $j \in \{1,2,3,4\}$ for the vertex $x_n \in \{a_n,b_n,c_n\}$ first, then $F_j(n+m+1,x_n)$ with arbitrary $m > 0$ in this section. The corresponding $G_j(n+1,x_n)$ and $H_j(n+1,x_n)$ with $j \in \{1,2,3,4\}$ will be used in the next section. Notice that $G_0(n+1,x_n) = H_0(n+1,x_n) = 0$ for $x_n \in \{a_n,b_n,c_n\}$ as these vertices are not outmost vertices of $SG(n+1)$.

For the Sierpinski gasket $SG(n+1)$, we know $f_j(n+1,a_n) = f_j(n+1,b_n) = f_j(n+1,c_n)$ and $h_j(n+1,a_n) = h_j(n+1,b_n) = h_j(n+1,c_n)$ with $j \in \{1,2,3,4\}$ because of rotation symmetry. From the definition of $g_j(n,x)$, we have $g_j(n+1,b_n) = g_j(n+1,c_n)$ but they are distinct from $g_j(n+1,a_n)$. The recursion relations for $f_j(n+1,a_n)$, $g_j(n+1,a_n)$, $g_j(n+1,b_n)$, $h_j(n+1,a_n)$ with $j \in \{1,2,3,4\}$, and the corresponding $F_j(n+1,a_n)$, $G_j(n+1,a_n)$, $G_j(n+1,b_n)$, $H_j(n+1,a_n)$ are lengthy and given in the appendix.

Next consider the Sierpinski gasket $SG(n+m+1)$ with $n \geq 0$ and $m > 0$. The left-hand-sides of Figs. 3-5 now represent $SG(n+m+1)$ with positive integer m, such that $x_n \in \{a_n,b_n,c_n\}$ locates within the lower-left triangle representing $SG(n+m)$ in the right-hand-sides of the figures. As the vertices are

denoted such that $x_{\tilde{\gamma}}$ and $\tilde{x}_{\tilde{\gamma}}$ are reflection of each other with respect to the extended line connecting o and c_0, we have $\tilde{a}_n = b_n$, $\tilde{b}_n = a_n$ and $\tilde{c}_n = c_n$. We obtain the following recursion relations for $j \in \{1,2,3,4\}$:

$$\begin{aligned} F_j(n+m+1,x_n) &= \frac{4f_j(n+m,x_n)f(n+m)g(n+m)}{6f(n+m)^2g(n+m)} \\ &\quad + \frac{[g_j(n+m,x_n)+g_j(n+m,\tilde{x}_n)]f(n+m)^2}{6f(n+m)^2g(n+m)} \\ &= \frac{2F_j(n+m,x_n)}{3} + \frac{G_j(n+m,x_n)+G_j(n+m,\tilde{x}_n)}{6}\,, \end{aligned} \tag{10}$$

and

$$G_j(n+m+1,x_n) = \frac{3F_j(n+m,x_n)}{5} + \frac{3G_j(n+m,x_n)}{10} + \frac{G_j(n+m,\tilde{x}_n)}{10}\,. \tag{11}$$

By symmetry, we know $F_j(n+m,\tilde{x}_n) = F_j(n+m,x_n)$ with $x_n \in \{a_n,b_n,c_n\}$ and $G_j(n+m,\tilde{c}_n) = G_j(n+m,c_n)$ for positive integer m. Let us define the 3×3 matrix

$$B'_j(n+m+1,n) = \begin{bmatrix} F_j(n+m+1,a_n) & G_j(n+m+1,a_n) & G_j(n+m+1,b_n) \\ F_j(n+m+1,b_n) & G_j(n+m+1,b_n) & G_j(n+m+1,a_n) \\ F_j(n+m+1,c_n) & G_j(n+m+1,c_n) & G_j(n+m+1,c_n) \end{bmatrix} \tag{12}$$

for non-negative integer n, m and $j \in \{1,2,3,4\}$. For $m=0$, $B'_j(n+1,n)$ has been obtained with elements given by (36)-(38) in the appendix. By (10) and (11), we have

$$B'_j(n+m+1,n) = B'_j(n+m,n)L'\,,$$

for any $m \geq 1$, where

$$L' = \begin{bmatrix} \frac{2}{3} & \frac{3}{5} & \frac{3}{5} \\ \frac{1}{6} & \frac{3}{10} & \frac{1}{10} \\ \frac{1}{6} & \frac{1}{10} & \frac{3}{10} \end{bmatrix}\,. \tag{13}$$

We arrive at

$$B'_j(n+m+1,n) = B'_j(n+1,n)L'^{m} \quad \text{for all } m \geq 0, n \geq 0\,. \tag{14}$$

Solving $B'_j(n+m+1,n)$ as in the proof of Theorem 3.1, its first column gives

$$\begin{aligned} \begin{bmatrix} F_j(n+m+1,a_n) \\ F_j(n+m+1,b_n) \\ F_j(n+m+1,c_n) \end{bmatrix} &= [\frac{9}{14} + \frac{5}{14}(\frac{1}{15})^m] \begin{bmatrix} F_j(n+1,a_n) \\ F_j(n+1,b_n) \\ F_j(n+1,c_n) \end{bmatrix} \\ &\quad + \frac{5}{28}[1-(\frac{1}{15})^m] \begin{bmatrix} G_j(n+1,a_n)+G_j(n+1,b_n) \\ G_j(n+1,b_n)+G_j(n+1,a_n) \\ 2G_j(n+1,c_n) \end{bmatrix}\,, \end{aligned}$$

and we have the following theorem using (36)-(38).

Theorem 4.1 *For the Sierpinski gasket $SG(n+m+1)$ with non-negative integer n and m,*

$$\begin{cases} F_1(n+m+1,a_n) = \frac{1815}{5488}\left(\frac{3}{5}\right)^n - \frac{99}{1372}\left(\frac{1}{25}\right)^n + \frac{55}{16464}\left(\frac{1}{375}\right)^n \\ \quad + \left(\frac{1}{15}\right)^m\left\{-\frac{121}{5488}\left(\frac{3}{5}\right)^n - \frac{22}{1029}\left(\frac{1}{25}\right)^n + \frac{185}{49392}\left(\frac{1}{375}\right)^n\right\}, \\ F_2(n+m+1,a_n) = \frac{121}{196} - \frac{825}{5488}\left(\frac{3}{5}\right)^n + \frac{171}{1372}\left(\frac{1}{25}\right)^n - \frac{55}{5488}\left(\frac{1}{375}\right)^n - \frac{121}{1176}\left(\frac{1}{15}\right)^n + \frac{1}{294}\left(\frac{1}{225}\right)^n \\ \quad + \left(\frac{1}{15}\right)^m\left\{\frac{55}{5488}\left(\frac{3}{5}\right)^n + \frac{38}{1029}\left(\frac{1}{25}\right)^n - \frac{185}{16464}\left(\frac{1}{375}\right)^n - \frac{143}{3528}\left(\frac{1}{15}\right)^n + \frac{11}{2646}\left(\frac{1}{225}\right)^n\right\}, \\ F_3(n+m+1,a_n) = \frac{33}{98} - \frac{855}{5488}\left(\frac{3}{5}\right)^n - \frac{45}{1372}\left(\frac{1}{25}\right)^n + \frac{55}{5488}\left(\frac{1}{375}\right)^n + \frac{11}{147}\left(\frac{1}{15}\right)^n - \frac{1}{147}\left(\frac{1}{225}\right)^n \\ \quad + \left(\frac{1}{15}\right)^m\left\{\frac{57}{5488}\left(\frac{3}{5}\right)^n - \frac{10}{1029}\left(\frac{1}{25}\right)^n + \frac{185}{16464}\left(\frac{1}{375}\right)^n + \frac{13}{441}\left(\frac{1}{15}\right)^n - \frac{11}{1323}\left(\frac{1}{225}\right)^n\right\}, \\ F_4(n+m+1,a_n) = \frac{9}{196} - \frac{135}{5488}\left(\frac{3}{5}\right)^n - \frac{27}{1372}\left(\frac{1}{25}\right)^n - \frac{55}{16464}\left(\frac{1}{375}\right)^n + \frac{11}{392}\left(\frac{1}{15}\right)^n + \frac{1}{294}\left(\frac{1}{225}\right)^n \\ \quad + \left(\frac{1}{15}\right)^m\left\{\frac{9}{5488}\left(\frac{3}{5}\right)^n - \frac{2}{343}\left(\frac{1}{25}\right)^n - \frac{185}{49392}\left(\frac{1}{375}\right)^n + \frac{13}{1176}\left(\frac{1}{15}\right)^n + \frac{11}{2646}\left(\frac{1}{225}\right)^n\right\}, \end{cases}$$

$$\begin{cases} F_1(n+m+1,c_n) = \frac{1089}{2744}\left(\frac{3}{5}\right)^n - \frac{22}{343}\left(\frac{1}{25}\right)^n + \frac{5}{8232}\left(\frac{1}{375}\right)^n \\ \quad + \left(\frac{1}{15}\right)^m\left\{-\frac{121}{1372}\left(\frac{3}{5}\right)^n - \frac{121}{4116}\left(\frac{1}{25}\right)^n + \frac{20}{3087}\left(\frac{1}{375}\right)^n\right\}, \\ F_2(n+m+1,c_n) = \frac{121}{196} - \frac{495}{2744}\left(\frac{3}{5}\right)^n + \frac{38}{343}\left(\frac{1}{25}\right)^n - \frac{5}{2744}\left(\frac{1}{375}\right)^n - \frac{55}{588}\left(\frac{1}{15}\right)^n \\ \quad + \left(\frac{1}{15}\right)^m\left\{\frac{55}{1372}\left(\frac{3}{5}\right)^n + \frac{209}{4116}\left(\frac{1}{25}\right)^n - \frac{20}{1029}\left(\frac{1}{375}\right)^n - \frac{22}{441}\left(\frac{1}{15}\right)^n + \frac{10}{1323}\left(\frac{1}{225}\right)^n\right\}, \\ F_3(n+m+1,c_n) = \frac{33}{98} - \frac{513}{2744}\left(\frac{3}{5}\right)^n - \frac{10}{343}\left(\frac{1}{25}\right)^n + \frac{5}{2744}\left(\frac{1}{375}\right)^n + \frac{10}{147}\left(\frac{1}{15}\right)^n \\ \quad + \left(\frac{1}{15}\right)^m\left\{\frac{57}{1372}\left(\frac{3}{5}\right)^n - \frac{55}{4116}\left(\frac{1}{25}\right)^n + \frac{20}{1029}\left(\frac{1}{375}\right)^n + \frac{16}{441}\left(\frac{1}{15}\right)^n - \frac{20}{1323}\left(\frac{1}{225}\right)^n\right\}, \\ F_4(n+m+1,c_n) = \frac{9}{196} - \frac{81}{2744}\left(\frac{3}{5}\right)^n - \frac{6}{343}\left(\frac{1}{25}\right)^n - \frac{5}{8232}\left(\frac{1}{375}\right)^n + \frac{5}{196}\left(\frac{1}{15}\right)^n \\ \quad + \left(\frac{1}{15}\right)^m\left\{\frac{9}{1372}\left(\frac{3}{5}\right)^n - \frac{11}{1372}\left(\frac{1}{25}\right)^n - \frac{20}{3087}\left(\frac{1}{375}\right)^n + \frac{2}{147}\left(\frac{1}{15}\right)^n + \frac{10}{1323}\left(\frac{1}{225}\right)^n\right\}. \end{cases}$$

Corollary 4.1 *For the Sierpinski gasket $SG(n+m+1)$ with non-negative integer n and m, the limiting probabilities are*

$$\lim_{n\to\infty} F_1(n+m+1,x_n) = 0\,, \qquad \lim_{n\to\infty} F_2(n+m+1,x_n) = \frac{121}{196}\,,$$
$$\lim_{n\to\infty} F_3(n+m+1,x_n) = \frac{33}{98}\,, \qquad \lim_{n\to\infty} F_4(n+m+1,x_n) = \frac{9}{196}\,,$$

where the vertex x_n can be either a_n, b_n or c_n.

It is intriguing to notice that in Theorem 4.1, $F_j(n+m+1,a_n)$ are distinct from $F_j(n+m+1,c_n)$ with $j \in \{1,2,3,4\}$, while they have the same value in the infinite n limit.

5 $F_j(n+m,x)$ for general $x_{\vec{\gamma}} \in V(SG(n))$ with $n \geq 0$, $m \geq 0$

Consider the Sierpinski gasket $SG(n+m)$ with $n \geq 0, m \geq 0$. We will derive in this section $F_j(n+m,x)$ with $j \in \{1,2,3,4\}$ for the general vertex $x_{\vec{\gamma}} \in V(SG(n))$ that has not been considered in previous sections. For the vertices inside the triangle with outmost vertices a_{n-1}, a_n and c_{n-1}, let us append subscripts in the notation such that $\vec{\gamma}_{n,s} = (\gamma_1 = n-1, \gamma_2, \cdots, \gamma_s)$ with $1 \leq s \leq n$ and $\gamma_k \in \{0,1,2\}$ for $k \in \{2,3,...,s\}$. The results obtained in section 4 correspond to the vertices with $s = 1$ and $n \geq 1$, and we will tackle the vertices with $s > 1$ here. Similar to the definition of the vertex $\tilde{x}_{\vec{\gamma}_{n,s}}$, let us define the vertex $\hat{x}_{\vec{\gamma}_{n,s}}$ as the reflection of $x_{\vec{\gamma}_{n,s}}$ with respect to the line connecting a_n and b_{n-1}. By definition, we have

$$\tilde{\tilde{x}}_{\vec{\gamma}_{n,s}} = x_{\vec{\gamma}_{n,s}}\,, \hat{\hat{x}}_{\vec{\gamma}_{n,s}} = x_{\vec{\gamma}_{n,s}}\,,$$

where x can be either a, b or c, and

$$F_j(n, x_{\vec{\gamma}_{n,s}}) = F_j(n, \tilde{x}_{\vec{\gamma}_{n,s}}) = F_j(n, \hat{x}_{\vec{\gamma}_{n,s}}) \,, \qquad H_j(n, x_{\vec{\gamma}_{n,s}}) = H_j(n, \tilde{x}_{\vec{\gamma}_{n,s}}) = H_j(n, \hat{x}_{\vec{\gamma}_{n,s}}) \,,$$

due to the symmetry of $SG(n)$. For $m \geq 0$, define the 3×5 matrix

$$\begin{aligned} &B_j(n+m, \vec{\gamma}_{n,s}) \\ &= \begin{bmatrix} F_j(n+m, a_{\vec{\gamma}_{n,s}}) & G_j(n+m, a_{\vec{\gamma}_{n,s}}) & G_j(n+m, \tilde{a}_{\vec{\gamma}_{n,s}}) & G_j(n+m, \hat{a}_{\vec{\gamma}_{n,s}}) & H_j(n+m, a_{\vec{\gamma}_{n,s}}) \\ F_j(n+m, b_{\vec{\gamma}_{n,s}}) & G_j(n+m, b_{\vec{\gamma}_{n,s}}) & G_j(n+m, \tilde{b}_{\vec{\gamma}_{n,s}}) & G_j(n+m, \hat{b}_{\vec{\gamma}_{n,s}}) & H_j(n+m, b_{\vec{\gamma}_{n,s}}) \\ F_j(n+m, c_{\vec{\gamma}_{n,s}}) & G_j(n+m, c_{\vec{\gamma}_{n,s}}) & G_j(n+m, \tilde{c}_{\vec{\gamma}_{n,s}}) & G_j(n+m, \hat{c}_{\vec{\gamma}_{n,s}}) & H_j(n+m, c_{\vec{\gamma}_{n,s}}) \end{bmatrix} . \end{aligned}$$

This is a generalization of $B'_j(n+m, n)$ in (12), which corresponds to the case with $s = 1$. By an argument similar to that of (14), we have

$$B_j(n+m, \gamma_{n,s}) = B_j(n, \gamma_{n,s}) L^m \tag{15}$$

for $m \geq 0$, where the 5×5 matrix

$$L = \begin{bmatrix} \frac{2}{3} & \frac{3}{5} & \frac{3}{5} & \frac{3}{5} & \frac{6}{25} \\ \frac{1}{6} & \frac{3}{10} & \frac{1}{10} & 0 & \frac{7}{50} \\ \frac{1}{6} & \frac{1}{10} & \frac{3}{10} & \frac{1}{10} & \frac{7}{50} \\ 0 & 0 & 0 & \frac{3}{10} & \frac{6}{25} \\ 0 & 0 & 0 & 0 & \frac{6}{25} \end{bmatrix}$$

is the generalization of L' in (13). It follows that the determination of $B_j(n, \vec{\gamma}_{n,s})$ for $s > 1$ will be sufficient.

Let us first consider the vertices with $s = 2$ and $\gamma_2 = 1$, namely, $\vec{\gamma}_{n,2} = (n-1, 1)$ with $n = 2, 3, \ldots$. We obtain the following equations (cf. Figs. 3-5):

$$\begin{cases} F_j(n, x_{(n-1,1)}) &= \frac{2F_j(n-1,x_{n-2})}{3} + \frac{G_j(n-1,x_{n-2})}{6} + \frac{G_j(n-1,\hat{x}_{n-2})}{6} \,, \\ G_j(n, x_{(n-1,1)}) &= \frac{3F_j(n-1,x_{n-2})}{5} + \frac{3G_j(n-1,x_{n-2})}{10} + \frac{G_j(n-1,\hat{x}_{n-2})}{10} \,, \\ G_j(n, \tilde{x}_{(n-1,1)}) &= \frac{3H_j(n-1,x_{n-2})}{10} + \frac{F_j(n-1,x_{n-2})}{10} + \frac{2G_j(n-1,\tilde{x}_{n-2})}{5} \\ &\quad + \frac{G_j(n-1,x_{n-2})}{10} + \frac{G_j(n-1,\hat{x}_{n-2})}{10} \,, \\ G_j(n, \hat{x}_{(n-1,1)}) &= \frac{3F_j(n-1,x_{n-2})}{5} + \frac{3G_j(n-1,\hat{x}_{n-2})}{10} + \frac{G_j(n-1,x_{n-2})}{10} \,, \\ H_j(n+1, x_{(n-1,1)}) &= \frac{6F_j(n-1,x_{n-2})}{25} + \frac{7G_j(n-1,x_{n-2})}{50} + \frac{6G_j(n-1,\tilde{x}_{n-2})}{25} \\ &\quad + \frac{7G_j(n-1,\hat{x}_{n-2})}{50} + \frac{6H_j(n-1,x_{n-2})}{25} \,, \end{cases} \tag{16}$$

where x can be either a, b or c. Define the 5×5 matrix

$$R = \begin{bmatrix} \frac{2}{3} & \frac{3}{5} & \frac{1}{10} & \frac{3}{5} & \frac{6}{25} \\ \frac{1}{6} & \frac{3}{10} & \frac{1}{10} & \frac{1}{10} & \frac{7}{50} \\ 0 & 0 & \frac{2}{5} & 0 & \frac{6}{25} \\ \frac{1}{6} & \frac{1}{10} & \frac{1}{10} & \frac{3}{10} & \frac{7}{50} \\ 0 & 0 & \frac{3}{10} & 0 & \frac{6}{25} \end{bmatrix} , \tag{17}$$

then (16) is equivalent to

$$B_j\big(n,(n-1,1)\big) = B_j\big(n-1,n-2\big)R\,. \tag{18}$$

For general $m \geq 0$, we have the following formula combining (15) and (18):

$$B_j\big(n+m,(n-1,1)\big) = B_j\Big(n,(n-1,1)\Big)L^m = B_j\big(n-1,n-2\big)RL^m\,. \tag{19}$$

As $F_j(n+m,x_{n-1,1}) = F_j(n+m,\tilde{x}_{n-1,1})$ for $x = a,b,c$, the first column of the matrix in (19) gives all $F_j(n+m,x_{\vec{\gamma}_{n,2}})$ in terms of the quantities for x_{n-2}.

Proposition 5.1 *For the Sierpinski gasket $SG(n+m)$ with $n \geq 2$, $m \geq 0$,*

$$\begin{bmatrix} F_j(n+m,a_{n-1,1}) \\ F_j(n+m,b_{n-1,1}) \\ F_j(n+m,c_{n-1,1}) \end{bmatrix} = \begin{bmatrix} F_j(n+m,\tilde{a}_{n-1,1}) \\ F_j(n+m,\tilde{b}_{n-1,1}) \\ F_j(n+m,\tilde{c}_{n-1,1}) \end{bmatrix} = B_j\big(n-1,n-2\big)RL^m e_1\,,$$

where

$$e_1 = (1,0,0,0,0)^T$$

and $j \in \{1,2,3,4\}$.

Move on to the general vertex $x_{\vec{\gamma}_{n+1,s}}$ inside the triangle with outmost vertices a_n, a_{n+1} and c_n for the Sierpinski gasket $SG(n+1)$, where $\vec{\gamma}_{n+1,s} = (n,1,\gamma_3,...,\gamma_s)$ with $\gamma_k \in \{0,1,2\}$, $k = 3,4,..,s$ and $3 \leq s \leq n+1$. As γ_3 can take three possible values, let us discuss them separately.

First consider the case with $\gamma_3 = 1$. The vertex $x_{\vec{\gamma}_{n+1,s}} = x_{(n,1,1,\gamma_4,...\gamma_s)}$ is located inside the triangle with outmost vertices $a_{n,1}$, a_{n+1} and $c_{n,1}$. Associate with this $x_{\vec{\gamma}_{n+1,s}}$ a vertex $x_{\vec{\gamma}^1_{n,s-1}}$, where $\vec{\gamma}^1_{n,s-1} = (n-1,1,\gamma_4,...,\gamma_s)$ has $s-1$ components. That is, $\vec{\gamma}^1_{n,s-1}$ is obtained from $\vec{\gamma}_{n+1,s}$ by taking out $\gamma_3 = 1$ and replacing $\gamma_1 = n$ by $n-1$. It can be seen that this vertex $x_{\vec{\gamma}^1_{n,s-1}}$ is located inside the triangle with outmost vertices a_{n-1}, a_n and c_{n-1}. Moreover, $x_{\vec{\gamma}^1_{n,s-1}}$ can be reached from $x_{\vec{\gamma}_{n+1,s}}$ by a horizontal translation with the distance from a_n to o. Particularly, if $s = 3$ such that $\vec{\gamma}_{n+1,s} = (n,1,1)$, then $\vec{\gamma}^1_{n,s-1} = (n-1,1)$. By the method obtaining (18), we have

$$B_j\Big(n+1,\vec{\gamma}_{n+1,s}\Big) = B_j\Big(n,\vec{\gamma}^1_{n,s-1}\Big)R \tag{20}$$

if $\gamma_3 = 1$.

Now consider the case with $\gamma_3 = 2$. The vertex $x_{\vec{\gamma}_{n+1,s}} = x_{(n,1,2,\gamma_4,...\gamma_s)}$ is located inside the triangle with outmost vertices $b_{n,1}$, $c_{n,1}$ and c_n. Associate with this $x_{\vec{\gamma}_{n+1,s}}$ a vertex $\tilde{y}_{\vec{\gamma}^2_{n,s-1}}$ where $\vec{\gamma}^2_{n,s-1} = (n-1,1,\gamma^2_4,...,\gamma^2_s))$ has $s-1$ components. Here $y = b$ when $x = a$ and vice versa, and $y = c$ when $x = c$. Namely, y is related to x with three possibilities: $(x,y) = (a,b)$, (b,a) and (c,c). Similarly, γ^2_k is related to γ_k with three possibilities: $(\gamma_k,\gamma^2_k) = (1,2)$, $(2,1)$ and $(0,0)$, where $k = 4,...,s$. Again, $\tilde{y}_{\vec{\gamma}^2_{n,s-1}}$ can be reached from $x_{\vec{\gamma}_{n+1,s}}$ by a horizontal translation with the distance from a_n to o. We have

$$B_j\Big(n+1,\vec{\gamma}_{n+1,s}\Big) = \tilde{B}_j\Big(n,\vec{\gamma}^2_{n,s-1}\Big)R\,, \tag{21}$$

where

$$\tilde{B}_j\Big(n,\vec{\gamma}^2_{n,s-1}\Big)$$

$$= \begin{bmatrix} F_j(n,\tilde{b}_{\vec{\gamma}^2_{n,s-1}}) & G_j(n,\tilde{b}_{\vec{\gamma}^2_{n,s-1}}) & G_j(n,\tilde{\tilde{b}}_{\vec{\gamma}^2_{n,s-1}}) & G_j(n,\hat{\tilde{b}}_{\vec{\gamma}^2_{n,s-1}}) & H_j(n,\tilde{b}_{\vec{\gamma}^2_{n,s-1}}) \\ F_j(n,\tilde{a}_{\vec{\gamma}^2_{n,s-1}}) & G_j(n,\tilde{a}_{\vec{\gamma}^2_{n,s-1}}) & G_j(n,\tilde{\tilde{a}}_{\vec{\gamma}^2_{n,s-1}}) & G_j(n,\hat{\tilde{a}}_{\vec{\gamma}^2_{n,s-1}}) & H_j(n,\tilde{a}_{\vec{\gamma}^2_{n,s-1}}) \\ F_j(n,\tilde{c}_{\vec{\gamma}^2_{n,s-1}}) & G_j(n,\tilde{c}_{\vec{\gamma}^2_{n,s-1}}) & G_j(n,\tilde{\tilde{c}}_{\vec{\gamma}^2_{n,s-1}}) & G_j(n,\hat{\tilde{c}}_{\vec{\gamma}^2_{n,s-1}}) & H_j(n,\tilde{c}_{\vec{\gamma}^2_{n,s-1}}) \end{bmatrix} \quad (22)$$

By symmetry, the columns in (22) can be replaced as

$$\begin{bmatrix} F_j(n,\tilde{b}_{\vec{\gamma}^2_{n,s-1}}) \\ F_j(n,\tilde{a}_{\vec{\gamma}^2_{n,s-1}}) \\ F_j(n,\tilde{c}_{\vec{\gamma}^2_{n,s-1}}) \end{bmatrix} = \begin{bmatrix} F_j(n,b_{\vec{\gamma}^2_{n,s-1}}) \\ F_j(n,a_{\vec{\gamma}^2_{n,s-1}}) \\ F_j(n,c_{\vec{\gamma}^2_{n,s-1}}) \end{bmatrix}, \quad \begin{bmatrix} H_j(n,\tilde{b}_{\vec{\gamma}^2_{n,s-1}}) \\ H_j(n,\tilde{a}_{\vec{\gamma}^2_{n,s-1}}) \\ H_j(n,\tilde{c}_{\vec{\gamma}^2_{n,s-1}}) \end{bmatrix} = \begin{bmatrix} H_j(n,b_{\vec{\gamma}^2_{n,s-1}}) \\ H_j(n,a_{\vec{\gamma}^2_{n,s-1}}) \\ H_j(n,c_{\vec{\gamma}^2_{n,s-1}}) \end{bmatrix},$$

and

$$\begin{bmatrix} G_j(n,\tilde{\tilde{b}}_{\vec{\gamma}^2_{n,s-1}}) \\ G_j(n,\tilde{\tilde{a}}_{\vec{\gamma}^2_{n,s-1}}) \\ G_j(n,\tilde{\tilde{c}}_{\vec{\gamma}^2_{n,s-1}}) \end{bmatrix} = \begin{bmatrix} G_j(n,b_{\vec{\gamma}^2_{n,s-1}}) \\ G_j(n,a_{\vec{\gamma}^2_{n,s-1}}) \\ G_j(n,c_{\vec{\gamma}^2_{n,s-1}}) \end{bmatrix}, \quad \begin{bmatrix} G_j(n,\hat{\tilde{b}}_{\vec{\gamma}^2_{n,s-1}}) \\ G_j(n,\hat{\tilde{a}}_{\vec{\gamma}^2_{n,s-1}}) \\ G_j(n,\hat{\tilde{c}}_{\vec{\gamma}^2_{n,s-1}}) \end{bmatrix} = \begin{bmatrix} G_j(n,\hat{b}_{\vec{\gamma}^2_{n,s-1}}) \\ G_j(n,\hat{a}_{\vec{\gamma}^2_{n,s-1}}) \\ G_j(n,\hat{c}_{\vec{\gamma}^2_{n,s-1}}) \end{bmatrix},$$

so that

$$\tilde{B}_j\big(n,\vec{\gamma}^2_{n,s-1}\big) = B_j\big(n,\vec{\gamma}^2_{n,s-1}\big)E_2\,,$$

where

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(21) can be rewritten as

$$B_j(n+1,\vec{\gamma}_{n+1,s}) = B_j(n,\vec{\gamma}^2_{n,s-1})E_2R \quad (23)$$

if $\gamma_3 = 2$.

Finally consider the case with $\gamma_3 = 0$. The vertex $x_{\vec{\gamma}_{n+1,s}} = x_{(n,1,0,\gamma_4,\ldots\gamma_s)}$ is located inside the triangle with outmost vertices a_n, $a_{n,1}$ and $b_{n,1}$. Associate with this $x_{\vec{\gamma}_{n+1,s}}$ a vertex $\bar{z}_{\vec{\gamma}^0_{n,s-1}}$ where $\vec{\gamma}^0_{n,s-1} = (n-1,1,\gamma^0_4,...,\gamma^0_s)$ has $s-1$ components. Here $z = c$ when $x = b$ and vice versa, and $z = a$ when $x = a$. Namely, z is related to x with three possibilities: $(x,z) = (a,a)$, (b,c) and (c,b). Similarly, γ^0_k is related to γ_k with three possibilities: $(\gamma_k,\gamma^0_k) = (1,0)$, $(0,1)$ and $(2,2)$, where $k = 4,...,s$. We use the notation such that the vertex $\bar{z}_{\vec{\gamma}^0_{n,s-1}}$ is the reflection of the vertex $x_{\vec{\gamma}_{n,s-1}}$ with respect to the line connecting a_{n-1} and b_n. It can be seen that $\bar{z}_{\vec{\gamma}^0_{n,s-1}}$ can be reached from $x_{\vec{\gamma}_{n+1,s}}$ by a horizontal translation with the distance from a_n to o. We have

$$B_j(n+1,\vec{\gamma}_{n+1,s}) = \bar{B}_j(n,\vec{\gamma}^0_{n,s-1})R\,, \quad (24)$$

where

$$\bar{B}_j\Big(n,\vec{\gamma}^0_{n,s-1}\Big)$$

$$= \begin{bmatrix} F_j(n,\bar{a}_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\bar{a}_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\tilde{\bar{a}}_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\hat{\bar{a}}_{\vec{\gamma}^0_{n,s-1}}) & H_j(n,\bar{a}_{\vec{\gamma}^0_{n,s-1}}) \\ F_j(n,\bar{c}_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\bar{c}_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\tilde{\bar{c}}_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\hat{\bar{c}}_{\vec{\gamma}^0_{n,s-1}}) & H_j(n,\bar{c}_{\vec{\gamma}^0_{n,s-1}}) \\ F_j(n,\bar{b}_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\bar{b}_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\tilde{\bar{b}}_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\hat{\bar{b}}_{\vec{\gamma}^0_{n,s-1}}) & H_j(n,\bar{b}_{\vec{\gamma}^0_{n,s-1}}) \end{bmatrix}$$

$$= \begin{bmatrix} F_j(n,a_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,a_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\hat{a}_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\tilde{a}_{\vec{\gamma}^0_{n,s-1}}) & H_j(n,a_{\vec{\gamma}^0_{n,s-1}}) \\ F_j(n,c_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,c_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\hat{c}_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\tilde{c}_{\vec{\gamma}^0_{n,s-1}}) & H_j(n,c_{\vec{\gamma}^0_{n,s-1}}) \\ F_j(n,b_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,b_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\hat{b}_{\vec{\gamma}^0_{n,s-1}}) & G_j(n,\tilde{b}_{\vec{\gamma}^0_{n,s-1}}) & H_j(n,b_{\vec{\gamma}^0_{n,s-1}}) \end{bmatrix}$$

by symmetry, so that

$$\bar{B}_j\big(n,\vec{\gamma}^0_{n,s-1}\big) = B_j\big(n,\vec{\gamma}^0_{n,s-1}\big)E_0 \ ,$$

where

$$E_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

(24) can be rewritten as

$$B_j\big(n+1,\vec{\gamma}_{n+1,s}\big) = B_j\big(n,\vec{\gamma}^0_{n,s-1}\big)E_0R \tag{25}$$

if $\gamma_3 = 0$.

Let $E_1 = I_{5\times 5}$ be the identity matrix. (20), (23) and (25) can be combined to give

$$B_j\big(n+1,\vec{\gamma}_{n+1,s}\big) = B_j\big(n,\vec{\gamma}^{\gamma_3}_{n,s-1}\big)E_{\gamma_3}R \ , \tag{26}$$

where $\gamma_3 \in \{0,1,2\}$. As $F_j(n+1,x_{\vec{\gamma}_{n+1,s}}) = F_j(n+1,\tilde{x}_{\vec{\gamma}_{n+1,s}})$ for $x = a,b,c$, the first column of the matrix in (26) gives $F_j(n+1,x_{\vec{\gamma}_{n+1,s}})$ for any vertex $x_{\vec{\gamma}_{n+1,s}}$ in terms of the quantities for $x_{\vec{\gamma}_{n,s-1}}$.

Proposition 5.2 *For the Sierpinski gasket $SG(n+1)$ with $n \geq 2$, consider the vertex $x_{\vec{\gamma}_{n+1,s}}$ where $\vec{\gamma}_{n+1,s} = (n,1,\gamma_3,...,\gamma_s)$ with $3 \leq s \leq n+1$ and $\gamma_k \in \{0,1,2\}$ for $k \in \{3,4,..,s\}$.*

$$\begin{bmatrix} F_j(n+1,a_{\vec{\gamma}_{n+1,s}}) \\ F_j(n+1,b_{\vec{\gamma}_{n+1,s}}) \\ F_j(n+1,c_{\vec{\gamma}_{n+1,s}}) \end{bmatrix} = B_j(n,\vec{\gamma}^{\gamma_3}_{n,s-1})E_{\gamma_3}Re_1 \ .$$

Using Theorems 3.1, 4.1 and Propositions 5.1, 5.2 repeatedly, $F_j(n+1,x_{\vec{\gamma}_{n+1,s}})$ for all the vertices of $SG(n+1)$ can be obtained.

6 Summation and average of $F_j(n,x)$ over all the vertices of $SG(n)$

It is worthwhile to derive the summation of $F_j(n,x)$ over all the vertices x of $SG(n)$, defined as

$$\Phi_j(n) = \sum_{x\in V(SG(n))} F_j(n,x) \ ,$$

and the average of $F_j(n,x)$ over all the vertices, defined as

$$\phi_j(n) = \frac{\Phi_j(n)}{v(SG(n))} = \frac{\Phi_j(n)}{\frac{3}{2}(3^n+1)} \ .$$

It is clear that for any non-negative integer n,

$$\sum_{j=1}^{4} \phi_j(n) = 1 \, .$$

For the vertices $x_{\vec{\gamma}_{n,s}}$ with $s = 1$, i.e. a_m, b_m and c_m, define their sum

$$X_j(n,m) = F_j(n,a_m) + F_j(n,b_m) + F_j(n,c_m) \, .$$

Similarly for the vertices with $s = 2$, define

$$Y_j(n,m') = F_j(n,a_{m',1}) + F_j(n,b_{m',1}) + F_j(n,c_{m',1}) \, ,$$

where $m \geq 0$, $m' \geq 1$ and n is larger than m and m'. By (15), we have

$$X_j(n,m) = (1,1,1)B_j(n,m)e_1 = (1,1,1)B_j(m+1,m)L^{n-m-1}e_1 \, , \tag{27}$$

$$\begin{aligned} Y_j(n,m') &= (1,1,1)B_j\big(n,(m',1)\big)e_1 = (1,1,1)B_j(m'+1,(m',1))L^{n-m'-1}e_1 \\ &= (1,1,1)B_j(m',m'-1)RL^{n-m'-1}e_1 \, . \end{aligned} \tag{28}$$

The first few $\Phi_j(n)$ are

$$\begin{aligned} \Phi_j(0) &= 3F_j(0,o) \, , \\ \Phi_j(1) &= 3F_j(1,o) + X_j(1,0) = 3F_j(1,o) + 2F_j(1,a_0) + F_j(1,c_0) \, , \end{aligned}$$

and

$$\begin{aligned} \Phi_j(2) &= 3F_j(2,o) + X_j(2,0) + X_j(2,1) + 2Y_j(2,1) \\ &= 3F_j(2,o) + (1,1,1)\Big\{B_j(1,0)L + B_j(2,1) + 2B_j(1,0)R\Big\}e_1 \, . \end{aligned}$$

The corresponding values for $j \in \{1,2,3,4\}$ are

$$\begin{aligned} &\Phi_1(0) = \frac{2}{3} \, , \qquad \Phi_2(0) = \frac{1}{3} \, , \qquad \Phi_3(0) = \Phi_4(0) = 0 \, , \\ &\Phi_1(1) = \frac{1}{2} \, , \qquad \Phi_2(1) = \frac{19}{54} \, , \qquad \Phi_3(1) = \frac{7}{54} \, , \qquad \Phi_4(1) = \frac{1}{54} \, , \\ &\Phi_1(2) = \frac{163}{450} \, , \qquad \Phi_2(2) = \frac{5257}{12150} \, , \qquad \Phi_3(2) = \frac{2203}{12150} \, , \qquad \Phi_4(2) = \frac{289}{12150} \, . \end{aligned}$$

For $n \geq 3$, we need the summation

$$M_j(n) = \sum_{s=3}^{n} \Big\{ \sum_{\gamma_s=0}^{2} \sum_{\gamma_{s-1}=0}^{2} \cdots \sum_{\gamma_3=0}^{2} B_j\big(n,\vec{\gamma}_{n,s}\big) \Big\}$$

for the vertices $x_{\vec{\gamma}_{n,s}}$ with $s \geq 3$. By (18) and (26), $M_j(3) = \sum_{\gamma_3=0}^{2} B_j(3, \vec{\gamma}_{3,3})$ is given by

$$\begin{aligned} M_j(3) &= \Big\{B_j\big(2, \vec{\gamma}^0_{2,2}\big)E_0 + B_j\big(2, \vec{\gamma}^1_{2,2}\big)E_1 + B_j\big(2, \vec{\gamma}^2_{2,2}\big)E_2\Big\}R \\ &= B_j\big(2, (1,1)\big)[E_0 + E_1 + E_2]R \\ &= B_j\big(1,0\big)R[E_0 + E_1 + E_2]R = B_j\big(1,0\big)RER\,, \end{aligned}$$

where

$$E = E_0 + E_1 + E_2 = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

The general expression for $n \geq 3$ is

$$\begin{aligned} &M_j(n+1) \\ &= \sum_{s=3}^{n+1}\Big\{\sum_{\gamma_s=0}^{2}\sum_{\gamma_{s-1}=0}^{2}\cdots\sum_{\gamma_3=0}^{2} B_j\big(n+1, \vec{\gamma}_{n+1,s}\big)\Big\} \\ &= \Big\{B_j\big(n, \vec{\gamma}^0_{n,2}\big)E_0 + B_j\big(n, \vec{\gamma}^1_{n,2}\big)E_1 + B_j\big(n, \vec{\gamma}^2_{n,2}\big)E_2\Big\}R \\ &\quad + \sum_{s=4}^{n+1}\Big\{\sum_{\gamma_s=0}^{2}\sum_{\gamma_{s-1}=0}^{2}\cdots\sum_{\gamma_4=0}^{2}\Big[B_j\big(n, \vec{\gamma}^0_{n,s-1}\big)E_0 + B_j\big(n, \vec{\gamma}^1_{n,s-1}\big)E_1 + B_j\big(n, \vec{\gamma}^2_{n,s-1}\big)E_2\Big]R\Big\} \\ &= \Big[B_j\big(n, (n-1,1)\big) + \sum_{s=3}^{n}\Big\{\sum_{\gamma_s=0}^{2}\sum_{\gamma_{s-1}=0}^{2}\cdots\sum_{\gamma_3=0}^{2} B_j\big(n, \vec{\gamma}_{n,s}\big)\Big\}\Big]ER \\ &= \Big\{B_j\big(n-1, n-2\big)R + M_j(n)\Big\}ER \\ &= B_j\big(n-1, n-2\big)RER + B_j\big(n-2, n-3\big)R(ER)^2 + M_j(n-1)(ER)^2 \\ &= \cdots \\ &= \sum_{m=1}^{n-1} B_j\big(m, m-1\big)R(ER)^{n-m}\,. \end{aligned} \tag{29}$$

For example,

$$\begin{aligned} \Phi_j(3) &= 3F_j(3,o) + X_j(3,0) + X_j(3,1) + X_j(3,2) \\ &\quad +2\big[Y_j(3,1) + Y_j(3,2) + (1,1,1)B_j\big(1,0\big)RERe_1\big] \\ &= 3F_j(3,o) + (1,1,1)\Big\{B_j(1,0)L^2 + B_j(2,1)L + B_j(3,2) \\ &\quad +2\big[B_j(1,0)RL + B_j(2,1)R + B_j(1,0)RER\big]\Big\}e_1 \\ &= 3F_j(3,o) + (1,1,1)\Big\{B_j(1,0)\big[L^2 + 2RL + 2RER\big] + B_j(2,1)\big[L + 2R\big] + B_j(3,2)\Big\}e_1\,. \end{aligned}$$

For general $n \geq 3$, we have

$$\Phi_j(n) = 3F_j(n,o) + \sum_{m=0}^{n-1} X_j(n,m) + 2\sum_{m=1}^{n-1} Y_j(n,m) + 2(1,1,1)\Big\{\sum_{m=3}^{n} M_j(m)L^{n-m}\Big\}e_1\,, \tag{30}$$

with $X_j(n,m)$ and $Y_j(n,m)$ given in (27) and (28), respectively. From (29), the summation in the last term of (30) is

$$\begin{aligned}
\sum_{m=3}^{n} M_j(m) L^{n-m} &= \sum_{m=3}^{n} \Big[\sum_{s=1}^{m-2} B_j(s,s-1) R (ER)^{m-1-s} \Big] L^{n-m} \\
&= \sum_{m=2}^{n-1} \Big[\sum_{s=1}^{m-1} B_j(s,s-1) R (ER)^{m-s} \Big] L^{n-1-m} \\
&= \sum_{s=1}^{n-2} B_j(s,s-1) \sum_{m=s+1}^{n-1} \Big[R (ER)^{m-s} L^{n-1-m} \Big] \\
&= \sum_{s=1}^{n-2} B_j(s,s-1) \sum_{m=1}^{n-s-1} \Big[R (ER)^{m} L^{n-1-m-s} \Big] ,
\end{aligned}$$

so that $\Phi_j(n)$ can be calculated exactly for any positive integer n.

Proposition 6.1 *For the Sierpinski gasket $SG(n)$ with $n \geq 3$, the summation of $F_j(n,x)$ over all the vertex is given by*

$$\begin{aligned}
\Phi_j(n) &= (1,1,1) \Bigg\{ \sum_{m=1}^{n} B_j(m,m-1) L^{n-m} + 2 \sum_{m=1}^{n-1} B_j\big(m,m-1\big) R L^{n-m-1} \\
&\quad + 2 \sum_{m=1}^{n-2} B_j(m,m-1) \sum_{s=1}^{n-m-1} \Big[R (ER)^{s} L^{n-1-m-s} \Big] \Bigg\} e_1 + 3 F_j(n,o) \\
&= (1,1,1) \Bigg\{ \sum_{m=1}^{n} B_j(m,m-1) L^{n-m} + 2 B_j\big(n-1,n-2\big) R \\
&\quad + 2 \sum_{m=1}^{n-2} B_j\big(m,m-1\big) \Big[\sum_{s=0}^{n-m-1} R (ER)^{s} L^{n-m-1-s} \Big] \Bigg\} e_1 + 3 F_j(n,o) \, . \qquad (31)
\end{aligned}$$

Let us consider limiting distribution

$$\lim_{n\to\infty} \phi_j(n) \equiv \phi_j$$

with $j \in \{1,2,3,4\}$. It is easy to see that the term $3F_j(n,o)$ in (31) can be neglected in the infinite n limit for ϕ_j, namely,

$$\lim_{n\to\infty} \frac{F_j(n,o)}{\frac{3}{2}(3^n+1)} = 0$$

as the value $F_j(n,o)$ is between 0 and 1 for the four possible j. Similarly, the values of the quantities $F_j(m,x_{m-1})$, $G_j(m,x_{m-1})$, $H_j(m,x_{m-1})$ with $x = a,b,c$ in the matrix $B_j(m,m-1)$ are between 0 and 1, and all the eigenvalues of R given in (17) are positive and less than or equal to 1, such that

$$\lim_{n\to\infty} \frac{(1,1,1) \Big\{ \sum_{m=1}^{n} B_j(m,m-1) L^{n-m} + 2 B_j\big(n-1,n-2\big) R \Big\} e_1}{\frac{3}{2}(3^n+1)} \leq \lim_{n\to\infty} \frac{3n+6}{\frac{3}{2}(3^n+1)} = 0 \, .$$

Therefore, only the double summation term in (31) gives a non-zero contribution to ϕ_j. Rewrite $ER = Q_1[D_1 + \bar{D}_1]Q_1^{-1}$ and $L = Q_2 D_2 Q_2^{-1}$, where

$$Q_1 = \begin{bmatrix} 159 & -87 & 0 & -3 & -3 \\ 38 & 14 & 1 & 2 & 1 \\ 38 & 14 & 0 & 2 & -4 \\ 38 & 14 & -1 & 2 & 1 \\ 15 & 45 & 0 & -3 & 5 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 18 & 0 & -27 & 0 & -2 \\ 5 & -1 & 98 & -1 & 1 \\ 5 & 0 & -32 & 1 & 1 \\ 0 & 1 & -52 & 0 & 0 \\ 0 & 0 & 13 & 0 & 0 \end{bmatrix},$$

and

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{25} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{D}_1 = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{10} & 0 & 0 & 0 \\ 0 & 0 & \frac{6}{25} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{15} \end{bmatrix},$$

then

$$\begin{aligned} \sum_{s=0}^{n-m-1} R(ER)^s L^{n-m-1-s} &= R \sum_{s=0}^{n-m-1} \left[Q_1(D_1^s + \bar{D}_1^s)Q_1^{-1}\right]\left[Q_2 D_2^{n-m-1-s} Q_2^{-1}\right] \\ &= R \sum_{s=0}^{n-m-1} \left[Q_1 D_1^s Q_1^{-1}\right]\left[Q_2 D_2^{n-m-1-s} Q_2^{-1}\right] \\ &\quad + R \sum_{s=0}^{n-m-1} \left[Q_1 \bar{D}_1^s Q_1^{-1}\right]\left[Q_2 D_2^{n-m-1-s} Q_2^{-1}\right]. \end{aligned} \tag{32}$$

The elements of $B_j(m, m-1)$ have been solved in (36)-(39). Define

$$\begin{aligned} Z_j(m) &= (1,1,1)B_j(m, m-1) \\ &= \begin{cases} \lambda_1^{(1)}(\frac{3}{5})^m + \lambda_2^{(1)}(\frac{1}{25})^m + \lambda_3^{(1)}(\frac{1}{375})^m \text{ for } j = 1\,, \\ \frac{363}{196}\lambda_0 + \lambda_1^{(2)}(\frac{3}{5})^m + \lambda_2^{(2)}(\frac{1}{25})^m + \lambda_3^{(2)}(\frac{1}{375})^m + \lambda_4^{(2)}(\frac{1}{15})^m + \lambda_5^{(2)}(\frac{1}{225})^m \text{ for } j = 2\,, \\ \frac{99}{98}\lambda_0 + \lambda_1^{(3)}(\frac{3}{5})^m + \lambda_2^{(3)}(\frac{1}{25})^m + \lambda_3^{(3)}(\frac{1}{375})^m + \lambda_4^{(3)}(\frac{1}{15})^m + \lambda_5^{(3)}(\frac{1}{225})^m \text{ for } j = 3\,, \\ \frac{27}{196}\lambda_0 + \lambda_1^{(4)}(\frac{3}{5})^m + \lambda_2^{(4)}(\frac{1}{25})^m + \lambda_3^{(4)}(\frac{1}{375})^m + \lambda_4^{(4)}(\frac{1}{15})^m + \lambda_5^{(4)}(\frac{1}{225})^m \text{ for } j = 4\,, \end{cases} \end{aligned} \tag{33}$$

where $\lambda_0 = (1,1,1,1,1)$ and

$$\begin{aligned} &\lambda_1^{(1)} = (\frac{605}{392}, \frac{121}{56}, \frac{121}{56}, \frac{121}{56}, \frac{1089}{392})\,, \quad \lambda_2^{(1)} = (\frac{-1375}{196}, \frac{-55}{28}, \frac{-55}{28}, \frac{-55}{28}, \frac{1221}{196})\,, \\ &\lambda_3^{(1)} = (\frac{3125}{392}, \frac{-375}{56}, \frac{-375}{56}, \frac{-375}{56}, \frac{585}{392})\,, \quad \lambda_1^{(2)} = (\frac{-275}{396}, \frac{-55}{56}, \frac{-55}{56}, \frac{-55}{56}, \frac{-495}{392})\,, \\ &\lambda_2^{(2)} = (\frac{2375}{196}, \frac{95}{28}, \frac{95}{28}, \frac{95}{28}, \frac{-2109}{196})\,, \quad \lambda_3^{(2)} = (\frac{-9375}{392}, \frac{1125}{56}, \frac{1125}{56}, \frac{1125}{56}, \frac{-1755}{392})\,, \\ &\lambda_4^{(2)} = (\frac{-1265}{196}, \frac{-187}{196}, \frac{-187}{196}, \frac{-187}{196}, \frac{891}{196})\,, \quad \lambda_5^{(2)} = (\frac{10}{49}, \frac{-240}{49}, \frac{-240}{49}, \frac{-240}{49}, \frac{54}{49})\,, \end{aligned}$$

$$\lambda_1^{(3)} = (\frac{-285}{392}, \frac{-57}{56}, \frac{-57}{56}, \frac{-57}{56}, \frac{-513}{392}) , \quad \lambda_2^{(3)} = (\frac{-625}{196}, \frac{-25}{28}, \frac{-25}{28}, \frac{-25}{28}, \frac{555}{196}) ,$$
$$\lambda_3^{(3)} = (\frac{-9375}{392}, \frac{-1125}{56}, \frac{-1125}{56}, \frac{-1125}{56}, \frac{1755}{392}) , \quad \lambda_4^{(3)} = (\frac{230}{49}, \frac{34}{49}, \frac{34}{49}, \frac{34}{49}, \frac{-162}{49}) ,$$
$$\lambda_5^{(3)} = (\frac{500}{49}, \frac{480}{49}, \frac{480}{49}, \frac{480}{49}, \frac{-108}{49}) , \quad \lambda_1^{(4)} = (\frac{-45}{392}, \frac{-9}{56}, \frac{-9}{56}, \frac{-9}{56}, \frac{-81}{392}) ,$$
$$\lambda_2^{(4)} = (\frac{-375}{196}, \frac{-15}{28}, \frac{-15}{28}, \frac{-15}{28}, \frac{333}{196}) , \quad \lambda_3^{(4)} = (\frac{-3125}{392}, \frac{375}{56}, \frac{375}{56}, \frac{375}{56}, \frac{-585}{392}) ,$$
$$\lambda_4^{(4)} = (\frac{345}{196}, \frac{51}{196}, \frac{51}{196}, \frac{51}{196}, \frac{-243}{196}) , \quad \lambda_5^{(4)} = (\frac{250}{49}, \frac{-240}{49}, \frac{-240}{49}, \frac{-240}{49}, \frac{54}{49}) .$$

Substituting (32) into (31), we get

$$\begin{aligned} \phi_j &= \lim_{n\to\infty} \frac{4}{3}(3^n+1)^{-1} \sum_{m=1}^{n-2} Z_j(m) R \Big\{ \sum_{s=0}^{n-m-1} \Big[Q_1 D_1^s Q_1^{-1}\Big] \Big[Q_2 D_2^{n-m-1-s} Q_2^{-1}\Big] \\ &\quad + \sum_{s=0}^{n-m-1} \Big[Q_1 \bar{D}_1^s Q_1^{-1}\Big] \Big[Q_2 D_2^{n-m-1-s} Q_2^{-1}\Big] \Big\} e_1 . \end{aligned} \tag{34}$$

As the eigenvalues of D_1 and D_2 are between 0 and 1, the first term in (34) makes no contributions since

$$\begin{aligned} 0 &< \sum_{m=1}^{n-2} Z_j(m) R \sum_{s=0}^{n-m-1} \Big[Q_1 D_1^s Q_1^{-1}\Big] \Big[Q_2 D_2^{n-m-1-s} Q_2^{-1}\Big] e_1 \\ &\le \sum_{m=1}^{n-2} Z_j(m) R \sum_{s=0}^{n-m-1} \Big[Q_1 I Q_1^{-1}\Big] \Big[Q_2 I Q_2^{-1}\Big] e_1 \le 3n^2 . \end{aligned}$$

Consider the second term in (34),

$$\begin{aligned} & \sum_{m=1}^{n-2} Z_j(m) R \sum_{s=0}^{n-m-1} \Big[Q_1 \bar{D}_1^s Q_1^{-1}\Big] \Big[Q_2 D_2^{n-m-1-s} Q_2^{-1}\Big] \\ &= \sum_{m=1}^{n-2} Z_j(m) R \Big[Q_1 \tilde{D}_1 Q_1^{-1}\Big] \sum_{s=0}^{n-m-1} 3^s \Big[Q_2 D_2^{n-m-1-s} Q_2^{-1}\Big] \\ &= \sum_{m=1}^{n-2} Z_j(m) R \Big[Q_1 \tilde{D}_1 Q_1^{-1}\Big] Q_2 D(n,m) Q_2^{-1} , \end{aligned}$$

where

$$\tilde{D}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and define

$$D(n,m) \equiv \sum_{s=0}^{n-m-1} 3^s D_2^{n-m-1-s}$$

$$
\begin{aligned}
&= \sum_{s=0}^{n-m-1} \begin{bmatrix} 3^s(\frac{6}{25})^{n-m-1-s} & 0 & 0 & 0 & 0 \\ 0 & 3^s(\frac{1}{15})^{n-m-1-s} & 0 & 0 & 0 \\ 0 & 0 & 3^s & 0 & 0 \\ 0 & 0 & 0 & 3^s(\frac{1}{5})^{n-m-1-s} & 0 \\ 0 & 0 & 0 & 0 & 3^s(\frac{3}{10})^{n-m-1-s} \end{bmatrix} \\
&= \begin{bmatrix} \frac{25[3^{n-m}-(\frac{6}{25})^{n-m}]}{69} & 0 & 0 & 0 & 0 \\ 0 & \frac{15[3^{n-m}-(\frac{1}{15})^{n-m}]}{44} & 0 & 0 & 0 \\ 0 & 0 & \frac{3^{n-m}-1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{5[3^{n-m}-(\frac{1}{5})^{n-m}]}{14} & 0 \\ 0 & 0 & 0 & 0 & \frac{10[3^{n-m}-(\frac{3}{10})^{n-m}]}{27} \end{bmatrix} \\
&\equiv 3^{n-m}\mathcal{D} + \mathcal{D}_2(n-m)\,,
\end{aligned}
$$

with

$$
\mathcal{D} = \begin{bmatrix} \frac{25}{69} & 0 & 0 & 0 & 0 \\ 0 & \frac{15}{44} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{14} & 0 \\ 0 & 0 & 0 & 0 & \frac{10}{27} \end{bmatrix},
$$

$$
\mathcal{D}_2(n-m) = \begin{bmatrix} \frac{-25(\frac{6}{25})^{n-m}}{69} & 0 & 0 & 0 & 0 \\ 0 & \frac{-(\frac{1}{15})^{n-m-1}}{44} & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{-(\frac{1}{5})^{n-m-1}}{14} & 0 \\ 0 & 0 & 0 & 0 & \frac{-(\frac{3}{10})^{n-m-1}}{9} \end{bmatrix}.
$$

Since the absolute values of all the eigenvalues of $\mathcal{D}_2(n-m)$ are less than one, we have

$$
\begin{aligned}
\phi_j &= \lim_{n\to\infty} \frac{4}{3}(3^n+1)^{-1} \sum_{m=1}^{n-2} \Big\{ 3^{n-m} Z_j(m) R Q_1 \tilde{D}_1 Q_1^{-1} Q_2 \mathcal{D} Q_2^{-1} \\
&\quad + Z_j(m) R Q_1 \tilde{D}_1 Q_1^{-1} Q_2 \mathcal{D}_2(n-m) Q_2^{-1} \Big\} e_1 \\
&= \lim_{n\to\infty} \frac{4}{3}(3^n+1)^{-1} \sum_{m=1}^{n-2} 3^{n-m} Z_j(m) \tilde{R} e_1\,, \qquad (35)
\end{aligned}
$$

where $\tilde{R} = R Q_1 \tilde{D}_1 Q_1^{-1} Q_2 \mathcal{D} Q_2^{-1}$. Substituting the expression of $Z_j(m)$ from (33) into (35), carrying out the summation and taking the infinite n limit, we arrive at

$$
\phi_j = \frac{4}{3} \times \begin{cases} \Big[\frac{\lambda_1^{(1)}}{4} + \frac{\lambda_2^{(1)}}{74} + \frac{\lambda_3^{(1)}}{1124}\Big] \tilde{R} e_1 & \text{for} \quad j=1\,, \\ \Big[\frac{363\lambda_0}{392} + \frac{\lambda_1^{(2)}}{4} + \frac{\lambda_2^{(2)}}{74} + \frac{\lambda_3^{(2)}}{1124} + \frac{\lambda_4^{(2)}}{44} + \frac{\lambda_5^{(2)}}{674}\Big] \tilde{R} e_1 & \text{for} \quad j=2\,, \\ \Big[\frac{99\lambda_0}{196} + \frac{\lambda_1^{(3)}}{4} + \frac{\lambda_2^{(3)}}{74} + \frac{\lambda_3^{(3)}}{1124} + \frac{\lambda_4^{(3)}}{44} + \frac{\lambda_5^{(3)}}{674}\Big] \tilde{R} e_1 & \text{for} \quad j=3\,, \\ \Big[\frac{27\lambda_0}{392} + \frac{\lambda_1^{(4)}}{4} + \frac{\lambda_2^{(4)}}{74} + \frac{\lambda_3^{(4)}}{1124} + \frac{\lambda_4^{(4)}}{44} + \frac{\lambda_5^{(4)}}{674}\Big] \tilde{R} e_1 & \text{for} \quad j=4\,. \end{cases}
$$

Tab. 1: Numerical values of $\phi_j(n)$ with $j \in \{1,2,3,4\}$, and the comparison of ϕ_j with f_j for the square lattice (sq). The last digits given are rounded off.

n	$\phi_1(n)$	$\phi_2(n)$	$\phi_3(n)$	$\phi_4(n)$
0	$\frac{2}{3}$ =0.6666666667	$\frac{1}{3}$ =0.3333333333	0	0
1	$\frac{1}{2}$ =0.5	$\frac{19}{54}$ =0.3518518519	$\frac{7}{54}$ =0.1296296296	$\frac{1}{54}$ =0.01851851852
2	$\frac{163}{450}$ =0.3622222222	$\frac{5257}{12150}$ =0.4326748971	$\frac{2203}{12150}$ =0.1813168724	$\frac{289}{12150}$ =0.02378600823
3	$\frac{143357}{472500}$ =0.3034010582	$\frac{17871899}{38272500}$ =0.4669645045	$\frac{7787951}{38272500}$ =0.2034868639	$\frac{1000733}{38272500}$ =0.02614757332
4	$\frac{24381607}{86484375}$ =0.2819192137	$\frac{30227565716}{63047109375}$ =0.4794441175	$\frac{13341669059}{63047109375}$ =0.2116142864	$\frac{1703683097}{63047109375}$ =0.02702238237
5	$\frac{39739246273}{144755859375}$ =0.2745259946	$\frac{51047283737324}{105527021484375}$ =0.4837366110	$\frac{22626394285676}{105527021484375}$ =0.2144132751	$\frac{2883432928358}{105527021484375}$ =0.02732411934
∞	$\frac{10957}{40464}$ =0.2707839067	$\frac{6626035}{13636368}$ =0.4859090778	$\frac{2943139}{13636368}$ =0.2158301243	$\frac{124895}{4545456}$ =0.02747689121
sq	f_1	f_2	f_3	f_4
	$\frac{8}{\pi^2} - \frac{16}{\pi^3}$ =0.2945449182	$\frac{8}{\pi} - \frac{36}{\pi^2} + \frac{48}{\pi^3}$ =0.4469901311	$2 - \frac{16}{\pi} + \frac{48}{\pi^2} - \frac{48}{\pi^3}$ =0.2223849831	$-1 + \frac{8}{\pi} - \frac{20}{\pi^2} + \frac{16}{\pi^3}$ =0.03607996755

The matrix productions can be done to give the following theorem.

Theorem 6.1 *Consider all the vertices of the Sierpinski gasket $SG(n)$ in the infinite n limit. The average probabilities that a vertex is connected by 1, 2, 3 or 4 bond(s) among all the spanning tree configurations are*

$$\phi_1 = \frac{10957}{40464} = 0.270783906682\cdots, \qquad \phi_2 = \frac{6626035}{13636368} = 0.485909077842\cdots,$$
$$\phi_3 = \frac{2943139}{13636368} = 0.215830124267\cdots, \qquad \phi_4 = \frac{124895}{4545456} = 0.0274768912073\cdots.$$

For the Sierpinski gasket $SG(n)$ in the infinite n limit, the average number of bonds connecting to a vertex among all the spanning tree configurations is equal to $\phi_1 + 2\phi_2 + 3\phi_3 + 4\phi_4 = 2$ as expected [Aldous(1990)]. We list the numerical values of $\phi_j(n)$ with $j \in \{1,2,3,4\}$ for $0 \le n \le 5$ and infinite n limit in Table 1. We find that $\phi_1(n)$ decreases monotonically as n increases, while $\phi_2(n)$, $\phi_3(n)$ and $\phi_4(n)$ increase monotonically. The values for $n = 5$ are already very close to ϕ_j in the infinite n limit with deviations about 1%.

It is interesting to compare the Sierpinski gasket $SG(n)$ in the infinite n limit with the infinite two-dimensional square lattice which is also a 4-regular lattice. For the square lattice, all the vertices are identical due to the translational invariant, and the probabilities that a vertex is connected by 1, 2, 3 or 4 bond(s) among all the spanning tree configurations have been determined exactly in [Manna et al.(1992)].

They are denoted by f_j with $j \in \{1,2,3,4\}$. As shown in Table 1, f_1, f_3 and f_4 are slightly larger than ϕ_1, ϕ_3, ϕ_4, respectively, while f_2 is smaller than ϕ_2.

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A Recursion relations for $F_j(n+1,a_n)$, $G_j(n+1,a_n)$, $G_j(n+1,b_n)$, $H_j(n+1,a_n)$ with $j \in \{1,2,3,4\}$

Using Figs. 3-5 for the vertex a_n or b_n, we obtain the following recursion relations

$$\begin{cases} f_1(n+1,a_n) &= 2f_1(n,o)g_0(n,b_n)f(n)\,, \\ f_2(n+1,a_n) &= 2f_2(n,o)g_0(n,b_n)f(n)+2f_1(n,o)\big[g_1(n,o)+g_1(n,b_n)\big]f(n) \\ &\quad +2f_1(n,o)^2g(n)\,, \\ f_3(n+1,a_n) &= 2f_2(n,o)[g_1(n,o)+g_1(n,b_n)]f(n)+4f_1(n,o)f_2(n,o)g(n) \\ &\quad +2f_1(n,o)[g_2(n,o)+g_2(n,b_n)]f(n)\,, \\ f_4(n+1,a_n) &= 2f_2(n,o)[g_2(n,o)+g_2(n,b_n)]f(n)+2f_2(n,o)^2g(n)\,, \end{cases}$$

$$\begin{cases} g_1(n+1,a_n) &= 2f_1(n,o)g_0(n,b_n)g(n)\,, \\ g_2(n+1,a_n) &= f_1(n,o)^2h(n)+g_1(n,o)^2f(n)+4f_1(n,o)g_1(n,o)g(n) \\ &\quad +2[f_1(n)g_1(n,b_n)+f_2(n)g_0(n,b_n)]g(n)\,, \\ g_3(n+1,a_n) &= 2f_2(n,o)f_1(n,o)h(n)+2g_1(n,o)g_2(n,o)f(n) \\ &\quad +4[f_2(n,o)g_1(n,o)+f_1(n,o)g_2(n,o)]g(n) \\ &\quad +2[f_2(n,o)g_1(n,b_n)+f_1(n,o)g_2(n,b_n)]g(n)\,, \\ g_4(n+1,a_n) &= f_2(n,o)^2h(n)+f_2(n,o)[2g_2(n,b_n)+4g_2(n,o)]g(n)+g_2(n,o)^2f(n)\,, \end{cases}$$

$$\begin{cases} g_1(n+1,b_n) &= f_1(n,o)h_0(n,o)f(n)+2f_1(n,o)g_0(n,b_n)g(n) \\ &\quad +2g_1(n,o)g_0(n,b_n)f(n)\,, \\ g_2(n+1,b_n) &= [f_2(n,o)h_0(n,o)+f_1(n,o)h_1(n,o)]f(n)+2f_1(n,o)g_1(n,o)g(n) \\ &\quad +g_1(n,o)^2f(n)+2[f_2(n,o)g_0(n,b_n)+f_1(n,o)g_1(n,b_n)]g(n) \\ &\quad +2[g_2(n,o)g_0(n,b_n)+g_1(n,o)g_1(n,b_n)]f(n)\,, \\ g_3(n+1,b_n) &= [f_2(n,o)h_1(n,o)+f_1(n,o)h_2(n,o)]f(n) \\ &\quad +2[f_2(n,o)g_1(n,o)+f_1(n,o)g_2(n,o)]g(n) \\ &\quad +2[f_2(n,o)g_1(n,b_n)+f_1(n,o)g_2(n,b_n)]g(n) \\ &\quad +2[g_2(n,o)g_1(n,b_n)+g_1(n,o)g_2(n,b_n)]f(n)+2g_1(n,o)g_2(n,o)f(n)\,, \\ g_4(n+1,b_n) &= f_2(n,o)h_2(n,o)f(n)+2g_2(n,o)g_2(n,b_n)f(n) \\ &\quad +2f_2(n,o)[g_2(n,o)+g_2(n,b_n)]g(n)+g_2(n,o)^2f(n)\,, \end{cases}$$

and

$$\left\{\begin{array}{rcl}
h_1(n+1,a_n) &=& 4f_1(n,o)h_0(n,o)g(n)+2f_1(n,o)g_0(n,b_n)h(n)\\
&&+4g_1(n,o)h_0(n,o)f(n)+8g_1(n,o)g_0(n,b_n)g(n)\,,\\
h_2(n+1,a_n) &=& 4[f_2(n,o)h_0(n,o)+f_1(n,o)h_1(n,o)]g(n)\\
&&+2[f_1(n,o)g_1(n,b_n)+f_2(n,o)g_0(n,b_n)]h(n)\\
&&+4[g_2(n,o)h_0(n,o)+g_1(n,o)h_1(n,o)]f(n)+2f_1(n,o)g_1(n,o)h(n)\\
&&+6g_1(n,o)^2g(n)+8[g_2(n,o)g_0(n,b_n)+g_1(n,o)g_1(n,b_n)]g(n)\,,\\
h_3(n+1,a_n) &=& 4[f_2(n,o)h_1(n,o)+f_1(n,o)h_2(n,o)]g(n)\\
&&+2[f_2(n,o)g_1(n,b_n)+f_1(n,o)g_2(n,b_n)]h(n)\\
&&+4[g_2(n,o)h_1(n,o)+g_1(n,o)h_2(n,o)]f(n)\\
&&+2[f_2(n,o)g_1(n,o)+f_1(n,o)g_2(n,o)]h(n)\\
&&+8[g_2(n,o)g_1(n,b_n)+g_1(n,o)g_2(n,b_n)]g(n)\\
&&+12g_1(n,o)g_2(n,o)g(n)\,,\\
h_4(n+1,a_n) &=& 4f_2(n,o)h_2(n,o)g(n)+2f_2(n,o)[g_2(n,b_n)+g_2(n,o)]h(n)\\
&&+4g_2(n,o)h_2(n,o)f(n)+8g_2(n,o)g_2(n,b_n)g(n)+6g_2(n,o)^2g(n)\,.
\end{array}\right.$$

Using the identity $3g(n)^2=f(n)h(n)$, it follows that

$$\left\{\begin{array}{rcl}
F_1(n+1,a_n) &=& \frac{F_1(n,o)G_0(n,b_n)}{3}\,,\\
F_2(n+1,a_n) &=& \frac{F_2(n,o)G_0(n,b_n)}{3}+\frac{F_1(n,o)^2}{3}+\frac{F_1(n,o)[G_1(n,o)+G_1(n,b_n)]}{3}\,,\\
F_3(n+1,a_n) &=& \frac{F_2(n,o)[G_1(n,o)+G_1(n,b_n)]}{3}+\frac{2F_1(n,o)F_2(n,o)}{3}+\frac{F_1(n,o)[G_2(n,o)+G_2(n,b_n)]}{3}\,,\\
F_4(n+1,a_n) &=& \frac{F_2(n,o)[G_2(n,o)+G_2(n,b_n)]}{3}+\frac{F_2(n,o)^2}{3}\,,
\end{array}\right. \tag{36}$$

$$\left\{\begin{array}{rcl}
G_1(n+1,a_n) &=& \frac{F_1(n,o)G_0(n,b_n)}{5}\,,\\
G_2(n+1,a_n) &=& \frac{3F_1(n,o)^2}{10}+\frac{F_1(n,o)[2G_1(n,o)+G_1(n,b_n)]}{5}+\frac{F_2(n,o)G_0(n,b_n)}{5}+\frac{G_1(n,o)^2}{10}\,,\\
G_3(n+1,a_n) &=& \frac{3F_2(n,o)F_1(n,o)}{5}+\frac{F_2(n,o)[2G_1(n,o)+G_1(n,b_n)]}{5}+\frac{G_2(n,o)G_1(n,o)}{5}\\
&&+\frac{F_1(n,o)[2G_2(n,o)+G_2(n,b_n)]}{5}\,,\\
G_4(n+1,a_n) &=& \frac{3F_2(n,o)^2}{10}+\frac{F_2(n,o)[2G_2(n,o)+G_2(n,b_n)]}{5}+\frac{G_2(n,o)^2}{10}\,,
\end{array}\right. \tag{37}$$

$$\left\{\begin{array}{rcl}
G_1(n+1,b_n) &=& \frac{3F_1(n,o)H_0(n,o)}{10}+\frac{[F_1(n,o)+G_1(n,o)]G_0(n,b_n)}{5}\,,\\
G_2(n+1,b_n) &=& \frac{3[F_2(n,o)H_0(n,o)+F_1(n,o)H_1(n,o)]}{10}+\frac{G_1(n,o)^2}{10}+\frac{F_1(n,o)[G_1(n,o)+G_1(n,b_n)]}{5}\\
&&+\frac{F_2(n,o)G_0(n,b_n)}{5}+\frac{G_0(n,b_n)G_2(n,o)+G_1(n,o)G_1(n,b_n)}{5}\,,\\
G_3(n+1,b_n) &=& \frac{3[F_2(n,o)H_1(n,o)+F_1(n,o)H_2(n,o)]}{10}\\
&&+\frac{[G_1(n,b_n)+G_1(n,o)]G_2(n,o)+G_1(n,o)G_2(n,b_n)}{5}\\
&&+\sum_{r=1}^{2}\frac{F_r(n,o)[G_{3-r}(n,o)+G_{3-r}(n,b_n)]}{5}\,,\\
G_4(n+1,b_n) &=& \frac{3F_2(n,o)H_2(n,o)}{10}+\frac{F_2(n,o)[G_2(n,o)+G_2(n,b_n)]+G_2(n,o)G_2(n,b_n)}{5}+\frac{G_2(n,o)^2}{10}\,,
\end{array}\right. \tag{38}$$

and

$$\left\{\begin{array}{lcl} H_1(n+1,a_n) & = & \frac{3F_1(n,o)[2H_0(n,o)+G_0(n,b_n)]}{25}+\frac{6G_1(n,o)H_0(n,o)}{25}+\frac{4G_1(n,o)G_0(n,b_n)}{25}\,, \\ H_2(n+1,a_n) & = & \frac{6[F_2(n,o)H_0(n,o)+F_1(n,o)H_1(n,o)]}{25}+\frac{3F_1(n,o)[G_1(n,o)+G_1(n,b_n)]}{25} \\ & & +\frac{3F_2(n,o)G_0(n,b_n)}{25}+\frac{4[G_2(n,o)G_0(n,b_n)+G_1(n,o)G_1(n,b_n)]}{25} \\ & & +\frac{3G_1(n,o)^2}{25}+\frac{6[H_0(n,o)G_2(n,o)+H_1(n,o)G_1(n,o)]}{25}\,, \\ H_3(n+1,a_n) & = & \frac{6[F_2(n,o)H_1(n,o)+F_1(n,o)H_2(n,o)]}{25}+\frac{3F_2(n,o)[G_1(n,o)+G_1(n,b_n)]}{25} \\ & & +\frac{3F_1(n,o)[G_2(n,o)+G_2(n,b_n)]}{25}+\frac{6[G_2(n,o)H_1(n,o)+G_1(n,o)H_2(n,o)]}{25} \\ & & +\frac{[4G_1(n,b_n)+6G_1(n,o)]G_2(n,o)}{25}+\frac{4G_1(n,o)G_2(n,b_n)}{25}\,, \\ H_4(n+1,a_n) & = & \frac{6F_2(n,o)H_2(n,o)}{25}+\frac{3F_2(n,o)[G_2(n,o)+G_2(n,b_n)]}{25}+\frac{6G_2(n,o)H_2(n,o)}{25} \\ & & +\frac{4G_2(n,o)G_2(n,b_n)}{25}+\frac{3G_2(n,o)^2}{25}\,. \end{array}\right. \tag{39}$$

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