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Asymptotics of the Stirling numbers of the first kind revisited: A saddle point approach

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Using the saddle point method, we obtain from the generating function of the Stirling numbers of the first kind $\left[\begin{matrix} n \\ j \end{matrix}\right]$ and Cauchy's integral formula, asymptotic results in central and non-central regions. In the central region, we revisit the celebrated Goncharov theorem with more precision. In the region $j = n - n^{\alpha}, \quad \alpha > 1/2$, we analyze the dependence of $\left[\begin{matrix} n \\ j \end{matrix}\right]$ on α.

Keywords: Stirling numbers, saddle point method.

To Philippe

1 Introduction

Let $\left[\begin{matrix} n \\ j \end{matrix}\right]$ be the Stirling number of the first kind (unsigned version). Their generating function is given by

$$\phi_n(z) = \prod_{0}^{n-1}(z+i) = \frac{\Gamma(z+n)}{\Gamma(z)}, \quad \phi_n(1) = n!.$$

In the sequel all asymptotics are meant for $n \to \infty$.

An asymptotic expansion for $j = \mathcal{O}(1)$ is given in Wilf [14], which has been extended to the range $j = \mathcal{O}(\ln n)$ by Hwang [6]. The generalized Stirling numbers have been considered by Tsylova [13] and Chelluri et al. [2]. The $q-$Stirling numbers are studied in Kyriakoussis and Vamvakari [9].

In Sec.2, we revisit the asymptotic expansions in the central region and in Sec.3, we analyse the non-central region $j = n - n^{\alpha}, \quad \alpha > 1/2$. We use Cauchy's integral formula and the saddle point method.

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2 Central region

Consider the random variable J_n, with probability distribution

$$\mathbb{P}[J_n = j] = Z_n(j),$$
$$Z_n(j) := \frac{\begin{bmatrix} n \\ j \end{bmatrix}}{n!}.$$

The mean and variance are given by

$$M := \mathbb{E}(J_n) = \sum_0^{n-1} \frac{1}{1+i} = H_n = \psi(n+1) + \gamma,$$
$$\sigma^2 := \mathbb{V}(J_n) = \sum_0^{n-1} \frac{i}{(1+i)^2} = \psi(1, n+1) + \psi(n+1) - \frac{\pi^2}{6} + \gamma,$$

where $\psi(x)$ is the digamma function, $\psi(k, x)$ is the kth polygamma function, and

$$M \sim \ln(n) + \gamma + \frac{1}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right),$$
$$\sigma^2 \sim \ln(n) - \frac{\pi^2}{6} + \gamma + \frac{3}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

It is convenient to set

$$A_n := \ln(n) - \frac{\pi^2}{6} + \gamma = \ln\left(ne^{\gamma - \pi^2/6}\right),$$

and to consider all our next asymptotics ($n \to \infty$) as functions of A_n. Of course, all asymptotics can be reformulated in terms of $\ln(n)$.

We have

$$M \sim A_n + \frac{\pi^2}{6} + \mathcal{O}\left(\frac{1}{n}\right),$$
$$\sigma^2 \sim A_n + \mathcal{O}\left(\frac{1}{n}\right).$$

A celebretated central limit theorem of Goncharov says that

$$J_n \sim \mathcal{N}\left(M, \sigma\right),$$

where $\mathcal{N}$ is the Gaussian distribution, with a rate of convergence $\mathcal{O}(1/\sqrt{\ln(n)})$. This can also be deduced from the Quasi-Power theorem of Hwang [7],[8].

In this Section, we want to obtain a more precise local limit theorem for J_n in terms of $x := \frac{J_n - M}{\sigma}$ and A_n. Actually, we obtain the following theorem, where we use $B_n := \sqrt{A_n}$ to simplify the expressions.

Theorem 2.1

$$Z_n(j) \sim \frac{1}{\sqrt{2\pi}B_n} e^{-x^2/2}\cdot$$
$$\cdot\left[1 + \frac{x^3/6 - x/2}{B_n} + \frac{3x^2/8 - x^4/6 - 1/12 + x^6/72}{B_n^2}\right.$$
$$\left.+\frac{-\pi^2 x^3/18 + 37x^5/240 - 355x^3/144 + x/8 - x^7/48 + x^9/1296 + \pi^2 x/6 - \zeta(3)x + \zeta(3)x^3/3}{B_n^3} + \dots\right].$$

Proof: By Cauchy's theorem,

$$Z_n(j) = \frac{1}{2\pi\mathbf{i}}\int_\Omega \frac{\phi_n(z)}{z^{j+1}n!}dz$$
$$= \frac{1}{2\pi\mathbf{i}}\int_\Omega e^{S(z)}dz,$$

where Ω is inside the analyticity domain of the integrand and encircles the origin and

$$S(z) = S_1(z) + S_2(z), \quad S_1(z) = \sum_{i=0}^{n-1}\ln(z+i)) - \ln(n!), \quad S_2(z) = -(j+1)\ln(z).$$

We will use the Saddle point method (for a good introduction to this method, see Flajolet and Sedgewick [3], ch.$VIII$). Set

$$S^{(i)} := \frac{d^i S}{dz^i}.$$

These derivatives can be expressed in terms of $\psi(k, z+n)$ and $\psi(k, z)$.

First we must find the solution of

$$S^{(1)}(\tilde{z}) = 0 \tag{1}$$

with smallest module.

Set $\tilde{z} := z^* - \varepsilon$, where $z^* = \lim_{n\to\infty}\tilde{z}$. Here, it is easy to check that $z^* = 1$. Set $j = M + x\sigma$, x fixed and $B_n := \sqrt{A_n}$.

This leads, to first order (keeping only the ε term in (1)), to

$$\varepsilon := \frac{-x}{B_n} + \frac{x^2-1}{B_n^2} + \dots + \frac{1}{n}\left(\frac{3x}{4B_n^3} + \dots\right) + \mathcal{O}\left(\frac{1}{n^2B_n^4}\right).$$

This shows that, asymptotically, ε is given by a series of powers of n^{-1}, where each coefficient is given by a series of powers of B_n^{-1}. To obtain more precision, we set again $j = M + x\sigma$, expand in powers of n^{-1}, and equate each coefficient to 0. . This leads to (here and in the following, we provide only a few terms but Maple knows more)

$$\varepsilon = \frac{-x}{B_n} - \frac{1}{B_n^2} + \frac{0}{B_n^3} + \dots + \frac{1}{n}\left(\frac{3x}{4B_n^3} + \frac{x^2+3/2}{B_n^4} + \dots\right) + \mathcal{O}\left(\frac{1}{n^2B_n^4}\right).$$

We have, with $\tilde{z} := z^* - \varepsilon = 1 - \varepsilon$,

$$Z_n(j) = \frac{1}{2\pi\mathbf{i}} \int_\Omega \exp\left[S(\tilde{z}) + S^{(2)}(\tilde{z})(z-\tilde{z})^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(z-\tilde{z})^l/l!\right] dz.$$

Note that the linear term vanishes. Set $z = \tilde{z} + \mathbf{i}\tau$. This gives

$$Z_n(j) = \frac{1}{2\pi} \exp[S(\tilde{z})] \int_{-\infty}^{\infty} \exp\left[S^{(2)}(\tilde{z})(\mathbf{i}\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(\mathbf{i}\tau)^l/l!\right] d\tau. \tag{2}$$

The justification of the integration procedure is given in the Appendix. Let us first analyze $S(\tilde{z})$. We obtain

$$\begin{aligned} S(\tilde{z}) = -x^2/2 + \frac{x^3/6 - x}{B_n} + \frac{-x^4/12 + x^2/2 - 1/2}{B_n^2} \\ + \frac{-x^3/3 + x^5/20 + x/2 - \pi^2 x^3/18 + \zeta(3)x^3/3}{B_n^3} + \ldots + \mathcal{O}\left(\frac{1}{nB_n^3}\right). \end{aligned}$$

Also,

$$\begin{aligned} S^{(2)}(\tilde{z}) &= B_n^2 - B_n x - 1 + x^2 + \ldots, \\ S^{(3)}(\tilde{z}) &= -2B_n^2 + 4B_n x - \pi^2/3 + 2\zeta(3) - 6x^2 + 4 + \ldots, \\ S^{(4)}(\tilde{z}) &= 6B_n^2 - 18B_n x + 36x^2 - 18 + \pi^2 - \pi^4/15 + \ldots, \\ S^{(l)}(\tilde{z}) &= \mathcal{O}\left(B_n^2\right), l \geq 5. \end{aligned}$$

We need these many terms in the following. Note that, with $z = \tilde{z}e^{\mathbf{i}\theta}$, this leads to

$$S^{(2)}(\tilde{z})\frac{(z-\tilde{z})^2}{2} \sim -\frac{1}{2}\ln(n)\theta^2. \tag{3}$$

We can now compute (2), for instance by using the classical trick of setting

$$S^{(2)}(\tilde{z})(i\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(i\tau)^l/l! = -u^2/2.$$

Computing τ as a truncated series in u, this gives, by inversion,

$$\tau = \left[u(1 + x/(2B_n) + \ldots) + u^2(\mathbf{i}/(3B_n) + \ldots) + u^3(-1/(36B_n^2) + \ldots)\right]/B_n + \ldots$$

Setting $d\tau = \frac{d\tau}{du}du$, expanding w.r.t. B_n and integrating on $[u = -\infty..\infty]$, this gives

$$\frac{1}{\sqrt{2\pi}B_n}\left[1 + \frac{x}{2B_n} + \frac{5/12 - x^2/8}{B_n^2} + \frac{x(8\pi^2 - 10 - 93x^2 - 48\zeta(3))}{48B_n^3} + \ldots\right].$$

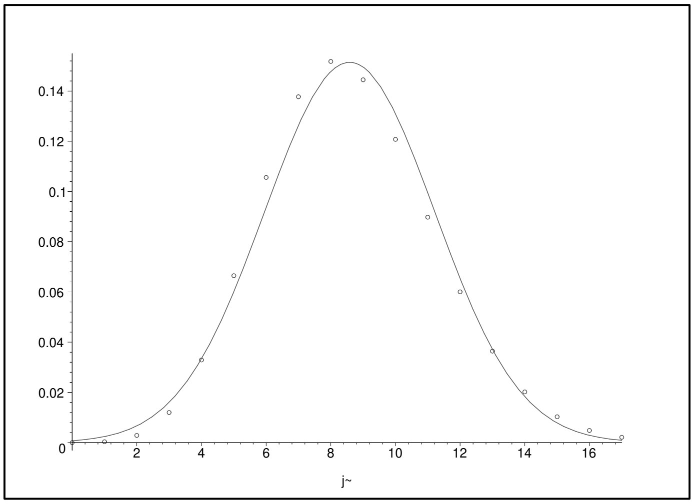

Fig. 1: Comparison between $Z_n(j)$ and $\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\left(\frac{j-M}{\sigma}\right)^2\Big/2\right]$, $n = 3000$

Finally (2) leads to

$$\begin{aligned}
Z_n(j) \sim{}& \frac{1}{\sqrt{2\pi}B_n}e^{-x^2/2}\cdot\\
&\cdot\exp\left[\frac{x^3/6-x}{B_n}+\frac{-x^4/12+x^2/2-1/2}{B_n^2}+\frac{-x^3/3+x^5/20+x/2-\pi^2x^3/18+\zeta(3)x^3/3}{B_n^3}+\dots\right]\cdot\\
&\cdot\left[1+\frac{x}{2B_n}+\frac{5/12-x^2/8}{B_n^2}+\frac{x(8\pi^2-10-93x^2-48\zeta(3))}{48B_n^3}+\dots\right],
\end{aligned}$$

or

$$\begin{aligned}
&Z_n(j) \sim R_1,\\
&\quad R_1 = \frac{1}{\sqrt{2\pi}B_n}e^{-x^2/2}\cdot\\
&\qquad\cdot\left[1+\frac{x^3/6-x/2}{B_n}+\frac{3x^2/8-x^4/6-1/12+x^6/72}{B_n^2}\right.\\
&\qquad\left.+\frac{-\pi^2x^3/18+37x^5/240-355x^3/144+x/8-x^7/48+x^9/1296+\pi^2x/6-\zeta(3)x+\zeta(3)x^3/3}{B_n^3}+\dots\right].
\end{aligned}$$

□

For $n = 3000$, a comparison between $Z_n(j)$ and $\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\left(\frac{j-M}{\sigma}\right)^2\Big/2\right]$ is given in Figure 1.

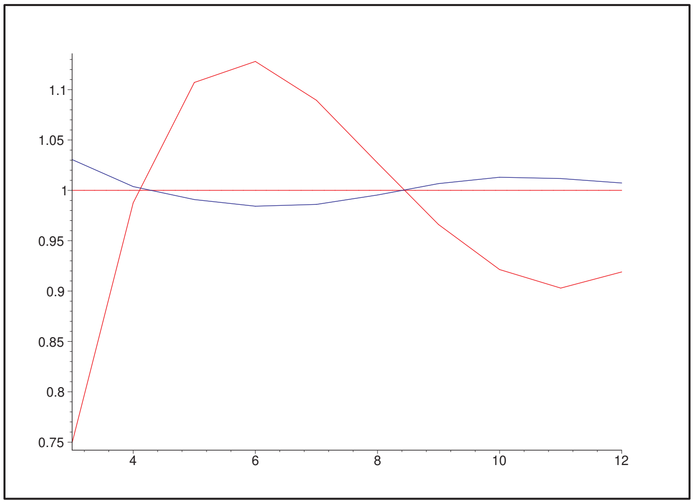

Fig. 2: $Z_n(j) \Big/ \left[\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\left(\frac{j-M}{\sigma}\right)^2 \Big/ 2\right]\right]$, color=red, $Z_n(j)/R_1$, color=blue, $n = 3000$

Of course, only few values of j are significant and also the quality of the Gaussian is low, all asymptotic expressions depend actually on powers of A_n^{-1}, but A_n is not large.

A comparison of $Z_n(j) \Big/ \left[\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\left(\frac{j-M}{\sigma}\right)^2 \Big/ 2\right]\right]$ with $Z_n(j)/R_1$, with 2 terms in R_1, is given in Figure 2.

The precision of R_1 is of order 10^{-2}. Using 3 terms in R_1 leads to a less good result: A_n is not large enough to take advantage of the $A_n^{-3/2}$ term: $A_n = 6.94$ here, we deal with asymptotic series, not necessarily convergent ones. More terms can be computed in R_1 (which is almost automatic with Maple).

3 Large deviation, $j = n - n^\alpha, \quad 1 > \alpha > 1/2$

The case $j = \mathcal{O}(n)$ is analyzed in Timashev [12]. But he obtains a series of powers of n^{-1}, determined by a power series of a certain function that depends on the solution of a given non-linear differential equation of the first order. The coefficients obey some linear recurrence relations in the complex plane. The case $j = n - c$, c constant, is considered in Grünberg [5]. As previous work for the case $j = n - n^\alpha$, let us mention Bender [1], Temme [11], Moser and Wyman [10] (see also the comments by Odlyzko in [4], p.1182). They all use, explicitly or not, the Saddle point method. For $\alpha < 1/2$, Moser and Wyman (6.9) give an explicit asymptotic expression. For $\alpha > 1/2$, they first compute in (4.52) the numerical solution zn of $S'(zn) = 0$ and give in (4.51) an asymptotic expression. This is rather precise: for $n = 50$, this gives a precision of order 10^{-4}. [1] and [11] also compute numerically zn.

However, all these results do not shed light on the dependence of $[z^j]\phi(z)$ on n^α. This is what we want to explicit in this Section. It appears that the range $\alpha > 1/2$ is more delicate than the other range.

Recall that

$$\phi_n(z) = \prod_0^{n-1}(z+i) = \frac{\Gamma(z+n)}{\Gamma(z)}.$$

We have

$$G_n(z) := \frac{\Gamma(z+n)}{\Gamma(z)z^{j+1}} = \exp[S(z)],$$

with

$$S(z) = S_1(z) + S_2(z), S_1(z) = \sum_0^{n-1} \ln(z+i), S_2(z) = -(j+1)\ln(z).$$

We first compute $\tilde{z}$ such that

$$S'(\tilde{z}) = 0. \tag{4}$$

We have

$$S'(z) = \psi(z+n) - \psi(z) - \frac{j+1}{z}.$$

Similarly (we need these expressions later on)

$$\begin{aligned}
S^{(2)}(z) &= \psi(1, z+n) - \psi(1, z) + \frac{j+1}{z^2}, \\
S^{(k)}(z) &= \psi(k-1, z+n) - \psi(k-1, z) + (-1)^k (k-1)! \frac{j+1}{z^k}.
\end{aligned}$$

Some experiments with some values for α ($\alpha = 5/8$ is a good choice) show that $\tilde{z}$ must be a combination of $x = n^\alpha$ and $y = n^{1-\alpha}$ and $x \gg y \gg 1$. Note that both x and y are large. We will derive series of powers of x^{-1}, where each coefficient is a series of powers of y^{-1}. We obtain the following theorem

Theorem 3.1

$$\begin{aligned}
[z^j]\phi_n(z) \sim \frac{1}{\sqrt{2\pi}} \frac{y^2\sqrt{x}}{2} \exp\Bigg[x \Bigg[1 - \ln(2) + 2\ln(y) + \ln(x) - \frac{2}{3y} - \frac{2}{9y^2} - \frac{44}{405y^3} - \frac{26}{405y^4} + \frac{40}{27y^5} \\
+ \frac{179968}{18225y^6} + \frac{4727552}{127575y^7} + \frac{3436796}{32805y^8} + \frac{5492621728}{22143375y^9} + \dots \Bigg] + \ln(2) - 2\ln(y) - \ln(x) \Bigg] \cdot \\
\cdot \Bigg[1 - \frac{3}{3y} - \frac{1}{18y^2} - \frac{1}{30y^3} + \frac{17207}{3240y^4} + \dots + \frac{1}{x}\left(-\frac{1}{12} + \frac{1}{36y} - \frac{35}{216y^2} + \frac{15029}{3240y^3} + \dots \right) \\
+ \frac{1}{x^2}\left(\frac{1}{288} - \frac{1}{864y} + \frac{3527}{5184y^2} + \dots \right) + \mathcal{O}\left(\frac{1}{x^3} \right) \Bigg].
\end{aligned}$$

Proof: Let us summarize the different steps of the proof. First we compute $\tilde{z}$ and $S(\tilde{z})$ as $S(\tilde{z}) = T_1 T_2$, where T_1 is the dominant term and T_2 is a series of powers of x^{-1}, where each coefficient is a series of powers of y^{-1}. We expand $T_3 := e^{T_2}$. Next the integration procedure leads to $\frac{y^2\sqrt{x}}{2}T_4$, where T_4 is again a series of powers of x^{-1}, where each coefficient is a series of powers of y^{-1}. We set $T_5 := \frac{1}{\sqrt{2\pi}}\frac{y^2\sqrt{x}}{2}e^{T_1}$. Finally, we obtain

$$[z^j]\phi_n(z) \sim T_5 T_3 T_4. \tag{5}$$

The first terms in the asymptotics of $\tilde{z}$ are easy to compute: set $\tilde{z} = n\beta$. Equation (4) leads to

$$\psi(n(1+\beta)) - \psi(n\beta) = \frac{1}{\beta} - \frac{1}{y\beta} + \frac{1}{n\beta}.$$

But $\psi(n) \sim \ln(n)$. So we have

$$\ln\left(1+\frac{1}{\beta}\right) \sim \frac{1}{\beta} - \frac{1}{y\beta} + \frac{1}{n\beta},$$

or

$$\frac{1}{\beta} - \frac{1}{2\beta^2} \sim \frac{1}{\beta} - \frac{1}{y\beta},$$

or $\beta \sim \frac{y}{2}$.

More generally, we have

$$\beta = \frac{y}{2}\left[1 + \frac{a_1}{y} + \frac{a_2}{y^2} + \frac{a_3}{y^3} + \ldots + \frac{1}{x}\left(1 + \frac{b_1}{y} + \frac{b_2}{y^2} + \ldots\right) + \frac{1}{x^2}\left(1 + \frac{c_1}{y} + \frac{c_2}{y^2} + \ldots\right) + \mathcal{O}\left(\frac{1}{x^3}\right)\right].$$

Note that $\frac{1}{y^3}$ can be of the same order than $\frac{1}{x}$, see below.

By bootstrapping, we obtain (we give the first terms)

$$\begin{aligned}\tilde{z} = \frac{ny}{2}&\left[1 - \frac{4}{3y} + \frac{2}{9y^2} + \frac{8}{135y^3} + \frac{8}{405y^4} + \frac{16}{1701y^5} + \frac{232}{45525y^6} + \frac{64}{18225y^7} + \ldots\right.\\ &+ \frac{1}{x}\left[1 - \frac{1}{y} + \frac{4}{9y^2} - \frac{16}{135y^3} + \ldots\right]\\ &+ \frac{1}{x^2}\left[1 - \frac{1}{y} + \frac{0}{y^2} + \ldots\right]\\ &+ \frac{1}{x^3}\left[1 + \ldots\right]\\ &\left.+\mathcal{O}\left(\frac{1}{x^4}\right)\right].\end{aligned} \tag{6}$$

Note that the choice of dominant terms in the bracket of (6) depends on α. For instance, for $\alpha = 3/4$, the dominant terms (in decreasing order) are

$$1, \frac{1}{y}, \frac{1}{y^2}, \left\{\frac{1}{x}, \frac{1}{y^3}\right\}, \left\{\frac{1}{xy}, \frac{1}{y^4}\right\}, \left\{\frac{1}{xy^2}, \frac{1}{y^5}\right\}, \left\{\frac{1}{x^2}, \frac{1}{xy^3}, \frac{1}{y^6}\right\}, \ldots$$

Now we must compute $S(\tilde{z})$ and its asymptotics. First we compute $\ln(\tilde{z}+i)$, take the asymptotics wrt x, sum on i, and again take the asymptotics wrt x (recall that $n = xy$). this leads to

$$\begin{aligned}S_1(\tilde{z}) = x&\left[(-\ln(2) + 2\ln(y) + \ln(x))y - \frac{1}{3} + \frac{4}{405y^2} + \frac{2}{405y^3} + \ldots\right] + y - \frac{2}{3} - \frac{2}{3y} - \frac{49}{135y^2} + \ldots\\ &+ \frac{1}{x}\left(\frac{y}{2} + \frac{1}{6y} + \ldots\right) + \frac{1}{x^2}\left(\frac{y}{3} + \ldots\right) + \mathcal{O}\left(\frac{y}{x^3}\right).\end{aligned}$$

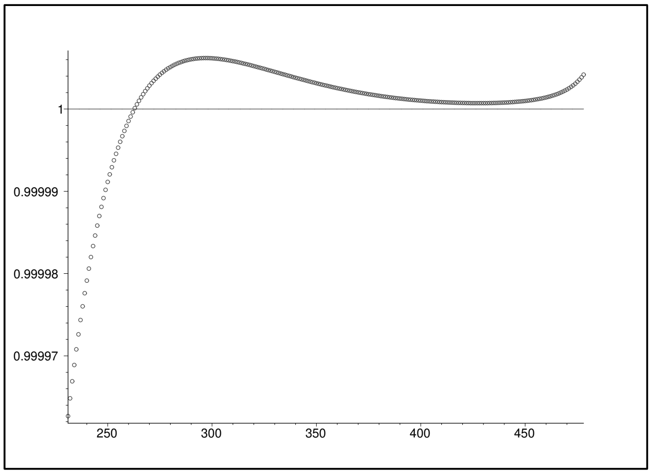

Fig. 3: $zn/\tilde{z}, n = 500$, as function of j, full range

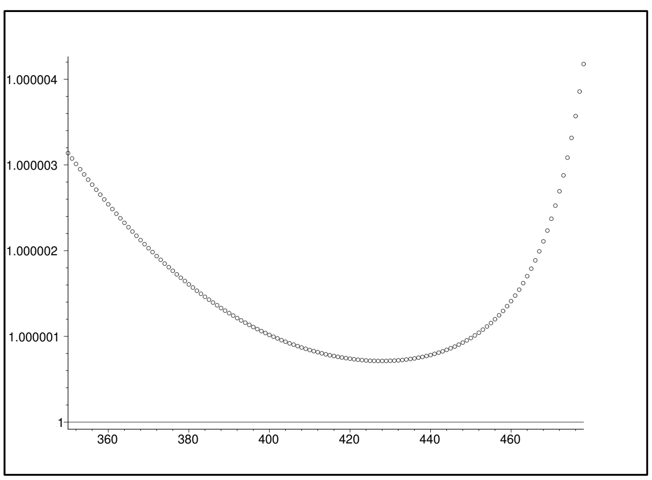

Fig. 4: $zn/\tilde{z}, n = 500$, as function of j, restricted range

Here we provide only a few terms but Maple knows more. Next

$$
\begin{aligned}
S_2(\tilde{z}) = x & \left[(\ln(2) - 2\ln(y) - \ln(x))y + \frac{4}{3} - \ln(2) + 2\ln(y) + \ln(x) - \frac{2}{3y} - \frac{94}{405y^2} + \dots\right] \\
& - y + \frac{2}{3} + \ln(2) - 2\ln(y) - \ln(x) + \frac{1}{y} + \frac{94}{135y^2} + \dots \\
& + \frac{1}{x}\left(\frac{y}{2} + \frac{1}{6y} + \dots\right) \\
& + \frac{1}{x^2}\left(\frac{y}{3} + \dots\right) \\
& + \mathcal{O}\left(\frac{y}{x^3}\right).
\end{aligned}
$$

So, finally

$$
\begin{aligned}
S(\tilde{z}) = S_1(\tilde{z}) + S_2(\tilde{z}) \sim x & \left[1 - \ln(2) + 2\ln(y) + \ln(x) - \frac{2}{3y} + \dots\right] \\
& + \ln(2) - 2\ln(y) - \ln(x) + \frac{1}{3y} + \frac{1}{3y^2} + \dots \\
& + \frac{1}{x}\left(-\frac{1}{2} + \frac{1}{3y} - \frac{1}{2y^2} + \dots\right) \\
& + \frac{1}{x^2}\left(-\frac{1}{6} + \frac{19}{18y^2} \dots\right) \\
& + \mathcal{O}\left(\frac{1}{x^3}\right).
\end{aligned}
$$

Now we split $S(\tilde{z})$ into two parts:

$$
\begin{aligned}
T_1 &= x\left[1 - \ln(2) + 2\ln(y) + \ln(x) - \frac{2}{3y} + \dots\right] + \ln(2) - 2\ln(y) - \ln(x), \\
T_2 &= \frac{1}{3y} + \frac{1}{3y^2} + \dots \\
& + \frac{1}{x}\left(-\frac{1}{2} + \frac{1}{3y} - \frac{1}{2y^2} + \dots\right) \\
& + \frac{1}{x^2}\left(-\frac{1}{6} - \frac{17}{18y^2} \dots\right) \\
& + \mathcal{O}\left(\frac{1}{x^3}\right).
\end{aligned}
$$

Note that the dominant term of T_1 is given by

$$T_1 \sim (2 - \alpha)n^{\alpha}\ln(n). \tag{7}$$

We obtain

$$\exp(S(\tilde{z})) = e^{T_1} e^{T_2} = e^{T_1} T_3,$$

with

$$\begin{aligned}
T_3 = e^{T_2} = {} & 1 + \frac{1}{3y} + \frac{7}{18y^2} + \frac{89}{270y^3} + \frac{18263}{3240y^4} + \frac{98009}{3240y^5} + \frac{9517337}{97200y^6} + \frac{491504273}{2041200y^7} + \dots \\
& + \frac{1}{x}\left(-\frac{1}{2} + \frac{1}{6y} - \frac{7}{12y^2} + \frac{2311}{540y^3} + \frac{112469}{6480y^4} + \frac{5137}{144y^5} + \dots\right) \\
& + \frac{1}{x^2}\left(-\frac{1}{24} - \frac{13}{72y} - \frac{557}{932y^2} + \dots\right) \\
& + \mathcal{O}\left(\frac{1}{x^3}\right).
\end{aligned}$$

Here we have given all terms compatible with the expansion (6). Also, with more precision,

$$\begin{aligned}
T_1 = {} & x\Bigg[1 - \ln(2) + 2\ln(y) + \ln(x) - \frac{2}{3y} - \frac{2}{9y^2} - \frac{44}{405y^3} - \frac{26}{405y^4} + \frac{40}{27y^5} \\
& + \frac{179968}{18225y^6} + \frac{4727552}{127575y^7} + \frac{3436796}{32805y^8} + \frac{5492621728}{22143375y^9} + \dots\Bigg] \\
& + \ln(2) - 2\ln(y) - \ln(x).
\end{aligned}$$

Now we must consider $S^{(k)}(\tilde{z})$. By direct expansion, this gives the following expressions (again we provide only the first few terms). We must use up to six derivatives to get a sufficient precision (of order x^{-2}) in the Saddle integrals.

$$\begin{aligned}
S^{(2)}(\tilde{z}) = {} & \frac{1}{x}\left[\frac{4}{y^4} + \frac{16}{3y^5} + \dots\right] \\
& + \frac{1}{x^2}\left[-\frac{12}{y^4} - \frac{40}{3y^5} + \dots\right] \\
& + \frac{1}{x^3}\left[\frac{12}{y^4} + \frac{8}{y^5} + \dots\right] \\
& + \frac{1}{x^4}\left[\frac{-4}{y^4} + \dots\right] \\
& + \mathcal{O}\left(\frac{1}{x^5 y^4}\right),
\end{aligned} \tag{8}$$

Note that, with $z = \tilde{z}e^{\mathbf{i}\theta}$, this leads to

$$S^{(2)}(\tilde{z})\frac{(z-\tilde{z})^2}{2} \sim -\frac{1}{2}n^{\alpha}\theta^2. \tag{9}$$

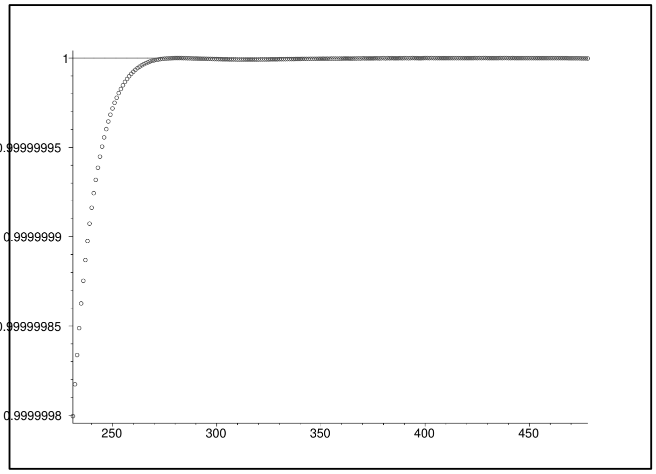

Fig. 5: $G_n(zn)/G_n(\tilde{z}), n = 500$, as function of j

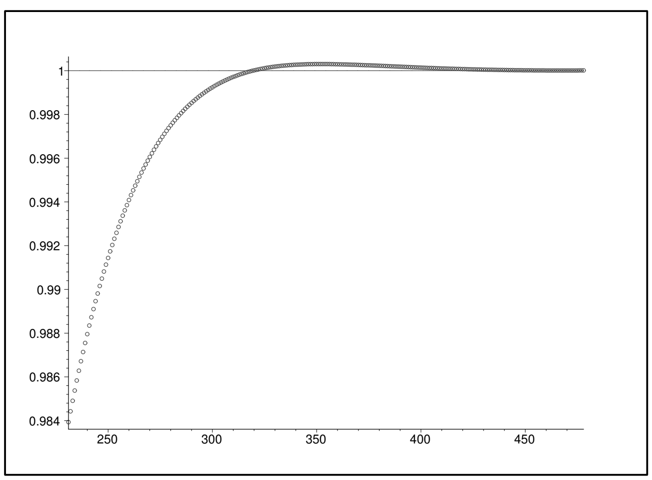

Fig. 6: The quotient of the expression (8) and $S^{(2)}(\tilde{z})$ as function of $j, n = 500$

$$
\begin{aligned}
S^{(3)}(\tilde{z}) = {} & \frac{1}{x^2}\left[-\frac{32}{y^6}+\dots\right] \\
& + \frac{1}{x^3}\left[\frac{128}{y^6}+\dots\right] \\
& + \frac{1}{x^4}\left[-\frac{192}{y^6}+\dots\right] \\
& + \frac{1}{x^5}\left[\frac{128}{y^6}+\dots\right] \\
& + \mathcal{O}\left(\frac{1}{x^6y^6}\right), \\
S^{(4)}(\tilde{z}) = {} & \frac{1}{x^3}\left[\frac{288}{y^8}+\dots\right] \\
& + \frac{1}{x^4}\left[-\frac{1440}{y^8}+\dots\right] \\
& + \frac{1}{x^5}\left[\frac{2880}{y^8}+\dots\right] \\
& + \frac{1}{x^6}\left[-\frac{2880}{y^8}+\dots\right] \\
& + \mathcal{O}\left(\frac{1}{x^7y^8}\right), \\
S^{(5)}(\tilde{z}) = {} & \frac{1}{x^4}\left[-\frac{3072}{y^{10}}+\dots\right] \\
& + \frac{1}{x^5}\left[\frac{18432}{y^{10}}+\dots\right] \\
& + \mathcal{O}\left(\frac{1}{x^6y^{10}}\right), \\
S^{(6)}(\tilde{z}) = {} & \frac{1}{x^5}\left[\frac{38400}{y^{12}}+\dots\right] \\
& + \frac{1}{x^6}\left[\frac{268800}{y^{12}}+\dots\right] \\
& + \mathcal{O}\left(\frac{1}{x^7y^{12}}\right).
\end{aligned}
$$

We proceed now as in Section 2. Again, the justification of the integration procedure is given in the Appendix. This leads to

$$
\tau = \frac{y^2\sqrt{x}}{2}\left[ua_1 + \frac{u^2a_2}{x^{1/2}} + \frac{u^3a_3}{x} + \frac{u^4a_4}{x^{3/2}} + \frac{u^5a_5}{x^2} + \mathcal{O}\left(\frac{u^6}{x^{5/2}}\right)\right].
$$

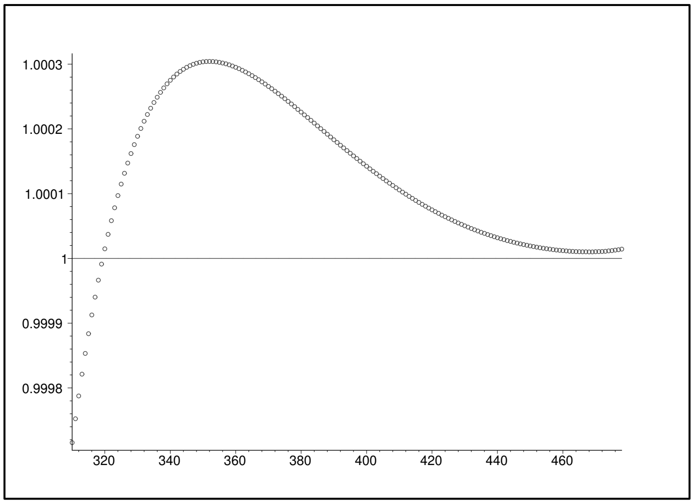

Fig. 7: The quotient of the expression (8) and $S^{(2)}(\tilde{z})$, as function of j, $n = 500$. Restricted range, $\alpha \leq .84$

We give only a_1:

$$a_1 = 1 - \frac{2}{3y} - \frac{2}{9y^2} + \ldots + \frac{1}{x}\left(\frac{3}{2} - \frac{4}{3y} + \ldots\right) + \frac{1}{x^2}\left(\frac{15}{8} - \frac{7}{4y} + \ldots\right) + \mathcal{O}\left(\frac{1}{x^3}\right).$$

This leads to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} \tau'(u) du = \frac{y^2\sqrt{x}}{2} T_4,$$

with

$$T_4 = 1 - \frac{2}{3y} - \frac{2}{9y^2} + \ldots + \frac{1}{x}\left(\frac{5}{12} - \frac{11}{18y} + \ldots\right) + \frac{1}{x^2}\left(\frac{73}{288} - \frac{133}{432y} + \ldots\right) + \frac{1}{x^3}\left(\frac{721}{576} + \ldots\right) + \mathcal{O}\left(\frac{1}{x^4}\right).$$

Set

$$T_5 := \frac{1}{\sqrt{2\pi}} \frac{y^2\sqrt{x}}{2} e^{T_1}.$$

This leads to

$$[z^j]\phi_n(z) \sim T_5 T_3 T_4. \tag{10}$$

We can of course combine T_3 and T_4:

$$T_6 := T_3 T_4 = 1 - \frac{3}{3y} - \frac{1}{18y^2} - \frac{1}{30y^3} + \frac{17207}{3240y^4} + \ldots + \frac{1}{x}\left(-\frac{1}{12} + \frac{1}{36y} - \frac{35}{216y^2} + \frac{15029}{3240y^3} + \ldots\right)$$
$$+ \frac{1}{x^2}\left(\frac{1}{288} - \frac{1}{864y} + \frac{3527}{5184y^2} + \ldots\right) + \mathcal{O}\left(\frac{1}{x^3}\right).$$

$\square$

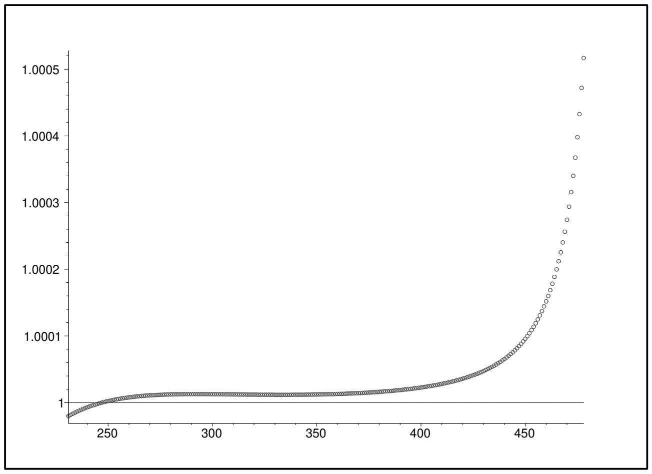

Fig. 8: The quotient $[z^j]\phi_n(z)/T_8$, two terms in T_4, as function of j, $n = 500$

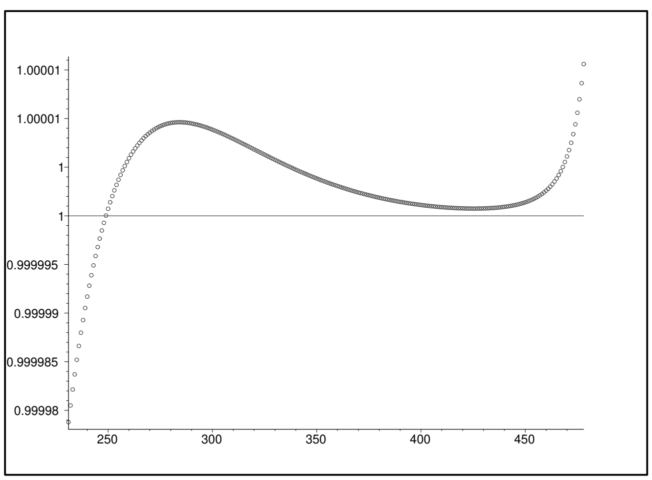

Fig. 9: The quotient $[z^j]\phi_n(z)/T_8$, three terms in T_4, as function of j, $n = 500$

Let us consider the precision of our asymptotics.

The quality of asymptotic (6) is given in Figure 3 and 4, for $n = 500$, and $x \in [\sqrt{n}, n^{0.9}]$ (first range) so that $y \in [n^{0.1}, \sqrt{n}]$. For some values of $j = n - x$, we show $\tilde{z}/zn$, where, as mentioned, zn is the numerical solution of $S'(zn) = 0$. In the full range $j \in [n - n^{0.9}, n - \sqrt{n}]$, the precision is of order 10^{-5}, in a restricted range, the precision is of order 10^{-6}.

Also a comparison of $G_n(\tilde{z})$ and $G_n(zn)$ is given in Figure 5, showing again a precision of order 10^{-6}.

To check the quality of asymptotic (6), we give in Figure 6 the comparison between the expression (8) and $S^{(2)}(\tilde{z})$. The precision is of of order 10^{-2}.

In a restricted range, given in Figure 7, the precision is of order 10^{-4}. $\alpha \leq 0.84$ in this range.

We have made several experiments with (10), with n up to 500. The result is unsatisfactory, only values of x of order $\sqrt{n}$ give reasonable results. Also using e^{T_2} instead of T_3 does not improve the precision. Actually, only very large values of n lead to good precision. So we turn to another formulation: instead of using $e^{T_1}T_3$ for $e^{S(\tilde{z})}$, we plug directly $\tilde{z}$ into $G_n(z)$, ie we set

$$T_7 = G_n(\tilde{z}),$$

leading to

$$[z^j]\phi_n(z) \sim \frac{1}{\sqrt{2\pi}} \frac{y^2\sqrt{x}}{2} T_7 T_4 =: T_8 \text{ say.}$$

For $n = 500$, using two and three terms in $T4$, we give in Figures 8 and 9, the quotient $[z^j]\phi_n(z)/T_8$. The precision is of order 10^{-5}.

4 Conclusion

Using an almost mechanized program in Maple, we have obtained some asymptotic expressions for Stirling numbers in central and non-central regions. We intend to use these techniques in other non-central ranges.

Acknowledgement
The pertinent comments of the referee led to improvements in the presentation.

A Appendix. Justification of the integration procedure

A.1 The central region

We proceed as in Flajolet and Sedgewick [3], ch.$VIII$. We can choose here $\tilde{z} = 1$. This leads, with $z = e^{\mathrm{i}\theta}$, to

$$S(z) \sim S_0(z) + \mathcal{O}\left(\sqrt{\ln(n)}\theta\right) + \text{ constant term,}$$

with

$$\begin{aligned}
S_0(z) &= \sum_{k=0}^{n-1} \ln[e^{\mathbf{i}\theta}+k] - H_n\mathbf{i}\theta \\
&\sim \sum_{k=0}^{n-1} \frac{1}{1+k}[e^{\mathbf{i}\theta}-1] - \frac{1}{2}\sum_{k=0}^{n-1}\left[\frac{1}{1+k}[e^{\mathbf{i}\theta}-1\right]^2 - H_n\mathbf{i}\theta + \mathcal{O}(\theta^3) \\
&\sim H_n[e^{\mathbf{i}\theta}-1-\mathbf{i}\theta] + \mathcal{O}(\theta^2).
\end{aligned}$$

Set

$$h(\theta) := e^{\mathbf{i}\theta} - 1 - \mathbf{i}\theta.$$

We have

$$h(\theta) \sim -\frac{\theta^2}{2},$$

which conforms to (3).

The function $h(\theta)$ is the same as in [3], Ex.$VIII.3$, which proves the validity of our integration procedure: we use here $H_n \sim \ln(n)$ instead of n. The complete asymptotic expansion is justified as in [3], Ex.$VIII.4$.

A.2 The non-central region

We choose here $\tilde{z} = \frac{ny}{2} = \frac{n^{2-\alpha}}{2} := \delta,$ say . We have

$$\begin{aligned}
&\frac{1}{2} < \alpha < 1, \\
&n^\alpha = \frac{n^2}{2\delta}, \\
&n^2 \gg \delta \gg n \gg n^\alpha.
\end{aligned}$$

Set $z = \delta e^{\mathbf{i}\theta}$, this leads, with Euler-Maclaurin formula, with the first correction (the other corrections are negligible), to

$$\begin{aligned}
S(z) &\sim \sum_{k=0}^{n-1} \ln\left[\delta e^{\mathbf{i}\theta}+k\right] - (n-n^\alpha)\mathbf{i}\theta - (n-n^\alpha)\ln(\delta) \\
&\sim \ln\left[n+\delta e^{\mathbf{i}\theta}\right]\left[n+\delta e^{\mathbf{i}\theta}\right] - n - \delta e^{\mathbf{i}\theta}\ln\left[\delta e^{\mathbf{i}\theta}\right] - \left[n - \frac{n^2}{2\delta}\right](\mathbf{i}\theta + \ln(\delta)) - \frac{1}{2}\ln\left[n+\delta e^{\mathbf{i}\theta}\right] + \frac{1}{2}\ln\left[\delta e^{\mathbf{i}\theta}\right].
\end{aligned}$$

Set now $n = \rho\delta, \rho = 2n^{\alpha-1} \ll 1$ and expand wrt ρ. This gives

$$\begin{aligned}
S(z) \sim\; &\rho\left[-\frac{1}{2}e^{-\mathbf{i}\theta}\right] \\
&+ \rho^2\left[\delta\frac{1+\mathbf{i}\theta e^{\mathbf{i}\theta}}{2e^{\mathbf{i}\theta}} + \frac{1}{4}e^{-2\mathbf{i}\theta} + \frac{1}{2}\delta\ln(\delta)\right] \\
&+ \rho^3\left[-\frac{\delta}{6}e^{-2\mathbf{i}\theta} - \frac{1}{6}e^{-3\mathbf{i}\theta}\right] \\
&+ \mathcal{O}(\delta\rho^4).
\end{aligned}$$

Note that the dominant constant contribution is given by $\frac{1}{2}\rho^2\delta\ln(\delta) = (2-\alpha)n^\alpha\ln(n)$, which conforms to (7). The first term gives a variable part $\mathcal{O}(\rho)$. The second term gives a variable part $2n^\alpha h(\theta) + \mathcal{O}(\rho^2)$, with

$$h(\theta) := \frac{1 + \mathbf{i}\theta e^{\mathbf{i}\theta}}{2e^{\mathbf{i}\theta}}.$$

The third term give $\mathcal{O}(n^{2\alpha-1}) \ll n^\alpha$. Note that $2n^\alpha h(\theta) \sim -\frac{1}{2}n^\alpha\theta^2$, which conforms to (9). The function $\left|e^{h(\theta)}\right| = e^{cos(\theta)/2}$ is unimodal with peak at 0 and $h(0) = 1/2$. Let us introduce a splitting value θ_0 such that $n^\alpha\theta_0^2 \to \infty, n^\alpha\theta_0^3 \to 0, n \to \infty$. For instance, we choose $\theta_0 = n^\beta, \beta = -\frac{5\alpha}{12}$. By unimodality property of the cosine, the tail integral

$$K_n^{(1)} := \int_{\theta_0}^{2\pi-\theta_0} e^{2n^\alpha(h(\theta)-1/2)}d\theta$$

is such that

$$\left|K_n^{(1)}\right| = \mathcal{O}\left(e^{n^\alpha[cos(\theta_0)-1]}\right) = \mathcal{O}\left(e^{-Cn^{\alpha/6}}\right)$$

for some $C > 0$. The tail integral is exponentially small.

As $h(\theta) \sim -\frac{\theta^2}{4}$, the central approximation and the tail completion are immediate.

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