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On neighbour-distinguishing colourings from lists

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An edge colouring of a graph is said to be neighbour-distinguishing if any two adjacent vertices have distinct sets of colours of their incident edges. In this paper the list version of the problem of determining the minimum number of colours in a neighbour-distinguishing colouring of a given graph is considered.

Keywords: set of colours, list of colours, neighbour-distinguishing edge colouring, list neighbour-dis- tinguishing index, general list neighbour-distinguishing index

1 Introduction

Let G be a finite simple graph, let C be a set of *colours* (in this paper we shall always suppose $C \subseteq \mathbb{N}$) and let $f : E(G) \to C$ be an edge colouring of G. The *colour set* of a vertex $v \in V(G)$ with respect to f is the set $S_f(v)$ of colours of edges incident to v. The colouring f is *neighbour-distinguishing* if it distinguishes any two adjacent vertices by their colour sets, *i.e.*, $S_f(u) \neq S_f(v)$ whenever $u, v \in V(G)$ and $uv \in E(G)$.

As usual, we are interested in the smallest number of colours in a neighbour-distinguishing colouring of G. If the optimisation runs over all proper colourings, that number is called the *neighbour-distinguishing index* of G, in symbols $\mathrm{ndi}(G)$. In the general case (when colourings are not required to be proper) the corresponding invariant is known as the *general neighbour-distinguishing index* of G and denoted by $\mathrm{gndi}(G)$.

The main goal of this paper is to analyse the list versions of the above problems. Denote by $\mathcal{P}_k(X)$ the set of all k-subsets of a set X and by $\mathcal{L}_k(G)$ the set of all mappings $L : E(G) \to \mathcal{P}_k(\mathbb{N})$. The set $L(e)$, $L \in \mathcal{L}_k(G)$, $e \in E(G)$, is the *list of available colours* for the edge e. Given $L \in \mathcal{L}_k(G)$, the graph G is L-*neighbour-distinguishable* provided there is a proper neighbour-distinguishing colouring $f : E(G) \to \mathbb{N}$ satisfying $f(e) \in L(e)$ for every $e \in E(G)$. The *list neighbour-distinguishing index* of G, in symbols $\mathrm{lndi}(G)$, is the minimum k such that for every $L \in \mathcal{L}_k(G)$ the graph G is L-neighbour-distinguishable.

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This invariant is a generalisation of the "classical" *edge-choosability* $\mathrm{ch}'(G)$ of G; for the definition of the edge-choosability as well as that of (*vertex-*) *choosability* $\mathrm{ch}(G)$ of G see the survey paper [8] by Tuza. Clearly, we have $\mathrm{lndi}(G) \geq \max(\mathrm{ndi}(G), \mathrm{ch}'(G))$. Recall that according to the List Colouring Conjecture $\mathrm{ch}'(G)$ should be equal to $\chi'(G)$, the *chromatic index* of G.

Remark. In the study of the neighbour-distinguishing index and the list neighbour-distinguishing index it is useful to realise that adjacent vertices of distinct degrees are trivially distinguished by their colour sets.

A graph G is said to be *neighbour-irregular* if any two of its vertices of the same degree are nonadjacent. The above remark immediately yields

Proposition 1 *If G is a neighbour-irregular graph, then* $\mathrm{ndi}(G) = \chi'(G)$ *and* $\mathrm{lndi}(G) = \mathrm{ch}'(G)$. □

Omitting the requirement that f should be proper we obtain the notions of a *generally L-neighbour-distinguishable* graph and the *general list neighbour-distinguishing index* $\mathrm{glndi}(G)$ of the graph G.

If G is a disconnected graph with components $G_1, \dots, G_q$, then evidently $\mathrm{lndi}(G) = \max(\mathrm{lndi}(G_i) : i = 1, \dots, q)$ and $\mathrm{glndi}(G) = \max(\mathrm{glndi}(G_i) : i = 1, \dots, q)$. Therefore, when searching for the neighbour-distinguishing index or general neighbour-distinguishing index, it is sufficient to restrict our attention to connected graphs.

Observe that $\mathrm{lndi}(K_1) = \mathrm{glndi}(K_1) = 0$ and that the graph K_2 does not have any neighbour-distinguishing colouring (either proper or not). So, we shall consider only connected graphs with at least three vertices.

In Section 2 we show that the list neighbour-distinguishing index is equal to the neighbour-distinguishing index for graphs in several classes, namely cycles, trees and unbalanced complete bipartite graphs. This leads us to

Conjecture 2 *If G is a connected graph with $|V(G)| \geq 3$, then* $\mathrm{lndi}(G) = \mathrm{ndi}(G)$.

Remark. From Proposition 1 it follows that for neighbour-irregular graphs Conjecture 2 and the List Colouring Conjecture are equivalent.

In Section 3 we prove that the general list neighbour-distinguishing index of a tree is at most 3 and we exhibit a tree T with $\mathrm{glndi}(G) > \mathrm{gndi}(G)$.

Given $p, q \in \mathbb{Z}$, let $[p, q] := \bigcup_{i=p}^{q} \{i\}$ be the *integer interval* bounded by p and q. The *concatenation* of finite sequences $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_n)$ is the sequence $AB := (a_1, \dots, a_m, b_1, \dots, b_n)$. Concatenating finite sequences is clearly associative and we shall use $\prod_{i=1}^{k} A^{(i)}$ for the result of (successive) concatenation of finite sequences $A^{(1)}, \dots, A^{(k)}$. So, if $A^{(i)} = (a_1^{(i)}, \dots, a_{m^{(i)}}^{(i)})$, then $\prod_{i=1}^{k} A^{(i)} = (a_1^{(1)}, \dots, a_{m^{(1)}}^{(1)}, \dots, a_1^{(k)}, \dots, a_{m^{(k)}}^{(k)})$. For $k \in \mathbb{Z}$ we denote by $(k)_2$ the (unique) $l \in [2, 3]$ satisfying $k \equiv l \pmod 2$.

Recall that a cycle C of length $l \geq 3$ in a graph G can be coded using a sequence $\prod_{i=1}^{l+1} (v_i)$ of its consecutive vertices so that $V(C) = \{v_i : i \in [1, l]\}$ and $E(C) = \{v_i v_{i+1} : i \in [1, l]\}$ (with $v_{l+1} = v_1$).

2 Proper colourings

The neighbour-distinguishing index of a graph was introduced by Zhang et al. in [9], where the authors found (among other things) the value of this invariant for cycles (C_n denotes an n-vertex cycle) and trees:

Proposition 3 *Let n be an integer, $n \geq 3$.*

(1) If $n = 5$*, then* $\mathrm{ndi}(C_n) = 5$.

(2) If $n \not\equiv 0 \pmod 3$*,* $n \neq 5$*, then* $\mathrm{ndi}(C_n) = 4$.

(3) If $n \equiv 0 \pmod 3$*, then* $\mathrm{ndi}(C_n) = 3$. □

Proposition 4 *If* T *is a tree with* $|V(T)| \geq 3$*, then* $\Delta(T) \leq \mathrm{ndi}(T) \leq \Delta(T) + 1$*. Moreover,* $\mathrm{ndi}(T) = \Delta(T) + 1$ *if and only if* T *has a pair of adjacent vertices of degree* $\Delta(T)$*.* □

In the above introductory paper it is conjectured that $\mathrm{ndi}(G) \leq \Delta(G) + 2$ for any connected graph $G \notin \{K_2, C_5\}$. The conjecture was confirmed by Balister et al. in [1] for bipartite graphs and for graphs G with $\Delta(G) = 3$. Edwards et al. in [2] showed even that $\mathrm{ndi}(G) \leq \Delta(G) + 1$ if G is bipartite, planar, and of maximum degree $\Delta(G) \geq 12$.

In this section we determine the list neighbour-distinguishing index for graphs in some classes with a simple structure.

2.1 Cycles

We suppose that $E(C_n) = \{e_i : i \in [1, n]\}$ and that e_i is adjacent to e_{i+1} for every $i \in [1, n]$ with indices taken modulo n (this usual convention will be used throughout the whole paper without explicitly mentioning it in what follows).

Proposition 5 *Let* n *be an integer,* $n \geq 3$*.*

(1) If $n = 5$*, then* $\mathrm{lndi}(C_n) = \mathrm{ndi}(C_n) = 5$.

(2) If $n \not\equiv 0 \pmod 3$*,* $n \neq 5$*, then* $\mathrm{lndi}(C_n) = \mathrm{ndi}(C_n) = 4$.

(3) If $n \equiv 0 \pmod 3$*, then* $\mathrm{lndi}(C_n) = \mathrm{ndi}(C_n) = 3$.

Proof: Because of Proposition 3 we shall be done by proving the inequality $\mathrm{lndi}(C_n) \leq \mathrm{ndi}(C_n)$ (the reverse inequality being obvious).

(1) The statement is trivial.

(2) It suffices to show that for any $L \in \mathcal{L}_4(C_n)$ there is a proper neighbour-distinguishing colouring $f : E(C_n) \to \mathbb{N}$ satisfying $f(e_i) \in L(e_i)$ for every $i \in [1, n]$.

If all lists are the same, the existence of f follows from the fact that $\mathrm{ndi}(C_n) = 4$. Otherwise there are two adjacent edges of C_n with distinct lists. Without loss of generality we may suppose that $L(e_1) \neq L(e_n)$, which implies $L(e_1) \setminus L(e_n) \neq \emptyset$. Now determine $f(e_i)$ in the following way: Choose $f(e_1) \in L(e_1) \setminus L(e_n)$ and $f(e_2) \in L(e_2) \setminus \{f(e_1)\}$. If $j \in [3, n-2]$ and $f(e_i)$ is known for all $i \in [1, j-1]$, take $f(e_j) \in L(e_j) \setminus \{f(e_{j-2}), f(e_{j-1})\}$. To finish, let $f(e_{n-1}) \in L(e_{n-1}) \setminus \{f(e_{n-3}), f(e_{n-2}), f(e_1)\}$ and $f(e_n) \in L(e_n) \setminus \{f(e_{n-2}), f(e_{n-1}), f(e_1), f(e_2)\}$ (which is possible because $f(e_1) \notin L(e_n)$).

(3) An edge colouring $f : E(C_n) \to \mathbb{N}$ is proper and neighbour-distinguishing if and only if $f(e_{i+1}) \neq f(e_i) \neq f(e_{i+2})$ for every $i \in [1, n]$. Let F_n be the graph with $V(F_n) = E(C_n)$ and $E(F_n) = \{e_i e_{i+1} : i \in [1, n]\} \cup \{e_i e_{i+2} : i \in [1, n]\}$. Thus F_n is the square of the line graph of C_n and, as the

line graph of C_n is C_n itself, $F_n \cong C_n^2$ (F_n is *isomorphic* to C_n^2). Therefore, $\mathrm{lndi}(C_n) = \mathrm{ch}(F_n) = \mathrm{ch}(C_n^2)$.

Since $n \equiv 0 \pmod 3$, say $n = 3k$, it is clear that the graph F_n is an edge-disjoint union of the cycle $(e_2)\prod_{j=1}^{k}(e_{3j-2}, e_{3j}, e_{3j+2})$ of length $3k$ and k vertex-disjoint cycles $(e_{3j-1}, e_{3j}, e_{3j+1}, e_{3j-1})$ of length 3 with $j \in [1, k]$. Graphs with such a structure were studied by Fleischner and Stiebitz in [3], where it is proved that $\mathrm{ch}(F_n) = 3$. So, by Proposition 3.(3), we have $\mathrm{lndi}(C_n) = \mathrm{ch}(F_n) = \mathrm{ndi}(C_n) = 3$. □

2.2 Trees

By Proposition 4, there are two categories of trees: *Category* k consists of trees T with $\mathrm{ndi}(T) = \Delta(T) - 1 + k$, $k = 1, 2$. Thus, a tree T is of Category 2 if and only if it has a pair of adjacent vertices of degree $\Delta(T)$.

Theorem 6 *If T is a tree with $|V(T)| \geq 3$, then* $\mathrm{lndi}(T) = \mathrm{ndi}(T)$.

Proof: Consider the following algorithm that enables us to find a neighbour-distinguishing colouring of a given tree T of a given category with colours belonging to lists of an appropriate cardinality.

Algorithm 1: ListNeighbourDistinguishing(T, k, L)

Input: A tree $T = (V, E)$ with $|V| \geq 3$ of Category k and a mapping $L : E \to \mathbb{N}$ that satisfies $|L(e)| = \Delta(T) - 1 + k$ for every $e \in E$.

Output: A proper neighbour-distinguishing colouring $f : E \to \mathbb{N}$ with $f(e) \in L(e)$ for every $e \in E$.

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do
    S(v) ← ∅ for every v ∈ V
    Pick z, t ∈ V so that zt ∈ E and deg_T(z) = 1, then choose f(zt) ∈ L(zt).
    S(z) ← {f(zt)}
    S(t) ← {f(zt)}
    X ← {z}
    while V \ X ≠ ∅ do
        Choose y ∈ V \ X so that there is (a unique) x ∈ X with xy ∈ E.
        if d = deg_T(y), N_T(y) = {x_i : i ∈ [1, d]} and N_T(y) ∩ X = {x_1} then
            // Denote neighbours of y by x_1, ..., x_d so that x_1 = x. //
            for i = 2 to d do
                // Find colours successively for yx_i, i = 2, ..., d. //
                L(yx_i) ← L(yx_i) \ {f(yx_j) : j ∈ [1, i − 1]}
                // Reduce the list for yx_i because f should be proper. //
                if i = d and S(x_1) \ S(y) = {c} then L(yx_d) ← L(yx_d) \ {c}
                // Reduce the list for yx_d since f should distinguish y from x_1. //
                Choose f(yx_i) ∈ L(yx_i).
                S(y) ← S(y) ∪ {f(yx_i)}
                S(x_i) ← S(x_i) ∪ {f(yx_i)}
                if deg_T(x_i) = 1 then X ← X ∪ {x_i}
        X ← X ∪ {y}
    return f(e) for every e ∈ E
    // Exhibit a required colouring. //
```

The algorithm builds step by step colour sets $S(v)$ of vertices $v \in V$ by assigning colours to all uncoloured edges incident to v. It starts with colouring an arbitrary pendant edge of T so that the (unique) pendant vertex of this edge is the first vertex of the set X of all vertices of T with currently finished colour sets. If $X \neq V$, the subgraph of T induced by all currently coloured edges has a leaf out of X (an *extension vertex*). A successive colouring of all edges (not coloured yet) incident to the extension vertex extends X by all pendant neighbours (out of X) of the extension vertex and, finally, by the extension vertex itself.

To see that the algorithm is correct it is sufficient to use the following two facts:

a) No list is reduced to the empty set. With $k = 1$ the worst case (the reduction to a singleton) can apply if either $\deg_T(y) = \Delta(T)$ or $\deg_T(x) = \Delta(T) - 1 = \deg_T(y)$. If $k = 2$, such a situation can appear provided that $\deg_T(x) = \Delta(T) = \deg_T(y)$.

b) If a vertex y is added to X and $X \cup \{y\} \neq V$, then the subgraph of T induced by all currently coloured edges is a subtree of T, all its non-leaves are in X and at least one of its leaves is not a leaf of the whole T (so that it can be chosen as a new y and guarantees that $d \geq 2$ in the corresponding **for** loop).

□

2.3 Unbalanced complete bipartite graphs

Unbalanced complete bipartite graphs also are in favour of Conjecture 2.

Theorem 7 *If m, n are positive integers with $m < n$, then* $\mathrm{lndi}(K_{m,n}) = \mathrm{ndi}(K_{m,n}) = n$.

Proof: It is well known that $\chi'(K_{m,n}) = n$. Since $K_{m,n}$ is neighbour-irregular, by Proposition 1 we have $\mathrm{ndi}(K_{m,n}) = \chi'(K_{m,n}) = n$.

On the other hand, Janssen in [7] proved that $\mathrm{ch}'(K_{m,n}) = n$; so, again by Proposition 1, $\mathrm{lndi}(K_{m,n}) = \mathrm{ch}'(K_{m,n}) = n$ and we are done. □

3 General colourings

The general neighbour-distinguishing index was introduced by Győri et al. in [4], where the authors proved that $\mathrm{gndi}(G) \leq 2\lceil \log_2 \chi(G) \rceil + 1$ for any graph G having no isolated edges. The result was improved by Győri and Palmer who proved in [5] that $\chi(G) \geq 3$ implies $\mathrm{gndi}(G) = \lceil \log_2 \chi(G) \rceil + 1$ (in the case $\log_2 \chi(G) \notin \mathbb{Z}$ the same expression was obtained independently by Horňák and Soták in [6]).

The analysis of the general list neighbour-distinguishing index seems to be a bit more complicated than that of the list neighbour-distinguishing index. Indeed, we know the exact values just for paths and cycles, for trees we only have an upper bound equal to 3. Also, an analogue of Conjecture 2 fails.

3.1 Paths and cycles

We shall see that, as in the case of proper colourings, when passing to the list version the value of the parameter does not change in the classes of paths and cycles.

Proposition 8 *Let n be an integer, $n \geq 3$.*

(1) If $n \equiv 1 \pmod 2$, then $\operatorname{glndi}(P_n) = \operatorname{gndi}(P_n) = 2$.

(2) If $n \equiv 0 \pmod 2$, then $\operatorname{glndi}(P_n) = \operatorname{gndi}(P_n) = 3$.

Proof: The values of $\operatorname{gndi}(P_n)$ come from [4] so that we only need to show that $\operatorname{glndi}(P_n) = \operatorname{gndi}(P_n)$.

An edge colouring $f : E(P_n) \to \mathbb{N}$ is neighbour-distinguishing if and only if $f(e_1) \neq f(e_2)$, $f(e_i) \neq f(e_{i+2})$ for every $i \in [1, n-3]$ and $f(e_{n-2}) \neq f(e_{n-1})$. Let G_n be the graph with $V(G_n) = E(P_n)$ and $E(G_n) = \{e_1e_2\} \cup \{e_ie_{i+2} : i \in [1, n-3]\} \cup \{e_{n-2}e_{n-1}\}$. If $n \geq 4$ ($n = 2k$ or $n = 2k+1$ with $k \in \mathbb{Z}$, $k \geq 2$), then $G_n \cong C_{n-1}$. Indeed, the following are cycles in G_n: $(e_1)\prod_{i=1}^{k-1}(e_{2i})[\prod_{i=1}^{k}(e_{2k+1-2i})]$ (if $n = 2k$) and $(e_1)\prod_{i=1}^{k}(e_{2i})[\prod_{i=1}^{k}(e_{2k+1-2i})]$ (if $n = 2k+1$). Therefore, $\operatorname{glndi}(P_n) = \operatorname{ch}(C_{n-1})$ and we are done by using the well-known fact that $\operatorname{ch}(C_l) = (l)_2$. Finally, $G_3 \cong P_2$, hence $\operatorname{glndi}(P_3) = \operatorname{ch}(P_2) = 2 = \operatorname{gndi}(P_3)$. □

Proposition 9 *Let n be an integer, $n \geq 3$.*

(1) If $n \equiv 0 \pmod 4$, then $\operatorname{glndi}(C_n) = \operatorname{gndi}(C_n) = 2$.

(2) If $n \not\equiv 0 \pmod 4$, then $\operatorname{glndi}(C_n) = \operatorname{gndi}(C_n) = 3$.

Proof: Similarly as in the preceding proposition, we only have to prove that $\operatorname{glndi}(C_n) = \operatorname{gndi}(C_n)$ (since $\operatorname{gndi}(C_n)$ is known from [4]).

An edge colouring $f : E(C_n) \to \mathbb{N}$ is neighbour-distinguishing if and only if $f(e_i) \neq f(e_{i+2})$ for every $i \in [1, n]$. Let H_n be the graph with $V(H_n) = E(C_n)$ and $E(H_n) = \{e_ie_{i+2} : i \in [1, n]\}$; then $\operatorname{glndi}(C_n) = \operatorname{ch}(H_n)$.

If n is odd, $n = 2k+1$, then $(e_{2k})\prod_{i=0}^{k}(e_{2i+1})[\prod_{i=1}^{k}(e_{2i})]$ is a cycle in H_n, $H_n \cong C_n$ and $\operatorname{glndi}(C_n) = \operatorname{ch}(C_n) = 3$. If n is even, $n = 2k$, $k \geq 3$, then $(e_{2k-2+j})\prod_{i=0}^{k-1}(e_{j+2i})$, $j = 1, 2$, is a cycle in H_n, hence $H_n \cong 2C_k$ and $\operatorname{glndi}(C_n) = \operatorname{ch}(2C_k) = \operatorname{ch}(C_k) = (k)_2$. Finally, $H_4 \cong 2P_2$, and $\operatorname{glndi}(H_4) = \operatorname{ch}(2P_2) = \operatorname{ch}(P_2) = 2$. □

3.2 Trees

The following two results are taken from [4].

Theorem 10 *If G is a connected bipartite graph with $|V(G)| \geq 3$, then* $2 \leq \operatorname{gndi}(G) \leq 3$. □

Proposition 11 *For any graph G the following statements are equivalent:*

(1) $\operatorname{gndi}(G) = 2$.

(2) G is bipartite and there is a bipartition $\{X_1 \cup X_2, Y\}$ of $V(G)$ such that $X_1 \cap X_2 = \emptyset$ and any vertex of Y has at least one neighbour in both X_1 and X_2. □

Proposition 12 *If T is a tree with $|V(T)| \geq 3$, then* $\operatorname{gndi}(T) = 2$ *if and only if the distance between any two leaves of T is even.*

Proof: If $\mathrm{gndi}(T) = 2$, by Proposition 11 we know that there is a bipartition $\{X_1 \cup X_2, Y\}$ of $V(T)$ such that $X_1 \cap X_2 = \emptyset$ and any vertex of Y has at least one neighbour in both X_1 and X_2. Thus $\deg_T(y) \geq 2$ for every $y \in Y$, all leaves of T are in $X_1 \cup X_2$ and the distance between any two of them is even.

Now suppose that the distance between any two leaves of T is even. Let $\{X, Y\}$ be the bipartition $\{X, Y\}$ of $V(T)$ with X containing all leaves of T. Pick $x_0 \in X$ and let $\{X_1, X_2\}$ be the decomposition of X determined as follows: $x \in X_i \overset{\text{df.}}{\Leftrightarrow} d_T(x_0, x) \equiv 2i \pmod 4$, $i = 1, 2$. Consider a vertex $y \in Y$ and its neighbour x' lying on the (unique) path joining y to x_0. Since y is not a leaf, it has a neighbour $x'' \neq x'$. Clearly, $d_T(x_0, x'') = d_T(x_0, x') + 2$, which means that one of x', x'' is in X_1 and the other in X_2. By Proposition 11 then $\mathrm{gndi}(T) = 2$. □

Theorem 13 *If T is a tree with $|V(T)| \geq 3$, then* $\mathrm{glndi}(T) \leq 3$.

Proof: We proceed by induction on the lexicographical ordering of pairs $(\mathrm{diam}(T), |V(T)|)$, where $\mathrm{diam}(T)$ is the diameter of T. From $|V(T)| \geq 3$ we know that $\mathrm{diam}(T) \geq 2$. If $\mathrm{diam}(T) = 2$, then T is a star and $\mathrm{glndi}(T) = 2$ (it is sufficient to colour the second coloured edge with a colour that is different from the first used colour).

So, suppose that $\mathrm{diam}(T) \geq 3$ and $\mathrm{glndi}(T') \leq 3$ for every tree T' with the pair $(\mathrm{diam}(T'), |V(T')|)$ lexicographically smaller than the pair $(\mathrm{diam}(T), |V(T)|)$. Let x be an endvertex (a leaf) of a longest path in T and let y be the unique neighbour of x. Further, let z be the unique non-leaf neighbour of y.

Consider a mapping $L \in \mathcal{L}_3(T)$, the tree $T' := T - x$ and the mapping $L' := L \setminus \{(xy, L(xy))\} \in \mathcal{L}_3(T')$. By the induction hypothesis there is a neighbour-distinguishing colouring $f' : E(T') \to \mathbb{N}$ with $f'(e) \in L'(e) = L(e)$ for every $e \in E(T')$. Let $f : E(T) \to \mathbb{N}$ be an extension of f' with $f(xy) \in L(xy)$ that has the following two properties: if $S_{f'}(z) \setminus S_{f'}(y) = \{c_1\}$, then $f(xy) \neq c_1$; if $S_{f'}(y) = \{c_2\}$ (which means that $\deg_{T'}(y) = 1$), then $f(xy) \neq c_2$. Since $|L(xy)| = 3$, f does exist. Clearly, f is neighbour-distinguishing, the tree T is L-neighbour-distinguishable and $\mathrm{glndi}(T) \leq 3$. □

The rest of this subsection is devoted to showing that an analogue of Conjecture 2, in which the list neighbour-distinguishing index is replaced by the general list neighbour-distinguishing index, fails.

Let T and L be the tree and the mapping from $\mathcal{L}_2(T)$, respectively, that are presented in Figure 1. The distance between any two leaves in T is even, hence, by Proposition 12, $\mathrm{gndi}(T) = 2$. Let us prove that T is not L-neighbour-distinguishable (so that, by Theorem 13, $\mathrm{glndi}(T) = 3$). Suppose, on the contrary, that f is a neighbour-distinguishing colouring of T with $f(e) \in L(e)$ for every $e \in E(T)$.

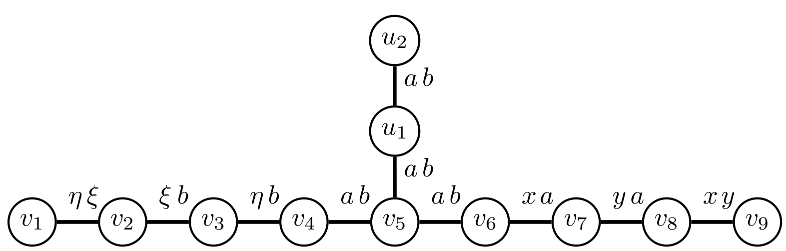

Fig. 1: A tree T with $2 = \mathrm{gndi}(T) < \mathrm{glndi}(T) = 3$.

Case 1: $|S_f(v_5)| = 1$

Suppose first that $S_f(v_5) = \{a\}$. Then $S_f(v_6) \neq S_f(v_7)$ implies $f(v_7v_8) = y$ and, consequently, $f(v_8v_9) = x$. Now, however, with $f(v_6v_7) = x$ we obtain $S_f(v_7) = \{x, y\} = S_f(v_8)$, while $f(v_6v_7) = a$ yields $S_f(v_5) = \{a\} = S_f(v_6)$, a contradiction in both cases. If $S_f(v_5) = \{b\}$, we get a contradiction by a "symmetrical" reasoning on the "left-hand side" of T.

Case 2: $S_f(v_5) = \{a, b\}$

Now it follows from $S_f(v_5) \neq S_f(u_1)$ and $L(v_5u_1) = L(u_1u_2) = \{a, b\} = S_f(v_5)$ that $|S_f(u_1)| = 1$. As a consequence of $d_T(u_2) = 1$ we have also $|S_f(u_2)| = 1$. Then, however, from $f(u_1u_2) \in S_f(u_1) \cap S_f(u_2)$ we obtain $S_f(u_1) = S_f(u_2)$, a contradiction.

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