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Series acceleration formulas for beta values

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We prove generating function identities producing fast convergent series for the sequences $\beta(2n+1)$, $\beta(2n+2)$ and $\beta(2n+3)$, where β is Dirichlet's beta function. In particular, we obtain a new accelerated series for Catalan's constant convergent at a geometric rate with ratio 2^{-10}, which can be considered as an analog of Amdeberhan-Zeilberger's series for $\zeta(3)$.

Keywords: Dirichlet's beta function, Catalan's constant, Apéry-like series, generating function, convergence acceleration, Markov-Wilf-Zeilberger method, Markov-WZ pair.

In honor of P. Flajolet's 60th birthday

1 Introduction

In this paper we continue our study on finding generalized identities [8, 9] that produce fast convergent series for some classical constants. In 1978, R. Apéry used the fast convergent series

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}} \tag{1}$$

first obtained by A. A. Markov [14] to derive the irrationality of $\zeta(3)$ [17]. This series converges at a geometric rate with ratio $1/4$. A general formula giving analogous series for all $\zeta(2n+3)$, $n \geq 0$, was proved by Koecher [11] (and independently in an expanded form by Leshchiner [12]). For $|a| < 1$, it has the form

$$\sum_{n=0}^{\infty}\zeta(2n+3)a^{2n} = \sum_{k=1}^{\infty}\frac{1}{k(k^2-a^2)} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}\,\frac{5k^2-a^2}{k^2-a^2}\prod_{m=1}^{k-1}\left(1-\frac{a^2}{m^2}\right). \tag{2}$$

Expanding the right-hand side of (2) by powers of a^2 and comparing coefficients of a^{2n} on both sides leads to the Apéry-like series for $\zeta(2n+3)$. In particular, comparing constant terms $(n = 0)$ recovers

Markov's formula (1) and comparing coefficients of a^2 and a^4 gives the following two formulas:

$$\zeta(5) = 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}} - \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^2},$$

$$\zeta(7) = 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}} - 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^2} + \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{m=1}^{k-1}\frac{1}{m^2}\sum_{j=1}^{m-1}\frac{1}{j^2},$$

respectively. In 1996, inspired by this result, J. Borwein and D. Bradley [4] applied extensive computer searches on the base of integer relations algorithms looking for additional zeta identities of this sort. This led to the discovery of the new identity

$$\zeta(7) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}} + \frac{25}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}, \tag{3}$$

which is simpler than Koecher's formula for $\zeta(7)$, and similar identities for $\zeta(9)$, $\zeta(11)$, $\zeta(13)$, etc. This allowed them to conjecture that certain of these identities, namely those for $\zeta(4n+3)$ are given by the following generating function formula [3]:

$$\sum_{n=0}^{\infty}\zeta(4n+3)a^{4n} = \sum_{k=1}^{\infty}\frac{k}{k^4-a^4} = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{\binom{2k}{k}(k^4-a^4)}\prod_{m=1}^{k-1}\left(\frac{m^4+4a^4}{m^4-a^4}\right), \quad |a|<1. \tag{4}$$

The validity of (4) was proved later by G. Almkvist and A. Granville [1] in 1999. Expanding the right-hand side of (4) in powers of a^4 gives a formula for $\zeta(4n+3)$, which for $n\ge 1$ contains fewer summations the the corresponding formula generated by (2). In particular, comparing constant terms gives (1) and comparing coefficients of a^4 yields (3).

There exists a bivariate unifying formula for identities (2) and (4)

$$\sum_{k=1}^{\infty}\frac{k}{k^4-x^2k^2-y^4} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k\binom{2k}{k}}\frac{5k^2-x^2}{k^4-x^2k^2-y^4}\prod_{m=1}^{k-1}\frac{(m^2-x^2)^2+4y^4}{m^4-x^2m^2-y^4}. \tag{5}$$

It was originally conjectured by H. Cohen and then proved by D. Bradley [7] and, independently, by T. Rivoal [18]. Their proof consists of reduction of (5) to a finite non-trivial combinatorial identity which can be proved on the basis of Almkvist and Granville's work [1]. Another proof of (5) based on application of WZ-pairs was given by the authors in [9]. Since

$$\sum_{k=1}^{\infty}\frac{k}{k^4-x^2k^2-y^4} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{n}\zeta(2n+4m+3)x^{2n}y^{4m}, \quad |x|^2+|y|^4<1, \tag{6}$$

the formula (5) generates Apéry-like series for all $\zeta(2n+4m+3)$, $n,m\ge 0$, convergent at the geometric rate with ratio $1/4$ and contains, as particular cases, both identities (2) and (4).

In [9], the authors showed that the generating function (6) also has a much more rapidly convergent representation, namely

$$\sum_{k=1}^{\infty}\frac{k}{k^4-x^2k^2-y^4} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}r(n)}{n\binom{2n}{n}}\frac{\prod_{m=1}^{n-1}((m^2-x^2)^2+4y^4)}{\prod_{m=n}^{2n}(m^4-x^2m^2-y^4)}, \tag{7}$$

where

$$r(n) = 205n^6 - 160n^5 + (32 - 62x^2)n^4 + 40x^2n^3 + (x^4 - 8x^2 - 25y^4)n^2 + 10y^4n + y^4(x^2 - 2).$$

The identity (7) produces accelerated series for all $\zeta(2n+4m+3)$, $n, m \geq$, convergent at the geometric rate with ratio 2^{-10}. In particular, if $x = y = 0$ we get Amdeberhan-Zeilberger's series [2] for $\zeta(3)$,

$$\zeta(3) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(205n^2 - 160n + 32)}{n^5\binom{2n}{n}^5}. \tag{8}$$

It is worth pointing out that both identities (5) and (7) were proved in [9] by using the same Markov-WZ pair (see also [10, p. 3] for the explicit expression), but with the help of different summation formulas.

A more general form of the bivariate identity (5) for the generating function

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+n}{n}(A_0\zeta(2n+4m+4) + B_0\zeta(2n+4m+3) + C_0\zeta(2n+4m+2))x^{2n}y^{4m}$$
$$= \sum_{k=1}^{\infty} \frac{A_0 + B_0k + C_0k^2}{k^4 - x^2k^2 - y^4}, \qquad |x|^2 + |y|^4 < 1,$$

where A_0, B_0, C_0 are arbitrary complex numbers, was proved in [9] by means of the Markov-Wilf-Zelberger theory. More precisely, one has

$$\sum_{k=1}^{\infty} \frac{A_0 + B_0k + C_0k^2}{k^4 - x^2k^2 - y^4} = \sum_{n=1}^{\infty} \frac{d_n}{\prod_{m=1}^{n}(m^4 - x^2m^2 - y^4)}, \tag{9}$$

where

$$d_n = \frac{(-1)^{n-1}B_0(5n^2 - x^2)}{2n\binom{2n}{n}} \prod_{m=1}^{n-1}((m^2 - x^2)^2 + 4y^4)$$
$$+ \frac{(40n+10)L_n + (35n^5 - 35n^3x^2 + 4n(3x^4 + 10y^4))L_{n-1}}{4(5n^2 - 2x^2)}$$

and L_n is a solution of a certain second order linear difference equation with polynomial coefficients in n and x, y with the initial values $L_0 = C_0$, $L_1 = (5 - 2x^2)A_0/15 + (5x^2 - 1 - 4(x^4 + 6y^4))C_0/30$. If we take $A_0 = C_0 = 0$, $B_0 = 1$ in (9), then $L_n = 0$ for all $n \geq 0$ and we get the bivariate identity (5). If $B_0 = C_0 = 0$, $A_0 = 1$, then we obtain

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+n}{n}\zeta(2n+4m+4)x^{2n}y^{4m}$$
$$= \frac{1}{4}\sum_{n=1}^{\infty} \frac{(40n+10)L_n + (35n^5 - 35x^2n^3 + 4n(3x^4 + 10y^4))L_{n-1}}{(5n^2 - 2x^2)\prod_{m=1}^{n}(m^4 - x^2m^2 - y^4)},$$

where the sequence L_n is defined recursively as above. A similar formula is valid for the sequence $\zeta(2n+4m+2)$.

First results related to generating function identities for even zeta values belong to Leshchiner [12] who proved (in an expanded form) that for $|a| < 1$,

$$\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{2n+1}}\right)\zeta(2n+2)a^{2n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-a^2} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\frac{3k^2+a^2}{k^2-a^2}\prod_{m=1}^{k-1}\left(1-\frac{a^2}{m^2}\right). \quad (10)$$

Comparing constant terms on both sides of (10) yields

$$\zeta(2) = 3\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}.$$

Since

$$\zeta(2n) = (-1)^{n-1}\frac{(2\pi)^{2n}}{2\cdot(2n)!}B_{2n},$$

where $B_{2n} \in \mathbb{Q}$ are the Bernoulli numbers generated by the exponential generating function

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty}B_n\frac{x^n}{n!},$$

formula (10) gives Apéry-like series for even powers π^{2n}, $n = 1, 2, \ldots$. In 2006, D. Bailey, J. Borwein and D. Bradley [5] proved another identity

$$\sum_{n=0}^{\infty}\zeta(2n+2)a^{2n} = \sum_{k=1}^{\infty}\frac{1}{k^2-a^2} = 3\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}(k^2-a^2)}\prod_{m=1}^{k-1}\left(\frac{m^2-4a^2}{m^2-a^2}\right).$$

It generates similar Apéry-like series for the numbers $\zeta(2n+2)$, which are not covered by Leshchiner's result (10). In the same paper [5], a generating function producing fast convergent series for the sequence π^{2n+4}, $n = 0, 1, 2, \ldots$, was found, which for $|a| < 1$, has the form

$$\sum_{n=0}^{\infty}\frac{(-1)^n(3^{-2n-3}-2B_{2n+4}(2^{2n+3}-1))\pi^{2n+4}}{4(2n+4)!}a^{2n} = \sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}(k^2-a^2)}\prod_{m=1}^{k-1}\left(1-\frac{a^2}{m^2}\right). \quad (11)$$

Here the left-hand side of (11) is a Maclaurin expansion of the function

$$\frac{\pi a\csc(\pi a)+3\cos(\pi a/3)-4}{4a^4}.$$

Comparing constant terms in (11) implies that

$$\zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}.$$

The identity (11) gives a formula for π^{2n+4} which for $n \ge 0$ involves fewer summations then the corresponding formula generated by (10).

In this paper, we consider values of Dirichlet's beta function, which for $\Re s > 0$ is defined by the series

$$\beta(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s}.$$

It is well-known due to Euler that for odd s, $\beta(s)$ is a rational multiple of π^s

$$\beta(2n+1) = \frac{(-1)^n E_{2n}}{2^{2n+2}(2n)!}\pi^{2n+1}, \qquad n = 0, 1, 2, \ldots, \tag{12}$$

where the integer coefficients E_{2n} are the even indexed Euler numbers defined by the exponential generating function

$$\frac{1}{\cosh(z)} = \frac{2e^z}{e^{2z}+1} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n = 1 - \frac{z^2}{2!} + \frac{5z^4}{4!} - \frac{61z^6}{6!} + \ldots.$$

For even s, no Euler-type formula is known, and $\beta(2)$ defines the well-known Catalan's constant

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965594\ldots.$$

In Sections 2, 3 we prove generating function identities producing fast convergent series for the sequences $\beta(2n+1)$ and $\beta(2n+2)$. In particular, we establish some analogs of identities (10) and (11) for the odd powers of $\pi : \pi^{2n+1}, n \geq 0$, and $\pi^{2n+3}, n \geq 0$. As a consequence of our results on even beta values, we derive the following nice formula for Catalan's constant:

$$G = \frac{1}{64} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 256^n q(n)}{\binom{8n}{4n}^2 \binom{2n}{n} n^3 (2n-1)(4n-1)^2(4n-3)^2}, \tag{13}$$

where

$$q(n) = 419840n^6 - 915456n^5 + 782848n^4 - 332800n^3 + 73256n^2 - 7800n + 315.$$

It can be considered as an analog of Amdeberhan-Zeilberger's series (8) for $\zeta(3)$, since the series on the right-hand side of (13) converges exponentially with ratio 2^{-10}.

2 Apéry-like series for $\beta(2n+1)$ and $\beta(2n+3)$.

We start by recalling several definitions and known facts related to the Markov-Wilf-Zeilberger theory (see [14, 15, 16]). A function $H(n,k)$, in the integer variables n and k, is called *hypergeometric* or *closed form (CF)* if the quotients

$$\frac{H(n+1,k)}{H(n,k)} \quad \text{and} \quad \frac{H(n,k+1)}{H(n,k)}$$

are both rational functions of n and k. A hypergeometric function that can be written as a ratio of products of factorials is called *pure-hypergeometric*. A pair of CF functions $F(n,k)$ and $G(n,k)$ is called a *WZ-pair* if

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k). \tag{14}$$

A *P-recursive* function is a function that satisfies a linear recurrence relation with polynomial coefficients. If for a given hypergeometric function $H(n,k)$, there exists a polynomial $P(n,k)$ in k of the form

$$P(n,k)=a_0(n)+a_1(n)k+\cdots+a_L(n)k^L,$$

for some non-negative integer L, and P-recursive functions $a_0(n),\dots,a_L(n)$ such that

$$F(n,k):=H(n,k)P(n,k)$$

satisfies (14) with some function G, then a pair (F,G) is called a *Markov-WZ pair* associated with the kernel $H(n,k)$ (MWZ-pair for short). We call $G(n,k)$ an *MWZ mate* of $F(n,k)$. If $L=0$, then (F,G) is simply a WZ-pair.

In 2005, M. Mohammed [15] showed that for any pure-hypergeometric kernel $H(n,k)$, there exists a non-negative integer L and a polynomial $P(n,k)$ as above such that $F(n,k)=H(n,k)P(n,k)$ has an MWZ mate $G(n,k)=F(n,k)Q(n,k)$, where $Q(n,k)$ is a ratio of two P-recursive functions. Paper [16] is accompanied by the Maple package MarkovWZ which, for a given $H(n,k)$ outputs the polynomial $P(n,k)$ and the $G(n,k)$ as above.

From relation (14) we get the following summation formulas.

Proposition A. [15, Theorem 2(b)] *Let (F,G) be an MWZ-pair. If $\lim\limits_{n\to\infty}F(n,k)=0$ for every $k\geq 0$, then*

$$\sum_{k=0}^{\infty}F(0,k)-\lim_{k\to\infty}\sum_{n=0}^{\infty}G(n,k)=\sum_{n=0}^{\infty}G(n,0), \tag{15}$$

whenever both sides converge.

Proposition B. [15, Cor. 2] *Let (F,G) be an MWZ-pair. If $\lim\limits_{k\to\infty}\sum\limits_{n=0}^{\infty}G(n,k)=0$, then*

$$\sum_{k=0}^{\infty}F(0,k)=\sum_{n=0}^{\infty}(F(n,n)+G(n,n+1)), \tag{16}$$

whenever both sides converge.

Formulas (15), (16) with an appropriate choice of MWZ-pairs can be used to convert a given hypergeometric series into a different rapidly converging one.

As usual, let $(\lambda)_\nu$ be the Pochhammer symbol (or the shifted factorial) defined by

$$(\lambda)_\nu=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}=\begin{cases}1, & \nu=0;\\ \lambda(\lambda+1)\dots(\lambda+\nu-1), & \nu\in\mathbb{N}.\end{cases}$$

In 1979, D. Leshchiner [12] proved accelerated series for the values $\beta(2n+1)$ in the spirit of Apéry's series (1). Namely, he showed that

$$\beta(2n+1)=\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{16^k}\sum_{\nu=0}^{n}\frac{(-1)^{n-\nu}A_\nu}{(2k+1)^{2\nu+1}}f_{k+1}^{n-\nu}, \qquad n=0,1,2,\dots \tag{17}$$

where $f_k^0 = 1$, $f_k^r = 0$ if $r < 0$ or $r \geq k$,

$$f_k^r = \sum_{0<l_1<\ldots<l_r<k} \prod_{j=1}^{r} \frac{1}{(2l_j-1)^2}, \qquad 1 \leq r \leq k-1,$$

and

$$A_\nu = \begin{cases} 3/4 & \text{if} \quad \nu = 0 \\ 1 & \text{if} \quad \nu > 0. \end{cases}$$

Formulas (12), (17) give the following Apéry-like series for odd powers of π :

$$\pi = 3\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k(2k+1)},$$

$$\pi^3 = 32\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k(2k+1)^3} - 24\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k(2k+1)} \sum_{m=0}^{k-1} \frac{1}{(2m+1)^2},$$

$$\pi^5 = \frac{1536}{5}\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k(2k+1)^5} - \frac{1536}{5}\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k(2k+1)^3} \sum_{m=0}^{k-1} \frac{1}{(2m+1)^2} + \frac{1152}{5}\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k(2k+1)} \sum_{m=0}^{k-1}\sum_{l=0}^{k-1} \frac{1}{(2m+1)^2(2l+1)^2}.$$

Using the generating function for odd beta values

$$\sum_{n=0}^{\infty} \beta(2n+1)a^{2n} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k(k+1/2)}{(k+1/2)^2 - a^2/4} \tag{18}$$

it is easily seen that formulas (17) are generated by the following identity.

Proposition 1 *For any complex a with $|a| < 1$ we have*

$$\sum_{n=0}^{\infty} \beta(2n+1)a^{2n} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^{2n+1}(2n+1)} \frac{3(2n+1)^2 + a^2}{(2n+1)^2 - a^2} \prod_{m=0}^{n-1} \left(1 - \frac{a^2}{(2m+1)^2}\right). \tag{19}$$

Proof. Identity (19) can easily be proved by the Markov-Wilf-Zeilberger method. Taking the kernel

$$H(n,k) = \frac{(-1)^k k!(n+k+1/2)}{(k+2n)!((n+k+1/2)^2 - a^2/4)}$$

and applying the Maple package MarkovWZ we get that

$$F(n,k) = \frac{2(-1)^k k!(2n+2k+1)}{(k+2n)!((2n+2k+1)^2 - a^2)} \left(\frac{1+a}{2}\right)_n \left(\frac{1-a}{2}\right)_n$$

and

$$G(n,k)=\frac{(-1)^k k!(3(2n+1)^2+a^2+4k(2+4n+k))}{2(k+2n+1)!((2n+2k+1)^2-a^2)}\left(\frac{1+a}{2}\right)_n\left(\frac{1-a}{2}\right)_n$$

give a WZ-pair, i.e.,

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k).$$

Now by Proposition A, we get

$$\sum_{k=0}^{\infty}F(0,k)=\sum_{n=0}^{\infty}G(n,0),$$

which implies (19). □

Lemma 1 *For any complex* a,

$$\sin\frac{\pi a}{6}=\frac{a}{2}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{16^n(2n+1)}\prod_{m=0}^{n-1}\left(1-\frac{a^2}{(2m+1)^2}\right). \tag{20}$$

Proof. Applying the transformation (see [6, §2.8]) in terms of the Gauss hypergeometric function

$$\sin az=a\sin z\cdot {}_2F_1\left(\frac{1+a}{2},\frac{1-a}{2};\frac{3}{2};\sin^2 z\right)$$

with $z=\pi/6$ we get the desired representation. □

Remark. Formula (20) can be considered as an expansion of the function $\sin(\pi a/6)$ into Newton's interpolation series with interpolation points at zero and odd integers.

Theorem 1 *Let* a *be a complex number, distinct from an odd integer. Then*

$$\frac{\pi}{4a^2\cos(\pi a/2)}-\frac{3}{2a^3}\sin\frac{\pi a}{6}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{16^n(2n+1)((2n+1)^2-a^2)}\prod_{m=0}^{n-1}\left(1-\frac{a^2}{(2m+1)^2}\right). \tag{21}$$

If $|a|<1$, *then the left-hand side of* (21) *has the following series expansion:*

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}((2n+3)9^{n+1}E_{2n+2}-1)\pi^{2n+3}}{4^{n+2}9^{n+1}(2n+3)!}a^{2n}=\frac{7\pi^3}{216}+\frac{253\pi^5}{77760}a^2+\dots.$$

In particular, we get

$$\pi^3=\frac{216}{7}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{16^n(2n+1)^3},$$

$$\pi^5=\frac{77760}{253}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{16^n(2n+1)^5}-\frac{77760}{253}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{16^n(2n+1)^3}\sum_{m=0}^{n-1}\frac{1}{(2m+1)^2}.$$

Proof. By Proposition 1 and Lemma 1 we get

$$\sum_{n=0}^{\infty}\beta(2n+1)a^{2n}=\frac{3}{2a}\sin\frac{\pi a}{6}+a^2\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{16^n(2n+1)((2n+1)^2-a^2)}\prod_{m=0}^{n-1}\left(1-\frac{a^2}{(2m+1)^2}\right).$$

On the other hand, by (18), we have

$$4\cdot\sum_{n=0}^{\infty}\beta(2n+1)a^{2n}=g\left(\frac{1-a}{2}\right)+g\left(\frac{1+a}{2}\right),$$

where $g(z)=\sum_{k=0}^{\infty}(-1)^k/(k+z)$. Using the reflection formula (see [6, §1.8])

$$g(z)+g(1-z)=\frac{\pi}{\sin(\pi z)}$$

we get

$$\sum_{n=0}^{\infty}\beta(2n+1)a^{2n}=\frac{\pi}{4\cos(\pi a/2)},$$

from which formula (21) follows. Expanding the left-hand side of (21), by (12), we get the required expansion. □

Comparing (21) and (19) we can easily see that these identities produce different series for the odd powers of π. The series generated by (21) are simpler in that sense they involve fewer summations then the corresponding series generated by (19). Moreover, the formulas obtained here lead to some non-trivial redundancy relations that can be written down explicitly by comparing (17) and the corresponding series for $\beta(2n+1)$, $n\ge 1$, given by (21). As an example, comparing series for π^3 yields the following redundancy formula:

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{16^n(2n+1)}\sum_{m=0}^{n-1}\frac{1}{(2m+1)^2}=\frac{1}{21}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{16^n(2n+1)^3}.$$

3 Generating function identities for even beta values.

In this section we derive series acceleration formulae for even beta values by application of the Markov-WZ method.

Theorem 2 *Let a and d be complex numbers such that $a^2\ne(d+n)^2$, $n=0,1,2,\dots$, and d distinct from zero and negative integers. Then*

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+d)^2-a^2}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(1+a)_n(1-a)_n(3n^2+2n(2d+1)+d^2+2d+a^2)}{(d)_{2n+2}((n+d)^2-a^2)}. \tag{22}$$

Proof. Considering the kernel

$$H(n,k)=\frac{(-1)^k(d)_k(n+k+d)}{(d)_{2n+k+1}((n+k+d)^2-a^2)}$$

we get that

$$F(n,k)=\frac{(-1)^k(d)_k(n+k+d)(1+a)_n(1-a)_n}{(d)_{2n+k+1}((n+k+d)^2-a^2)}$$

and

$$G(n,k)=\frac{(-1)^k(d)_k(1+a)_n(1-a)_n}{2(d)_{2n+k+2}((n+k+d)^2-a^2)}\times \\ \times(3n^2+2n(2d+1)+d^2+2d+a^2+k^2+k(2d+4n+2))$$

give a WZ-pair, and by Proposition A, we obtain

$$\sum_{k=0}^{\infty}F(0,k)=\sum_{n=0}^{\infty}G(n,0)$$

implying (22). □

In particular, from (22) if $d=1$ we get Leshchiner's identity (10). If $d=1/2$ we get the following identity generating accelerated series for even beta values.

Corollary 1 *Let a be a complex number with $|a|<1$. Then*

$$\sum_{n=0}^{\infty}\beta(2n+2)a^{2n}=\frac{1}{8}\sum_{n=1}^{\infty}\frac{16^n(12n^2-8n+1+a^2)}{\binom{4n}{2n}\binom{2n}{n}n^2((2n-1)^2-a^2)}\prod_{m=1}^{n-1}\left(1-\frac{a^2}{4m^2}\right).$$

In particular, if $a=0$ we get

$$G=\frac{1}{8}\sum_{n=1}^{\infty}\frac{16^n(6n-1)}{\binom{4n}{2n}\binom{2n}{n}n^2(2n-1)}.$$

Theorem 3 *Let a be a complex number with $|a|<1$. Then*

$$\sum_{n=0}^{\infty}\beta(2n+2)a^{2n}=\frac{1}{64}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}256^n(40n^2-24n+3-a^2)}{\binom{4n}{2n}^2\binom{2n}{n}n^3(2n-1)}\frac{\prod_{m=1}^{n-1}(1-a^2/(4m^2))}{\prod_{m=1}^{2n}(1-a^2/(2m-1)^2)}.$$

In particular,

$$G=\frac{1}{64}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}256^n(40n^2-24n+3)}{\binom{4n}{2n}^2\binom{2n}{n}n^3(2n-1)}. \tag{23}$$

Proof. Starting with the kernel

$$H(n,k)=(-1)^k\frac{\left(\frac{1+a}{2}\right)_k\left(\frac{1-a}{2}\right)_k}{\left(\frac{1-a}{2}\right)_{2n+k+1}\left(\frac{1+a}{2}\right)_{2n+k+1}}$$

and applying the Maple package MarkovWZ [16], we see that

$$F(n,k)=\frac{(-1)^{n+k}\left(\frac{1+a}{2}\right)_k\left(\frac{1-a}{2}\right)_k\left(1+\frac{a}{2}\right)_n\left(1-\frac{a}{2}\right)_n(2n)!}{\left(\frac{1-a}{2}\right)_{2n+k+1}\left(\frac{1+a}{2}\right)_{2n+k+1}}$$

and

$$G(n,k)=\frac{(-1)^{n+k}\left(\frac{1+a}{2}\right)_k\left(\frac{1-a}{2}\right)_k\left(1+\frac{a}{2}\right)_n\left(1-\frac{a}{2}\right)_n(2n)!}{8\left(\frac{1-a}{2}\right)_{2n+k+2}\left(\frac{1+a}{2}\right)_{2n+k+2}}\times$$
$$\times\left(40n^2+56n+19-a^2+24nk+16k+4k^2\right)$$

is a Markov-WZ pair. By using Proposition A, we get

$$\sum_{k=0}^{\infty}F(0,k)=\sum_{n=0}^{\infty}G(n,0),$$

which implies the required statement. □

The generalized identities of Corollary 1 and Theorem 3 generate fast convergent series for $\beta(2n+2)$ convergent at a geometric rate with ratio $1/4$. Formula (23) was earlier found by Lupas in [13]. The next corollary gives much faster convergent series at the geometric rate with ratio $2^2/3^6=4/729=0.005\ldots$.

Corollary 2 *Let a be a complex number with $|a|<1$. Then*

$$\sum_{k=0}^{\infty}\beta(2k+2)a^{2k}=\frac{1}{16}\sum_{n=1}^{\infty}\frac{256^n p(n,a)}{n^3(2n-1)^3\binom{6n}{3n}\binom{6n}{4n}\binom{4n}{2n}}\frac{\prod_{m=1}^{n-1}(1-a^2/(4m^2))}{\prod_{m=n}^{3n}(1-a^2/(2m-1)^2)},$$

where $p(n,a)=580n^4-764n^3+344n^2-61n+15/4-a^2(18n^2-15n+4-a^2/4)$.
In particular,

$$G=\frac{1}{64}\sum_{n=1}^{\infty}\frac{256^n(580n^2-184n+15)}{n^3(2n-1)\binom{6n}{3n}\binom{6n}{4n}\binom{4n}{2n}}.$$

Proof. By applying Proposition B to the MWZ pair from the proof of Theorem 3 we get

$$\sum_{k=0}^{\infty}F(0,k)=\sum_{n=0}^{\infty}(F(n,n)+G(n,n+1)),$$

or

$$\sum_{k=0}^{\infty}\beta(2k+2)a^{2k}=\frac{1}{32}\sum_{n=0}^{\infty}\frac{\left(\frac{1+a}{2}\right)_n\left(\frac{1-a}{2}\right)_n\left(1+\frac{a}{2}\right)_n\left(1-\frac{a}{2}\right)_n(2n)!\,p(n+1,a)}{\left(\frac{1+a}{2}\right)_{3n+3}\left(\frac{1-a}{2}\right)_{3n+3}},$$

where $p(n,a)$ is defined as above. After simplifying and replacing Pochhammer's symbols by binomial coefficients we get the desired identity. □

The next theorem gives even much faster convergent series at the geometric rate with ratio 2^{-10}.

Theorem 4 *Let a be a complex number with $|a|<1$. Then*

$$\sum_{k=0}^{\infty}\beta(2k+2)a^{2k}=\frac{1}{64}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}256^n q(n,a)}{\binom{8n}{4n}^2\binom{2n}{n}n^3(2n-1)(4n-1)^2(4n-3)^2}\frac{\prod_{m=1}^{n-1}\left(1-a^2/(4m^2)\right)}{\prod_{m=2n-1}^{4n}\left(1-a^2/(2m-1)^2\right)},$$

where

$$q(n,a)=419840n^6-915456n^5+n^4(782848-15872a^2)-n^3(332800-25600a^2)$$
$$+n^2(73256-14992a^2+232a^4)-n(7800-3760a^2+184a^4)+315-359a^2+45a^4-a^6.$$

In particular,

$$G=\frac{1}{64}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}256^n q(n)}{\binom{8n}{4n}^2\binom{2n}{n}n^3(2n-1)(4n-1)^2(4n-3)^2},$$

where

$$q(n)=419840n^6-915456n^5+782848n^4-332800n^3+73256n^2-7800n+315.$$

Proof. We first observe that the generating function of even beta values can be written in the form

$$\sum_{k=0}^{\infty}\beta(2k+2)a^{2k}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2-a^2}=\sum_{k=0}^{\infty}\left(\frac{1}{(4k+1)^2-a^2}-\frac{1}{(4k+3)^2-a^2}\right)$$
$$=\frac{1}{32}\sum_{k=0}^{\infty}\frac{2k+1}{((k+1/4)^2-a^2/16)((k+3/4)^2-a^2/16)}.$$

Define the function

$$H(n,k)=\frac{\left(\frac{1+a}{4}\right)_k\left(\frac{1-a}{4}\right)_k\left(\frac{3+a}{4}\right)_k\left(\frac{3-a}{4}\right)_k(n+1+2k)}{\left(\frac{1+a}{4}\right)_{n+k+1}\left(\frac{1-a}{4}\right)_{n+k+1}\left(\frac{3+a}{4}\right)_{n+k+1}\left(\frac{3-a}{4}\right)_{n+k+1}}.$$

Then using the Maple package MarkovWZ we obtain that

$$F(n,k)=\frac{(-1)^n}{16^n}(2n+1)!\left(1+\frac{a}{2}\right)_n\left(1-\frac{a}{2}\right)_n H(n,k)$$

and

$$G(n,k)=\frac{(-1)^n(2n)!\,H(n,k)}{16^{n+1}(n+1+2k)}\left(1+\frac{a}{2}\right)_n\left(1-\frac{a}{2}\right)_n(40n^2+56n+19-a^2+16k(k+3n+2))$$

is a Markov-WZ pair corresponding to the kernel $H(n,k)$. Now by Proposition B, we have

$$\sum_{n=0}^{\infty}(F(n,n)+G(n,n+1))=\sum_{k=0}^{\infty}F(0,k)=\sum_{k=0}^{\infty}H(0,k)=32\sum_{k=0}^{\infty}\beta(2k+2)a^{2k}$$

or

$$\sum_{k=0}^{\infty}\beta(2k+2)a^{2k}$$
$$=\frac{1}{32}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{16^{n+1}}\frac{\left(1+\frac{a}{2}\right)_n\left(1-\frac{a}{2}\right)_n\left(\frac{1+a}{4}\right)_n\left(\frac{1-a}{4}\right)_n\left(\frac{3+a}{4}\right)_n\left(\frac{3-a}{4}\right)_n}{\left(\frac{1+a}{4}\right)_{2n+2}\left(\frac{1-a}{4}\right)_{2n+2}\left(\frac{3+a}{4}\right)_{2n+2}\left(\frac{3-a}{4}\right)_{2n+2}}\,q(n+1,a),$$

where $q(n,a)$ is defined as above. After simplifying and replacing Pochhammer's symbols by binomial coefficients we get the required identity.

Note that application of Proposition A to the Markov-WZ pair found above recovers Theorem 3. □

In conclusion we note that the different approaches described in the introduction for studying and finding various Apéry-like series for zeta values are applicable to the Dirichlet beta function as well. Hence, it would be of interest to find more general identities for beta values by means of bivariate generating functions as it was done in the case of the zeta function (see for instance identities (5), (7)).

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