Discrete Mathematics and Theoretical Computer Science DMTCS vol. **12:2**, 2010, 249–262

Asymptotic enumeration of orientations

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received July 7, 2009, *revised March 3, 2010*, *accepted March 31, 2010*.

We find the asymptotic number of 2-orientations of quadrangulations with n inner faces, and of 3-orientations of triangulations with n inner vertices. We also find the asymptotic number of prime 2-orientations (no separating quadrangle) and prime 3-orientations (no separating triangle). The estimates we find are of the form $c \cdot n^{-\alpha}\gamma^n$, for suitable constants c, α, γ, with $\alpha = 4$ for 2-orientations and $\alpha = 5$ for 3-orientations. The proofs are based on singularity analysis of D-finite generating functions, using the Fuchsian theory of complex linear differential equations.

Keywords: Orientation, Quadrangulation, Triangulation, D-finite generating function.

To Philippe

1 Introduction

In a previous paper (5) we studied 2-orientations of quadrangulations and 3-orientations of triangulations from an enumerative point of view, focusing on bijections with other combinatorial objects, particularly trees and permutations. In this paper we focus on asymptotic counting.

A quadrangulation Q is a plane graph in which every face is a quadrangle. Since every face is bounded by an even circuit, a quadrangulation is bipartite; we always consider quadrangulations with a fixed 2-coloring of the vertices. We distinguish two non-adjacent black vertices s and t on the outer face. An orientation of the edges of Q is a *2-orientation* if every vertex, except s and t, has outdegree two. From the count of edges it follows that s and t are sinks in every 2-orientation.

Let q_n be the number of 2-orientations among all quadrangulations with n inner faces. It is well known (8) that q_n also counts bipolar orientations on rooted planar maps with n edges (the root going from the source to the sink), which has been shown by R. Baxter (2) —using functional equations on generating functions— to satisfy the formula

$$q_n = \frac{1}{\binom{n+1}{1}\binom{n+1}{2}} \sum_{r=0}^{n-1} \binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2}.$$

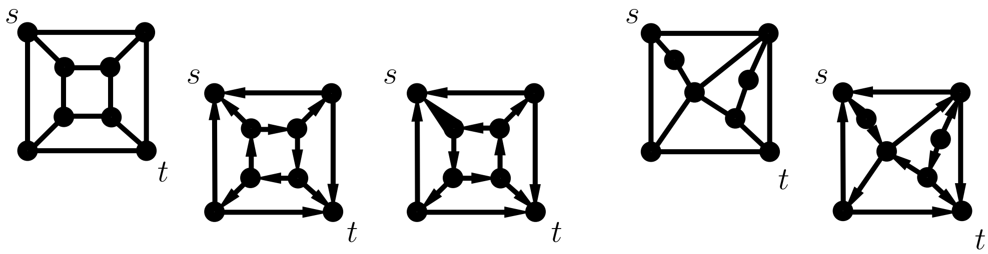

Fig. 1: Two quadrangulations and their 2-orientations.

Using standard tools for estimating combinatorial sums we find in Theorem 1 that

$$q_n \sim \frac{2^5}{\pi\sqrt{3}} n^{-4} 8^n,$$

where $a_n \sim b_n$ means $\lim a_n/b_n = 1$.

A quadrangulation is called *prime* if it has at least 3 inner faces and any of its 4-cycles is the boundary of a face. Accordingly, a 2-orientation of a prime quadrangulation is called a *prime 2-orientation*. The smallest prime quadrangulation is the cube, which has 5 inner faces, and has two distinct 2-orientations. Let p_n be the number of prime 2-orientations with n inner faces; p_n also counts bipolar orientations on rooted *3-connected* maps with n edges (again the root going from the source to the sink). This follows from specializing the bijection between 2-orientations and bipolar orientations to prime quadrangulations (indeed rooted prime quadrangulations with n faces correspond to rooted 3-connected maps with n edges (14)).

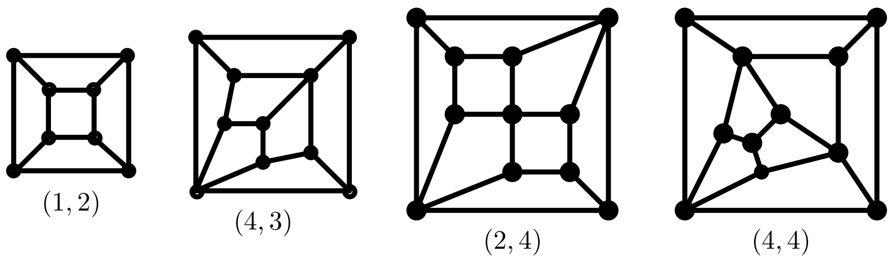

Fig. 2: All prime quadrangulations with 5, 7 and 8 inner faces. A pair (a, b) below a quadrangulation Q indicates that Q allows a non-isomorphic choices of s, t on the outer face and it has b 2-orientations for each choice. This yields $p_5 = 2$, $p_7 = 12$ and $p_8 = 24$.

The sequence (p_n) starts from $n = 5$ as $2, 0, 12, 24, 116, 418, 1722, \ldots$. We show in Theorem 2 that the number of prime 2-orientations is asymptotically

$$p_n \sim c \cdot n^{-4} \gamma^n,$$

where ≈ 0.51 and $\gamma \approx 5.52$ are constants (rounded to the digits shown) which can be computed analytically. For the proof we find first an equation linking the generating functions $P(z) = \sum p_n z^n$ and $Q(z) = \sum q_n z^n$, which reflects the decomposition of arbitrary quadrangulations into prime quadrangulations.

Given the form of the q_n, the series $Q(z)$ is D-finite, that is, is the solution of a linear differential equation with polynomial coefficients. Using techniques from differential equations, we find the singular expansion of $Q(z)$ near its dominant singularity $1/8$ and then, using functional inversion, the singular expansion of $P(z)$. Finally transfer theorems give the desired estimate.

We carry over the same program for 3-orientations. Consider a plane triangulation T, that is, a maximal plane graph, with n vertices and three special vertices a_1, a_2, a_3 in clockwise order around the outer face. An orientation of the inner edges of T is a *3-orientation* if every inner vertex has outdegree three. From the count of edges it follows that the special vertices a_i are sinks in every 3-orientation.

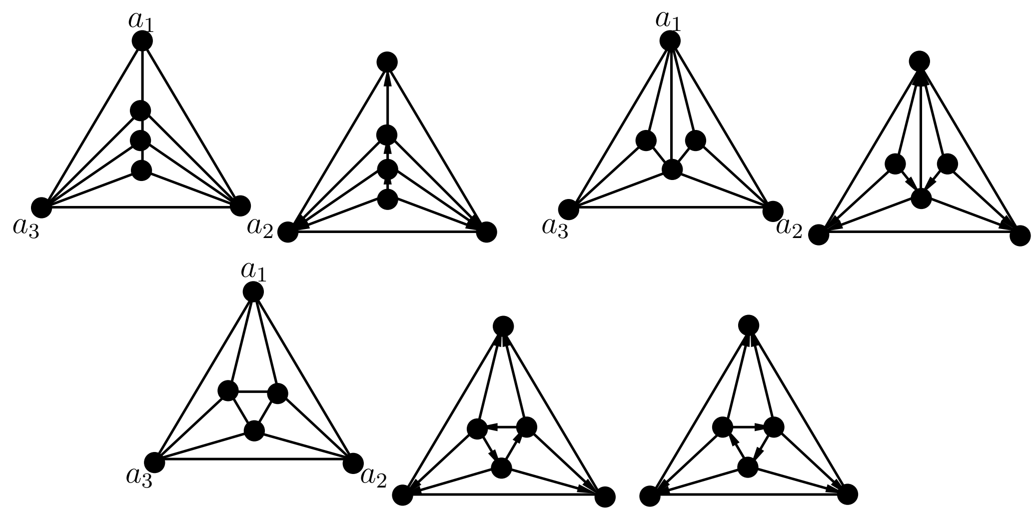

Fig. 3: Three triangulations and their 3-orientations.

The number t_n of 3-orientations among all triangulations with n inner vertices was shown in (4) (see (3; 5) for alternative proofs) to be equal to $c_{n+2}c_n - c_{n+1}^2$, where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is a Catalan number. From this it follows easily, as shown in Theorem 3, that

$$t_n \sim \frac{24}{\pi} n^{-5} 16^n.$$

A 3-orientation is called *prime* if the underlying triangulation is 4-connected, that is, it has no separating triangle. Let s_n be the number of prime 3-orientations. The sequence (s_n) starts from $n = 1$ as $1, 0, 2, 9, 57, \ldots$. We show in Theorem 4 that the number of prime 3-orientations is asymptotically

$$s_n \sim c \cdot n^{-5} \gamma^n,$$

where in this case $c \approx 2.60$ and $\gamma \approx 13.71$. The proof is again based on a functional equation between $T(z) = \sum t_n z^n$ and $S(z) = \sum s_n z^m$, and on singular expansions and transfer theorems for D-finite functions.

The following table summarizes the results in this paper.

Class of orientations	Growth constant	Subexponential order
2-orientations	8	n^{-4}
prime 2-orientations	5.519568931499...	n^{-4}
3-orientations	16	n^{-5}
prime 3-orientations	13.712839314390...	n^{-5}

It is worth noticing that the subexponential order for the enumeration of structured maps (rooted maps endowed with a specific combinatorial structure, such as an orientation or a spanning tree) is *not universal*, being n^{-4} for 2-orientations and n^{-5} for 3-orientations (another example is the subexponential term n^{-3} for the enumeration of rooted maps endowed with a spanning tree (13)). This is in contrast to the asymptotic enumeration of rooted maps, where the subexponential order $n^{-5/2}$ occurs systematically (see, for instance, (1)).

For more general classes of orientations and related problems, we direct the reader to (6).

2 Preliminaries

To obtain asymptotic estimates we take advantage of the framework of *Analytic Combinatorics*, as developed in the book by Flajolet and Sedgewick (7). Singularity analysis makes it possible to estimate the coefficients f_n from the behavior of the generating function

$$f(z) = \sum_n f_n z^n$$

at its dominant singularity. We recall that a dominant singularity is one of smallest modulus. By Pringsheim's theorem, the radius of convergence $\rho > 0$ is always a dominant singularity (7).

More precisely, assume that $f(z)$ is analytically continuable to a so-called Δ*-domain for* ρ, of the form

$$\Omega = \{|z| < \rho(1+\delta),\ \operatorname{Arg}(z-\rho) \in [\phi, 2\pi - \phi]\},$$

for some $\delta > 0$ and $0 < \phi < \pi/2$. Then the following *transfer rules* hold, where $\Gamma(x)$ is Euler's gamma function.

Transfer rules. For $\alpha \notin \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}_{\geq 0}$ we have

$$f(z) \underset{z\to\rho}{\sim} (1 - z/\rho)^{\alpha} \quad \Rightarrow \quad f_n \underset{n\to\infty}{\sim} \frac{1}{\Gamma(-\alpha)} n^{-\alpha-1}\rho^{-n}, \tag{1}$$

$$f(z) \underset{z\to\rho}{\sim} (z/\rho - 1)^{r} \log(z/\rho - 1) \quad \Rightarrow \quad f_n \underset{n\to\infty}{\sim} r! n^{-r-1}\rho^{-n}. \tag{2}$$

Thus if we can show that $f(z)$ in a Δ-domain at its dominant singularity behaves like the left-hand side of one of the transfer rules, then we get a precise estimate for its coefficients.

The transfer rules also apply when there are finitely many singularities $\zeta_1, \ldots, \zeta_\ell$ on the boundary of the disk of convergence, provided analytic continuation holds in a Δ-domain of the form $\zeta_i \cdot \Omega$ around each singularity. The contribution of each singularity is given by the transfer rules (with ρ replaced by ζ_i) and the asymptotics for f_n is obtained by adding up the contributions of all singularities. As we are going to see, one of the problems we have to face is to rule out the possibility of other singularities besides the obvious one.

We also need some elements of the theory of linear differential equations in the complex plane. Our basic reference in this topic is the classical book by Ince (11) (see (10) for a more modern treatment). Let

$$p_0(z)\frac{d^n f(z)}{z^n} + p_1(z)\frac{d^{n-1}f(z)}{z^{n-1}} + \cdots + p_{n-1}(z)\frac{df(z)}{dz} + p_n(z)f(z) = 0, \tag{3}$$

be a homogeneous linear differential equation whose coefficients $p_i(z)$ are polynomials in z. Then the only finite singularities of a solution $f(z)$ which can occur are the zeroes of the leading coefficient $p_0(z)$; notice in particular that there are no movable singularities, that is, singularities which depend on the initial conditions. Let $\mathcal{S}$ be the set of zeroes of $p_0(z)$. Then the analytic solutions of (3) in the neighbourhood of any point of $\mathbb{C}\backslash\mathcal{S}$ form a vector space of dimension n over $\mathbb{C}$. This implies that any solution of (3) is analytically continuable along any path avoiding $\mathcal{S}$.

The analytic theory of linear differential equations further extends to deal with singularities. However in this case one has to add further restrictions. If one wishes that the solution through a singular point is well behaved, one has to impose the condition of the singular point being *regular*. A necessary and sufficient condition for regularity is the following. Rewrite (3) as

$$\frac{d^n f(z)}{z^n} + P_1(z)\frac{d^{n-1}f(z)}{z^{n-1}} + \cdots + P_{n-1}(z)\frac{df(z)}{dz} + P_n(z)f(z) = 0, \tag{4}$$

where the P_i are now rational functions. A singular point z_0 is *regular* if

$$P_i = O((z-z_0)^{-i}), \qquad \text{for } i = 1, \ldots, n.$$

That is, P_i has a pole at z_0 of order at most i. The terminology does not mean that the solution is analytic, but refers to the fact that the possible divergence at the singularity is at most polynomial.

If z_0 is a regular singularity, then it can be shown that there exists a basis of solutions $B_1(z), \ldots, B_n(z)$ that are linear combinations of elements of the the form

$$Z^\mu (\log Z)^k s(Z),$$

where $Z = z - z_0$ and $s(Z)$ is analytic at zero, and the μ are complex numbers. Hence the solutions are analytic in a slit domain $\{|Z| < \delta\}\backslash\{Z \geq 0\}$. Moreover, the $B_i(z)$ have singular expansions at z_0 that are explicitly computable (11, Chap.16).

We summarize the previous discussion as follows.

Lemma 1 *A sufficient condition for a singular point z_0 of Equation (3) to be regular is that $P_i = O((z-z_0)^{-i})$, for $i = 1, \ldots, n$. In this case there exists a basis of solutions that are linear combinations of elements of the form $Z^\mu(\log Z)^k s(Z)$, where $Z = z - z_0$ and $s(Z)$ is analytic at zero.*

3 Counting two-orientations

Let q_n be the number of 2-orientations on quadrangulations with n inner faces, which satisfies (2)

$$q_n = \frac{1}{\binom{n+1}{1}\binom{n+1}{2}} \sum_{r=0}^{n-1} \binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2}.$$

This expression is a sum involving binomial coefficients, and there are well established techniques for estimating this kind of sums (15).

Theorem 1 (asymptotic number of 2-orientations) *The number q_n of 2-orientations on quadrangulations with n inner faces satisfies asymptotically*

$$q_n \sim \frac{2^5}{\pi\sqrt{3}} n^{-4} 8^n. \tag{5}$$

Proof: Let $b(n,r) := \binom{n}{r}\binom{n}{r+1}\binom{n}{r+2}$. The asymptotic study of a sum such as $\sum_r b(n,r)$ is classically done in four steps.

1. Locate the index $r_{\max}$ of the largest summand $b(n,r)$; in our case $r_{\max} = \lfloor n/2 \rfloor$ up to shifting by one or two.
2. Compute the asymptotics of $b(n,r)$ around $r_{\max}$. Using Stirling's formula we find
$$b(n, r_{\max} + r) = (2/\pi)^{3/2} 8^n n^{-3/2} \exp(-6r^2/n)(1 + O(1/n)) \text{ uniformly for } r \le n^{2/3}.$$
3. Select a window to focus on, and prove that the summand is negligible outside the window. We find
$$b(n, r_{\max} + r) = O(8^n \exp(-6n^{1/3})) \text{ uniformly for } |r| \ge n^{2/3},$$
so that the sum outside the window $|r - r_{\max}| \le n^{2/3}$ is negligible in front of $b(n, r_{\max})$.
4. Show that the sum in the window converges to an integral. We find
$$\sum_{r=-n^{2/3}}^{n^{2/3}} b(n, r_{\max} + r) \sim (2/\pi)^{3/2} 8^n n^{-3/2} \sum_{r=-n^{2/3}}^{n^{2/3}} \exp(-6r^2/n)$$
$$\sim (2/\pi)^{3/2} 8^n n^{-3/2} \left(\sqrt{n} \int_{\mathbb{R}} \exp(-6x^2) \mathrm{d}x \right).$$
Since the integral equals $\sqrt{\pi/6}$, we finally obtain $\sum_r b(n,r) \sim 2/(\pi\sqrt{3}) 8^n n^{-1}$. This yields the estimate of q_n as claimed, since $q_n \sim 2n^{-3} \sum_r b(n+1, r)$. □

Recall that a 2-orientation is called *prime* if the underlying quadrangulation has at least 3 inner faces and has no separating quadrangle. Our aim is to find an asymptotic estimate for p_n.

The asymptotic number of prime 2-orientations is obtained by singularity analysis of the corresponding generating function. Here is an outline of the strategy we follow:

1. Find an equation relating the generating function $Q(z) = \sum_n q_n z^n$ for 2-orientations and the generating function $P(w) = \sum_n p_n w^n$ for prime 2-orientations, counted with respect to the number of inner faces (Lemma 2).

2. Find the singular expansion of $Q(z)$ at the dominant singularity $1/8$ (Lemma 3), using both the property that $Q(z)$ is D-finite (solution of a linear differential equation with polynomial coefficients) and the asymptotic estimate of $q_n = [z^n]Q(z)$, obtained in Theorem 1.

3. From the singular expansion of $Q(z)$ and the equation relating $Q(z)$ and $P(w)$, determine the dominant singularity ρ of $P(w)$, and the singular expansion of $P(w)$ at ρ (Lemma 4). And show that ρ is the unique singularity of $P(w)$ on the disk $\{w; |w| = \rho\}$ (Lemma 5).

4. Using the transfer rules, translate the singular expansion of $P(w)$ to an asymptotic estimate of the coefficients p_n (Theorem 2).

All calculations have been done with the help of the computer algebra system Maple (12). The package DETOOLS has been particularly helpful to deal with the singularity analysis of D-finite series.

Lemma 2 *Let $Q(z) = \sum_n q_n z^n$ and $P(w) = \sum_n p_n w^n$ be the series counting respectively 2-orientations and prime 2-orientations with respect to the number of inner faces. Then*

$$Q(z) = z + \frac{2Q(z)^2}{1 + Q(z)} + P(Q(z)). \tag{6}$$

Proof: We rely on a well-known decomposition of a quadrangulation into prime components (14), reformulated here for 2-orientations. First, define a diagonal of a quadrangulation Q as a path of two inner edges connecting two opposite outer vertices of Q. A quadrangulation is said to be *black-diagonal* if it has a diagonal connecting the two outer black vertices. Let $D(z) = \sum_n d_n z^n$ be the series counting black-diagonal 2-orientations with respect to the number of inner faces. To construct a black-diagonal 2-orientation in a unambiguous way, take a quadrangle $\mathcal{Q}$ and draw $k \geq 1$ diagonals connecting the pair of black vertices of $\mathcal{Q}$. Then insert $k + 1$ non black-diagonal 2-orientations in the slots created by the k diagonals. This construction yields

$$D(z) = \sum_{k \geq 1} (Q(z) - D(z))^{k+1} = \frac{(Q(z) - D(z))^2}{1 - (Q(z) - D(z))},$$

so that $D(z) = Q(z)^2/(1 + Q(z))$. Next, define a non-diagonal quadrangulation as a quadrangulation having no diagonal, and define a non-diagonal 2-orientation as a 2-orientation of a non-diagonal quadrangulation. A non-diagonal 2-orientation with at least two inner faces is obtained from a prime 2-orientation by substituting every inner quadrangle (necessarily a face) by an arbitrary 2-orientation: this gives the term $P(Q(z))$, since substitution of combinatorial objects corresponds to substitution in generating functions (we are using throughout the *symbolic approach* for working with generating functions, as in (7)). The expression (6) for $Q(z)$ is simply obtained by summing the possible cases: the first term stands for the 2-orientation with no inner vertex, the second term for diagonal 2-orientations, and the third term for non-diagonal 2-orientations with at least two inner faces. $\square$

Lemma 3 (singular behavior of $Q(z)$) *The series $Q(z)$ is analytically continuable to the whole complex plane slit by the half-lines $z \geq 1/8$ and $z \leq -1$, and has the following singular expansion as z tends to $1/8$,*

$$Q(z) = Z^3 \log(Z)\alpha(Z) + \beta(Z) \quad \text{with } Z = z - 1/8, \tag{7}$$

where $\alpha(Z)$ and $\beta(Z)$ are analytic at 0 and their expansions start as

$$\alpha(Z) = 2^{13}/(3^{3/2}\pi) + O(Z), \qquad \beta(Z) = Q(1/8) + Q'(1/8)Z + O(Z^2).$$

Proof: First, it is easy to show that q_n is P-recursive, that is, satisfies a linear recurrence with polynomial coefficients. Indeed q_n is obtained from binomial coefficients using operators (sum, termwise product, ...) that are well-known to keep the property of P-recursiveness, see (7, App. B.4). It turns out that the q_n satisfy a second-order linear recurrence:

$$(n+3)(n+2)\,q_n - \left(7\,n^2 + 7\,n - 2\right) q_{n-1} - 8\,(n-1)(n-2)\,q_{n-2} = 0.$$

Multiplying by z^n and summing over n, the previous equation translates to the following differential equation satisfied by $Q(z) = \sum_n q_n z^n$:

$$\left(-8\,z^4 - 7\,z^3 + z^2\right)\frac{d^2}{dz^2} f(z) + \left(-16\,z^3 - 28\,z^2 + 6\,z\right)\frac{d}{dz} f(z) + (-12\,z + 6)\, f(z) - 12\,z = 0.$$

Remark. The recurrence was computed using the function `sumrecursion` from the Maple package `sumtools`; it is also listed in entry A00181 from the On-Line Encyclopedia of Integer Sequences. The differential equation is obtained using the function `rectodiffeq` from the Maple package `gfun`.

As mentioned in Section 2, we need a *homogenous* equation normalized in $d^3 f/dz^3$. This is achieved upon replacing the left-hand side $L(z)$ of the previous equation by $zL'(z) - L(z)$, and normalizing. The result is

$$\frac{d^3}{dz^3} f(z) + \frac{40\,z^2 + 42\,z - 7}{z(z+1)(8z-1)}\frac{d^2}{dz^2} f(z) + \frac{32z^2 + 40z - 6}{z^2(z+1)(8z-1)}\frac{d}{dz} f(z) + \frac{6}{z^3(z+1)(8z-1)} f(z) = 0. \tag{8}$$

The singularities of the coefficients (poles) are in $\mathcal{S} := \{0, -1, 1/8\}$. As discussed earlier the analytic solutions of (8) in the neighbourhood of any point of $\mathbb{C}\backslash\mathcal{S}$ form a vector space of dimension 3 over $\mathbb{C}$. This implies that any solution of (8) is analytically continuable along any path avoiding $\mathcal{S}$. In particular, the solution $Q(z) = \sum_n q_n z^n$, which is clearly analytic at 0, is continuable to the whole complex plane slit by the half-lines $z \geq 1/8$ and $z \leq -1$. This is a Δ-domain for the dominant singularity $1/8$, hence transfer theorems will apply.

For $i \in \{1, 2, 3\}$, the coefficient for the derivative of order $3 - i$ in (8) is $O((8z-1)^{-i})$, so that $1/8$ is a regular singularity. Hence there exists a basis of solutions $B_1(z), B_2(z), B_3(z)$ of (8) that are analytic in a slit domain $\{|Z| < \delta\}\backslash\{Z \geq 0\}$, where we set $Z = z - 1/8$. Moreover, singular expansions at $1/8$ of the basis functions are explicitly computable.

All calculations done, we find that two of the basis solutions are analytic and the third one has a (convergent) singular expansion of the form $f(Z)Z^3\log(Z) + g(Z)$ for some functions $f(Z)$ and $g(Z)$

analytic at 0. Therefore, as a linear combination of the basis solutions, $Q(z)$ has also a singular expansion of the form

$$Q(z) = \log(Z) \sum_{k \geq 3} a_k Z^k + \sum_{k \geq 0} b_k Z^k, \quad \text{with } Z = z - 1/8,$$

holding in a slit neighborhood of $1/8$. Notice that $b_0 = Q(1/8)$ and $b_1 = Q'(1/8)$.

Remark. The basis solutions are computed using the function `formal_sol` from the Maple package `DEtools`. At first sight it appears that only one of the solutions is analytic, but the two solutions containing logarithmic terms can be combined into a second analytic solution.

Let us look at the singular part. Clearly $Q(z)$ is singular at $1/8$, otherwise the growth rate of q_n would be larger than 8, and Theorem 1 shows that it is equal to 8. Hence there exists a smallest integer $k \geq 3$ such that $a_k \neq 0$. According to the transfer rule (2),

$$q_n \underset{n \to \infty}{\sim} a_k k! 8^{-k} n^{-k-1} 8^n.$$

Comparing with the asymptotics of q_n obtained in Theorem 1, we obtain $k = 3$ and $a_3 = 2^{13}/(3^{3/2}\pi)$. This concludes the proof. $\square$

Lemma 4 (singular expansion of $P(w)$) *The generating function $P(w) = \sum_n p_n w^n$ counting prime 2-orientations has radius of convergence $\rho = Q(1/8) \approx 0.18$, and the leading singular term of $P(w)$ at ρ is*

$$C(w - \rho)^3 \log(w - \rho), \quad \text{where} \quad C = 2^{13}/(Q'(1/8)^4 3^{3/2} \pi),$$

holding in a "Δ-neighborhood" for ρ, of the form

$$\{|w - \rho| < \delta',\ \mathrm{Arg}(w - \rho) \in [\phi, 2\pi - \phi]\},$$

with $\delta' > 0$ and $\phi < \pi/2$.

Proof: Let $F(w)$ be the inverse function of $Q(z)$. Setting $w = Q(z)$ in Equation (6), we get

$$P(w) = w - 2w^2/(1 + w) - F(w).$$

Since $Q(1/8) < 1$ and the only singularity of $2w^2/(1 + w)$ is at $w = -1$, the radius of convergence ρ of $P(w)$ is the same as that of $F(w)$. Since the series $Q(z)$ has positive coefficients, it defines an increasing function. It follows that the radius of convergence of the inverse $F(w)$ is $\rho = Q(1/8) < +\infty$.

The singular expansion of $F(w)$ at ρ is obtained iteratively from the one of its inverse $Q(z)$ at $1/8$, given by Equation (7), using first $w - \rho \sim Q'(1/8) \cdot (z - 1/8)$ and then repeatedly bootstrapping. The leading singular term $\lambda(z - 1/8)^3 \log(z - 1/8)$ of $Q(z)$, with $\lambda = 2^{13}/(3^{3/2}\pi)$, yields a term

$$-\lambda / Q'(1/8)^4 (w - \rho)^3 \log(w - \rho)$$

in the expansion of $P(w)$, which is the leading singular term.

Moreover, the singular expansion of $P(w)$ at ρ holds in a neighborhood Ω of ρ, where Ω is the image of a slit neighborhood $\{|z - 1/8| < \delta\} \backslash \{z \geq 1/8\}$, with $\delta > 0$. Since $\rho'(w)$ converges to a positive constant at $w = \rho$, it is locally conformal at ρ. Hence Ω contains a "Δ-neighborhood" for ρ, of the form

$$\{|w - \rho| < \delta',\ \mathrm{Arg}(w - \rho) \in [\phi, 2\pi - \phi]\},$$

with $\delta' > 0$ and $\phi < \pi/2$. Finally, since ρ is the smallest positive singularity of $P(w)$, Pringsheim's theorem ensures that ρ is the radius of convergence of $P(w)$. □

Lemma 5 (uniqueness of dominant singularity of $P(w)$) *The series $P(w)$ has no singularity other from ρ on the closed disk $\{w; |w| \leq \rho\}$, that is, $P(w)$ is analytically continuable around any $w \neq \rho$ with $|w| \leq \rho$.*

Proof: First, since ρ is the radius of convergence of $P(w)$, $P(w)$ is analytic on the open disk $\{w; |w| < \rho\}$. Recall that $Q(z)$ and $Q'(z)$ converge at $1/8$, since the coefficients of $Q(z)$ are asymptotically equivalent to $cn^{-4}8^n$. In addition, as we have seen $Q(1/8)$ is equal to ρ. Since $F(Q(z)) = z$, we have $F'(Q(z))Q'(z) = 1$, so $F'(\rho) = 1/Q'(\rho)$ and $P'(\rho) = 1 - 4\rho/(1+\rho) + 2\rho^2/(1+\rho)^2 - F'(\rho)$. In other words $P'(\rho)$ is finite, which ensures that $P'(w)$ (and similarly $P(w)$) is absolutely convergent on the closed disk $\{w; |w| \leq \rho\}$. We claim that, for $w_0 \in \mathcal{C}(0,\rho)\backslash\{\rho\}$, Q is analytic at $z_0 := F(w_0)$, i.e., z_0 is not in $\mathcal{S} := \{-1, 1/8\}$. The case $z_0 = -1$ is excluded easily. Indeed

$$|F(w)| = |w(1-w)/(1+w) - P(w)| \leq |w(1-w)/(1+w)| + |P(w)|,$$

hence for $|w| = \rho$ (and using the fact that $P(w)$ has nonnegative coefficients),

$$|F(w)| \leq \rho\frac{1+\rho}{1-\rho} + P(\rho) \approx 0.1873 + 0.0006 < 1,$$

computing $P(\rho)$ as $\rho(1-\rho)/(1+\rho) - F(\rho) = \rho(1-\rho)/(1+\rho) - 1/8$.

Excluding $z_0 = 1/8$ requires a bit more care. Notice that the function

$$|w - 2w^2/(1+w)| = |w||(1-w)/(1+w)|$$

has its unique minimum on $\mathcal{C}(0,\rho)$ at ρ, and $|P(w)|$ has its unique maximum on $\mathcal{C}(0,\rho)$ at ρ. Therefore, for $w \in \mathcal{C}(0,\rho)\backslash\{\rho\}$,

$$\begin{aligned} |z| = |F(w)| &= |w(1-w)/(1+w) - P(w)| \geq |w(1-w)/(1+w)| - |P(w)| \\ &> \frac{\rho(1-\rho)}{1+\rho} - P(\rho) = F(\rho) = 1/8, \end{aligned}$$

that is, $|z| > 1/8$. Hence $Q(z)$ is analytic at z_0.

Now, take a path $\mathcal{P}_{w_0}$ from 0 to w_0 that avoids the preimages (for $Q(z)$) of the singularities $\{-1, 1/8\}$; and let $\mathcal{Q}_{z_0}$ be the image of $\mathcal{P}_{w_0}$ by $P(w)$. Since $\mathcal{Q}_{z_0}$ avoids its singularities, $Q(z)$ is analytically continuable along $\mathcal{Q}_{z_0}$. In addition, all along $\mathcal{P}_{w_0}$, there holds the equation $Q(P(w)) = w$. Hence $P(w)$ is the local inverse of $Q(z)$ all along the path $\mathcal{P}_{w_0}$. In particular $Q(z)$ is the local inverse of $P(w)$ at the end of the path, i.e., at the point w_0. Since $Q(z)$ is analytic at $z_0 = P(w_0)$, we conclude that $P(w)$ is analytic around w_0. □

From the singular expansion of $P(w)$, the transfer rule (2) yields directly the following asymptotic estimate for the number of prime 2-orientations.

Theorem 2 (asymptotic number of prime 2-orientations) *Let p_n be the number of prime 2-orientations with n inner quadrangles. Then*

$$p_n \sim c \cdot n^{-4}\gamma^n, \tag{9}$$

where $c = Q(1/8)^3 2^{14}/(Q'(1/8)^4\sqrt{3}\pi) \approx 0.5097001$ and $\gamma = Q(1/8)^{-1} \approx 5.5195689$.

4 Counting three-orientations

We now focus on the enumeration of 3-orientations (equivalently, of Schnyder woods) on triangulations. Recall that the number of 3-orientations with n inner vertices is $c_{n+2}c_n - c_{n+1}^2$, where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is a Catalan number.

Theorem 3 (asymtotic number of 3-orientations) *Let t_n denote the number of 3-orientations with a fixed outer triangle and n inner vertices. Then*

$$t_n \sim \frac{24}{\pi} n^{-5} 16^n. \tag{10}$$

Proof: From the formula above we get $t_n = 6(2n)!(2n+2)!/(n!(n+1)!(n+2)!(n+3)!)$. Applying Stirling's estimate, the estimate for t_n follows easily. □

Our study continues with the asymptotic enumeration of so-called *prime 3-orientations*, which play a similar role for 3-orientations as prime 2-orientations for 2-orientations. Precisely, a *prime triangulation* is defined as a triangulation with at least one inner vertex and where each 3-cycle delimits a face (these are called *simple* by Tutte (16) and correspond exactly to the triangulations that are 4-connected); and a *prime 3-orientation* is a 3-orientation on a prime triangulation. The first prime triangulation is the tetrahedron, with has 4 vertices. As in Section 3, we aim at finding an asymptotic estimate for the number s_n of prime 3-orientations with n inner vertices. (The coefficients s_n start from $n = 1$ as $1, 0, 2, 9, 57, 400, 3066$.)

Our study follows the same lines as for prime 2-orientations, except that we have to deal with two dominant singularities (instead of a unique one for prime 2-orientations) when analyzing the generating functions counting 3-orientations and prime 3-orientations.

Lemma 6 *Let $T(z) = \sum_n t_n z^{2n+1}$ and $S(w) = \sum s_n w^{2n+1}$ be the series counting respectively 3-orientations and prime 3-orientations with respect to the number of inner faces (due to Euler relation, the number of inner faces is of the form $2n+1$, with n the number of inner vertices). Then*

$$T(z) = z + S(T(z)). \tag{11}$$

Proof: A 3-orientation with at least one inner vertex is obtained from a prime 3-orientation by substituting every inner face by an arbitrary 3-orientation: this gives the term $P(T(z))$, while the term z takes account of the empty triangle. This is a classical approach introduced by Tutte (16). □

Lemma 7 (singular behavior of $T(z)$) *The generating function $T(z)$ has two dominant singularities: $1/4$ and $-1/4$. In addition, $T(z)$ is analytically continuable to the whole complex plane slit by the two half-lines $z \geq 1/4$ and $z \leq -1/4$; and the following singular expansions hold at $\pm 1/4$:*

$$T(z) = \pm(Z^4 \log(Z) S_1(Z) + S_2(Z)), \text{ with } Z = \pm z - 1/4, \tag{12}$$

where $S_1(Z)$ and $S_2(Z)$ are analytic at 0 and their expansions start as

$$S_1(Z) = \frac{1}{2^6 \pi} + O(Z), \quad S_2(Z) = T(1/4) + T'(1/4)Z + O(Z^2).$$

Proof: We set $t(y) := \sum_n t_n y^n$, so that $T(z) = zt(z^2)$. Hence singularities and singular expansions of $T(z)$ are directly obtained from those of $t(y)$. the study of $t(y)$ is completely similar to the one of the generating function $Q(z)$ counting 2-orientations. First we observe that $t_n/t_{n-1} = 4(4n^2-1)/((n+3)(n+2))$, so that t_n is P-recursive (of order 1). The recurrence turns to a linear differential equation satisfied by $t(y)$, which reduces to the homogenous equation

$$\frac{d^3}{dy^3} f(y) + 8\,\frac{(12\,y-1)}{y\,(16\,y-1)}\frac{d^2}{dy^2} f(y) + 12\,\frac{(-1+9\,y)}{y^2\,(16\,y-1)}\frac{d}{dy} f(y) + \frac{12}{y^2\,(16\,y-1)} f(y) = 0. \tag{13}$$

The only poles are 0 and $1/16$. As a solution of (13), $t(y)$, which is clearly analytic at 0, is analytically continuable along any path avoiding $1/16$, in particular $t(y)$ is continuable to the complex plane slit by $z \geq 1/16$. The singularity analysis of $t(y)$ is done in a completely similar way as for $Q(z)$ (Lemma 3). We find that two of the basis solutions of (13) are analytic at $1/16$ and the third one admits a singular expansion of the form $\alpha(Z)Z^4\log(Z) + \beta(Z)$ for some analytic functions $\alpha(Z)$ and $\beta(Z)$. Therefore, $t(z)$ has also a singular expansion of the form

$$t(y) = \log(Y)\sum_{k\geq 4} a_k Y^k + \sum_{k\geq 0} b_k Y^k, \quad \text{with } Y = y - 1/16,$$

Clearly $b_0 = t(1/16)$ and $b_1 = t'(1/16)$. According to the transfer rules 2, compared with the asymptotic of t_n obtained in Theorem 1, we obtain $a_4 = 1/\pi$. The singularities and singular expansions of $T(z)$ immediately follow, as $T(z) = zt(z^2)$. □

Lemma 8 (singular behavior of $S(w)$) *The generating function $S(w) = \sum_n s_n w^{2n+1}$ has two dominant singularities at $\pm\rho$, with $\rho = T(1/4)$, and is analytically continuable to a domain of the form $\{|z| < \rho(1+\delta),\ \mathrm{Arg}(z-1/4) \in [\phi, 2\pi-\phi],\ \mathrm{Arg}(-z-1/4) \in [\phi, 2\pi-\phi]\}$ (Δ-domain with two singularities), for some $\delta > 0$ and $\phi \in (0, \pi/2)$. The leading singular terms of $S(w)$ at $\pm\rho$ is*

$$\pm C \cdot W^4 \log(W), \quad \text{with } W = \pm w - \rho \ \text{ and } C = 1/(2^6\pi T'(1/4)^5).$$

Proof: According to Equation (11), the functional inverse $U(w)$ of $T(z)$ satisfies $U(w) = w - S(w)$. Hence the radius of convergence ρ of $S(w)$ is the same as that of $U(w)$. Since $U(w)$ is the inverse of an increasing function $T(z)$, it follows that $\rho = T(1/4)$. The singular expansion of $U(w)$ at ρ is easily obtained from that of $T(z)$ at $1/4$: the leading singular term $\lambda(z-1/4)^4\log(z-1/4)$ of $T(z)$, with $\lambda = 1/(2^6\pi)$, yields a term $-\lambda T'(1/4)^{-5}(w-\rho)^4\log(w-\rho)$ in the singular expansion of $U(w)$, which is the leading singular term. As argumented in the proof of Lemma 4, the singular expansion of $U(w)$ at ρ holds in a "Δ-neighborhood" of ρ. As $U(w) = -U(-w)$, the singular expansion of U at $-\rho$ is the opposite of the one at ρ, and holds in a Δ-neighborhood of $-\rho$. It remains to show that $S(w)$ is analytic on $\mathcal{C}(0,\rho)\backslash\{\pm\rho\}$. We proceed similarly as in Lemma 5. First we check that, for $w_0 \in \mathcal{C}(0,\rho)\backslash\{\pm\rho\}$, the function $T(z)$ is analytic at $z_0 := U(w_0)$, i.e., $z_0 \neq \pm 1/4$; this follows from $|z_0| = |w_0 - S(w_0)| \geq |w_0| - |S(w_0)| > \rho - S(\rho) = 1/4$, so $z_0 \neq \pm 1/4$. Finally, by the same arguments as in Lemma 5, we prove that $T(z)$ is the local inverse of $S(w)$ around w_0. □

From the transfer rule (2) applied to $S(w)$ (there are two dominant singularities $\pm\rho$, whose contributions are added up), we obtain the following asymptotic estimate for $s_n = [w^{2n+1}]S(w)$.

Theorem 4 (asymptotic number of prime 3-orientations) *Let s_n be the number of prime 3-orientations with n inner vertices. Then*

$$s_n \sim c \cdot n^{-5} \gamma^n, \tag{14}$$

where $c = 3 \cdot 2^9 \cdot T(1/4)^3/(T'(1/4)^5\pi) \approx 2.5976882$ and $\gamma = T(1/4)^{-2} \approx 13.7128393$.

Acknowledgements

The authors thank Philippe Flajolet for a very helpful discussion on the asymptotics of D-finite series.

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