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On certain non-unique solutions of the Stieltjes moment problem

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We construct explicit solutions of a number of Stieltjes moment problems based on moments of the form $\rho_1^{(r)}(n) = (2rn)!$ and $\rho_2^{(r)}(n) = [(rn)!]^2$, $r = 1, 2, \ldots$, $n = 0, 1, 2, \ldots$, *i.e.* we find functions $W_{1,2}^{(r)}(x) > 0$ satisfying $\int_0^\infty x^n W_{1,2}^{(r)}(x)dx = \rho_{1,2}^{(r)}(n)$. It is shown using criteria for uniqueness and non-uniqueness (Carleman, Krein, Berg, Gut, Pakes, Stoyanov) that for $r > 1$ both $\rho_{1,2}^{(r)}(n)$ give rise to non-unique solutions. Examples of such solutions are constructed using the technique of the inverse Mellin transform supplemented by a Mellin convolution. We outline a general method of generating non-unique solutions for moment problems generalizing $\rho_{1,2}^{(r)}(n)$, such as the product $\rho_1^{(r)}(n) \cdot \rho_2^{(r)}(n)$ and $[(rn)!]^p$, $p = 3, 4, \ldots$.

Keywords: Classical moment problem, Stieltjes moment problem, Mellin transform

1 Introduction

This paper concerns solutions of the Stieltjes moment problem [1, 2], *i.e.* positive functions $W(x)$ which satisfy the infinite set of equations

$$\int_0^\infty x^n W(x)dx = \rho(n), \qquad n = 0, 1, 2, \ldots. \tag{1}$$

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We previously met this problem when considering the properties of the so-called *(generalized) coherent states* (CS) of Quantum Mechanics [(i)]
defined, for complex $z \in \mathbb{D} \subset \mathbb{C}$ and positive numbers $\rho(n)$, $n = 0, 1, 2, \ldots$, as

$$|z\rangle = \mathcal{N}^{-1/2}(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho(n)}} |n\rangle \tag{2}$$

where $|n\rangle$'s, $n = 0, 1, 2, \ldots$, form an orthonormal complete basis in the Hilbert space $\mathcal{H}$, $\langle m|n\rangle = \delta_{mn}$, $\sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1}$ [(ii)]. These CS are required to be normalizable, *i.e.* $\sum_{n=0}^{\infty} |z|^{2n}/\rho(n) = \mathcal{N}(|z|^2)$ has a non-zero radius of convergence R, continuous in the label z, *i.e.*, $z' \to z$ implies $|z'\rangle \to |z\rangle$, and to satisfy the property of *Resolution of Unity* with a measure $\mu(|z|^2) > 0$ for $z \in \mathbb{D} \subset \mathbb{C}$, where $\mathbb{D}$ is a disc of radius R centered in $z = 0$,

$$\frac{1}{\pi} \int_{\mathbb{D}\subset\mathbb{C}} d\,\mathrm{Re}(z)\, d\,\mathrm{Im}(z)\, \mu(|z|^2)\, |z\rangle\langle z| = \mathbf{1} = \sum_{n=0}^{\infty} |n\rangle\langle n| \,, \tag{3}$$

which is essentially equivalent to Eq. (1) for $W(x) = \mu(|z|^2)/\mathcal{N}(|z|^2)|_{|z|^2=x}$ [3, 4]. The property of *Resolution of Unity* plays an important role in various applications of CS, in particular in the construction of the so-called Bargmann representation in quantum mechanics within which quantum states $|f\rangle = \sum_{n=0}^{\infty} f_n|n\rangle$, $\sum_{n=0}^{\infty} |f_n|^2 = 1$ are represented as entire functions [5, 6]

$$|f\rangle \to f_B(z) = \mathcal{N}^{1/2}(|z|^2)\langle z^*|f\rangle = \sum_{n=0}^{\infty} \frac{f_n z^n}{\sqrt{\rho(n)}} \tag{4}$$

and the scalar product of two states $\langle g|f\rangle = \sum_{n=0}^{\infty} g_n^* f_n$ is given in terms of so defined Bargmann functions by

$$\langle g|f\rangle = \frac{1}{\pi} \int_{\mathbb{D}\subset\mathbb{C}} d\,\mathrm{Re}(z)\, d\,\mathrm{Im}(z)\, g_B^*(z) f_B(z) \frac{\mu(|z|^2)}{\mathcal{N}(|z|^2)}. \tag{5}$$

Taking $|g\rangle = |m\rangle$ and $|f\rangle = |n\rangle$ we see that solutions to the moment problem Eq. (1) provide us with explicit forms of the scalar product in the Bargmann representation generated by the CS of Eq. (2). Standard CS, usually called Glauber or harmonic oscillator CS, for which $\rho(n) = n!$, lead to $\mu(|z|^2) = 1$, $\mathcal{N}(|z|^2) = e^{-|z|^2}$ and, consequently, $W(x) = e^{-x}$. *Generalized* CS lead to $\rho(n)$'s other than $n!$ [7] and the solutions of Eq.(1), if any, must be studied in each individual case separately [8, 9, 10, 11]. An efficient

(i) Coherent states were originally proposed by Erwin Schrödinger in the early days of Quantum Mechanics to describe wave packets obeying the time evolution close to the classical motion (E. Schrödinger, "Der Stetige Übergang von der Mikro- zur Makromechanik", Naturwissenschaften **14** (1926) 664–666). They were reintroduced more than half a century ago in seminal papers by R. J. Glauber and others: R. J. Glauber, "The quantum theory of optical coherence", Phys. Rev. **130** (1963) 2529–2539, E. C. G. Sudarshan, "Equivalence of semiclassical and quantum mechanical description of statistical light beams", Phys. Rev. Lett. **10** (1963) 277–279, and J. R. Klauder, "Continuous-representation theory. I. Postulates of continuous-representation theory", J. Math. Phys. **4** (1963) 1055–1058; "Continuous-representation theory. II. Generalized relation between quantum and classical dynamics", J. Math. Phys. **4** (1963) 1058–1073. Such states are nowadays an important tool widely used in quantum optics and in general investigations of quantization.

(ii) Here we follow Dirac's notation universally used in quantum physics: elements of a Hilbert space are denoted by *kets* $|\cdot\rangle$, their Hermitean conjugates by *bras* $\langle\cdot|$, projection operators as $|\cdot\rangle\langle\cdot|$ and the scalar product as $\langle\cdot|\cdot\rangle$.

way to tackle the problem is to use the inverse Mellin transform method which allows one to establish many solutions of Eq. (1), either by analytic methods [12] or by extensive use of available tables [13, 14]. As a byproduct of this method we have established that, for a large number of combinatorial sequences such as Bell and Catalan numbers, *etc.*, the corresponding sequences $\rho(n)$ are solutions of the moment problem Eq. (1) [15, 16]. Likewise, sequences arising in theory of ordering of differential operators [17] solve appropriate moment problems too [18].

We want also to emphasize that any investigation of the moment problem is deeply rooted in a classical problem of statistics, namely that of the unique or non-unique determination of a probability distribution from its moments. We will link to it in Section 4; here we only remark that this problem was fully treated more than one hundred years ago by T. J. Stieltjes [19], recalled by M. G. Krein in the middle of the XXth century [20] and is still the subject of extensive research which in recent years has led to significant progress in its understanding [21, 22, 23, 24, 25, 26, 27, 28, 30, 29, 31, 32, 33, 34]. Statistical aspects of the moment problem have an analogue in quantum physics: according to its standard interpretation the probability that the state $|n\rangle$ appears in the coherent state $|z\rangle$ of Eq.(2) is

$$P_n(z) = |\langle n|z\rangle|^2 = \frac{|z|^{2n}}{\mathcal{N}(|z|^2)\rho(n)} \tag{6}$$

and, if $\rho(n)$ satisfy Eq. (1) with $W(x) = \mu(|z|^2)/\mathcal{N}(|z|^2)|_{|z|^2=x}$. Then for all $n = 0, 1, 2, ...$ we have

$$\int\limits_{\mathbb{D}\subset\mathbb{C}} d\,\mathrm{Re}(z)\, d\,\mathrm{Im}(z)\; \mu(|z|^2) P_n(z) = 1, \tag{7}$$

which is equivalent to the Resolution of Unity of Eq.(3), asserting the completeness of so defined CS. The physical interpretation and consequences of uniqueness or non-uniqueness of the measure $\mu(|z|^2)$ satisfying Eq.(7) is however beyond the scope of the current paper and will be treated elsewhere [35].

This admixture of quantum-mechanical, analytical, combinatorial and statistical features deserves a deeper study which we intend to pursue. Our paper is partly expository in character, and has the following structure: first we establish a link between the Mellin transform and the moment problem; next, in Section 3, we provide principal solutions to two moment problems, termed toy models. In Section 4 we discuss criteria for the uniqueness of solutions of the moment problem. Section 5 is devoted to the explicit construction of non-unique solutions of the toy-models. In Section 5, some generalizations of model sequences, together with their solutions, are reviewed. Section 6 is devoted to a discussion and conclusions.

We dedicate this paper to Philippe Flajolet on the occasion of his 60th birthday. His pioneering applications of Mellin transform asymptotics to the analysis of combinatorial structures [36, 37, 38] have been a source of inspiration for us.

2 The Mellin transform *versus* moment problem

The Mellin transform of a function $f(x)$ of the real variable x is defined for complex s by the following relation

$$\mathcal{M}[f(x); s] = \int_0^\infty x^{s-1} f(x) dx \equiv f^*(s) \tag{8}$$

(in this definition $f^*(s)$ is not the complex conjugate of $f(s)$!), and its inverse $\mathcal{M}^{-1}$ is defined by

$$\mathcal{M}^{-1}[f^*(s); x] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) x^{-s} ds. \tag{9}$$

See Ref. [39] for a discussion of the dependence of $f(x)$ on the real constant c. Among the many relations satisfied by the Mellin transform we shall mainly use the following

$$\mathcal{M}[x^b f(ax^h); s] = \frac{1}{h} a^{-\frac{s+b}{h}} f^*\left(\frac{s+b}{h}\right), \quad a, h > 0. \tag{10}$$

If $\mathcal{M}[f(x); s] = f^*(s)$ and $\mathcal{M}[g(x); s] = g^*(s)$, then

$$\mathcal{M}^{-1}[f^*(s) g^*(s); x] = \int_0^\infty f\left(\frac{x}{t}\right) g(t) \frac{dt}{t} = \int_0^\infty g\left(\frac{x}{t}\right) f(t) \frac{dt}{t} \tag{11}$$

which is called the Mellin convolution property. Note that if in Eq. (11) both $f(x)$ and $g(x)$ are positive for $x > 0$ then $\int_0^\infty f(\frac{x}{t}) g(t) \frac{dt}{t}$ is also positive for $x > 0$. This means that the Mellin convolution preserves positivity, an essential property when considering the moment problem.

Eq.(1), if rewritten for $n = s - 1$ as

$$W(x) = \mathcal{M}^{-1}[\rho(s-1); x], \tag{12}$$

is, if $W(x) > 0$, a solution of a Stieltjes moment problem. Thus according to Eq. (12) one can solve the Stieltjes moment problem by performing the inverse Mellin transform on the moment sequence and checking if the resulting function is positive. All the solutions in the sequel have been obtained using Eqs. (12), (10) and (11). Note that $W(x)$ obtained *via* Eq. (12) may not be the only solution of Eq.(1). We call $W(x) > 0$ obtained *via* Eq. (12) from Eq. (1) the *principal solution* (iii).

(iii) Sets of positive functions consisting of 'center of the class' modulated by an oscillatory function orthogonal to all polynomials, *i.e.,* sets of positive functions sharing the same (Stieltjes) moments, called a 'Stieltjes class', were first discussed in [31] and then, in various aspects, constructed and analysed in [28, 32, 33].

3 Principal solutions of the moment problems

Let us consider two sequences of integers given by

$$\rho_1^{(r)}(n) = (2rn)!, \quad n = 0, 1, 2, \ldots, \quad r = 1, 2, \ldots. \tag{13}$$

$$\rho_2^{(r)}(n) = [(rn)!]^2, \quad n = 0, 1, 2, \ldots, \quad r = 1, 2, \ldots. \tag{14}$$

In the following we shall obtain the solutions of the Stieltjes moment problem for the moment sequences given by Eqs. (13) and (14), *i.e.* the functions $W_{1,2}^{(r)}(x) > 0$ satisfying

$$\int_0^\infty x^n W_{1,2}^{(r)}(x) dx = \rho_{1,2}^{(r)}(n). \tag{15}$$

From now on we shall refer to the model problems $\rho_{1,2}^{(r)}(n)$ as *toy models* **TM1** and **TM2**, respectively.

a) **TM1**: we begin by obtaining $W_1^{(1)}(x)$. Note that $(2n)! = \Gamma(2n+1) = \Gamma(2(s-\frac{1}{2}))$. We now apply Eq.(10) with $a = 1$, $b = -\frac{1}{2}$ and $h = \frac{1}{2}$. We subsequently use $\mathcal{M}^{-1}[\Gamma(s); x] = e^{-x}$ which gives

$$\int_0^\infty x^n W_1^{(1)}(x) dx = \int_0^\infty x^n \left[\frac{e^{-\sqrt{x}}}{2\sqrt{x}}\right] dx = (2n)!, \quad n = 0, 1, 2, \ldots. \tag{16}$$

In the same spirit we observe that $(2rn)! = \Gamma\left(2r\left(n + \frac{1}{2r}\right)\right) = \Gamma\left(2r\left(s - \frac{2r-1}{2r}\right)\right)$. Upon using Eq. (10) but now with $a = 1$, $b = -\frac{2r-1}{2r}$ and $h = \frac{1}{2r}$ one obtains

$$\int_0^\infty x^n W_1^{(r)}(x) dx = \int_0^\infty x^n \left[\frac{1}{2rx^{\frac{2r-1}{2r}}} e^{-x^{\frac{1}{2r}}}\right] dx = (2rn)!, \quad n = 0, 1, 2, \ldots. \tag{17}$$

This means that $W_1^{(r)}(x) > 0$ is the *principal solution* of the moment problem Eq. (17).

b) **TM2**: we begin by deriving $W_2^{(1)}(x) = \mathcal{M}^{-1}[\Gamma^2(s); x]$ and employ the Mellin convolution Eq. (11). By using the Sommerfeld representation of the modified Bessel function of second kind $K_0(x)$ [40] one obtains

$$\int_0^\infty x^n W_2^{(1)}(x) dx = \int_0^\infty x^n \left[2K_0(2x^{\frac{1}{2}})\right] dx = (n!)^2, \quad n = 0, 1, 2, \ldots. \tag{18}$$

Subsequently note that $[(rn)!]^2 = [\Gamma\left(r\left(n + \frac{1}{r}\right)\right)]^2 = [\Gamma\left(r\left(s - \frac{r-1}{r}\right)\right)]^2$ and again apply Eq.(10) with $a = 1$, $b = -\frac{r-1}{r}$ and $h = \frac{1}{r}$. The result is

$$\int_0^\infty x^n W_2^{(r)}(x) dx = \int_0^\infty x^n \left[\frac{2}{rx^{\frac{r-1}{r}}} K_0(2x^{\frac{1}{2r}})\right] dx = [(rn)!]^2, \quad n = 0, 1, 2, \ldots. \tag{19}$$

Since $K_0(t) > 0$ for $t > 0$, $W_2^{(r)}(x) > 0$ is the *principal solution* of the moment problem Eq. (19).

We emphasize that although the inverse Mellin transform technique deserves to be better known to broader physics community, it is standard in probability [23, 41, 42].

4 Criteria of uniqueness and non-uniqueness of the Stieltjes moment problem

As we remarked in the Introduction it was realized from the very beginning of the history of the moment problem that its solutions may not be unique; *i.e.* for a given moment sequence there may exist more than one solution. Stieltjes himself gave an example of a non-unique solution of a problem leading to what turned out to be the lognormal distribution [19]. This example, being about the only one available, was quoted repeatedly in the literature. As recently as twenty years ago new non-unique solutions of other types of problems have been constructed. Of special interest were Stieltjes moment problems arising in probability theory, related to investigation of probability distributions not determined by their moments. Consequently, the subject was further developed and systematized, largely due to the comprehensive work of Berg [21, 23], Berg and Pedersen [22], Gut [24], Lin [25], Pakes with coworkers [26, 27, 28], Stoyanov [29, 30, 31], Stoyanov and Tolmatz [32, 33] and others. For recent extension of Ref. [21] see Ostrovska and Stoyanov [34].

From the practical point of view one needs criteria to decide whether the pursuit of non-unique solutions is reasonable. Such criteria are either based on the $\rho(n)$'s alone or on the solution $W(x)$ alone, or on both $\rho(n)$ and $W(x)$, see below. From now on we assume that all the $\rho(n)$'s are finite and that $W(x)$ is continuous. We now give , in a somewhat condensed form, a list of such criteria.

C1 Carleman uniqueness criterion (T. Carleman, 1922, [1]) This is based on the properties of the $\rho(n)$'s alone and is:

If $S = \sum_{n=1}^{\infty}[\rho(n)]^{-\frac{1}{2n}} = \infty$*, then the solution is unique.*

This criterion does not imply that if $S < \infty$ then the solution is non-unique. In fact it is possible to construct models for which $S < \infty$ and solutions are still unique [29, 43].

C2 Krein's non-uniqueness criterion (M. G. Krein, around 1950, [20])

This is based entirely on the solution $W(x)$ and does not involve the moments.

If $\int_0^{\infty} \frac{-\ln[W(x^2)]}{1+x^2}dx < \infty$*, i.e., the so-called Krein integral exists, then the solution is non-unique.*

C3 Converse Carleman criterion for non-uniqueness (A. Pakes, 2001, [26], A. Gut, 2002, [24])

This is based on $\rho(n)$'s and $W(x)$.

If there exists $x' \geq 0$ *such that for* $x > x'$, $0 < W(x) < \infty$ *and* $\psi(y) = -\ln[W(e^y)]$ *is convex in* (y', ∞)*, where* $y' = \ln(x')$*, and if, in addition,* $S = \sum_{n=1}^{\infty}[\rho(n)]^{-\frac{1}{2n}} < \infty$*, then the solution* $W(x)$ *is non-unique.*

See [21, 22, 24, 26, 27, 29] for various refinements of these criteria.

5 Construction of non-unique solutions for TM1 and TM2

We first apply the criterion C1 to sequences $\rho_{1,2}^{(r)}(n)$ (the logarithmic test of divergence of the series S is conclusive) and conclude that for $r = 1$ both solutions $W_1^{(1)}(x) = \frac{e^{-\sqrt{x}}}{2\sqrt{x}}$ and $W_2^{(1)}(x) = 2K_0(2x^{\frac{1}{2}})$ are unique. For $r > 1$ we find that $\sum_{n=1}^{\infty}[\rho_{1,2}^{(r)}(n)]^{-\frac{1}{2n}}$ is convergent. Application of criterion C2 gives convergence of the Krein integral, showing that the solutions of **TM1** and **TM2** are non-unique. These findings are confirmed by use of criterion C3: convexity of $\psi_{1,2}^{(r)}(x) = -\ln[W_{1,2}^{(r)}(e^x)]$ is proved as it is equivalent to $e^{x/r} > 0$ for $x > 0$, (**TM1**) and $K_1(2e^{x/r}) - K_0(2e^{x/r}) > 0$ for $x > 0$, (**TM2**). The quest for non-unique solutions for $r > 1$ is then well founded. We remark that the phenomenon of switching from unique to non-unique solutions can be also explained by methods exposed by Pakes et al. in Ref.[27], which would also permit the study of r non-integer case.

We know of no general method to construct such solutions. However we have proposed a procedure based on the application of inverse Mellin transform which generates the required solutions [9, 44] which we now briefly expose.

The first step is to construct, within the framework of a given set of $\rho(n)$'s, a family of functions $\omega_k(x)$, parametrized by a constant k (to be defined below), such that *all* their moments vanish, *i.e.*

$$\int_0^\infty x^n \omega_k(x)dx = \int_0^\infty x^{s-1}\omega_k(x)dx = 0, \quad n = 0, 1, 2, \ldots, \quad s = 1, 2, \ldots. \tag{20}$$

The Mellin transform of $\omega_k(x)$, $\omega_k^\star(s)$, vanishes for $s = 1, 2, \ldots$. Such functions are orthogonal to all polynomials and play an important role in the study of integral transforms [45]. For our purposes we choose a particular method of producing the functions $\omega_k^\star(x)$:

$$\int_0^\infty x^n \omega_k(x)dx = \rho_{1,2}^{(r)}(n) \cdot h_k(n), \tag{21}$$

or equivalently

$$\int_0^\infty x^{s-1}\omega_k(x)dx = \rho_{1,2}^{(r)}(s-1) \cdot h_k(s-1), \tag{22}$$

where $h_k(s)$ is any holomorphic function vanishing for $s = 1, 2, \ldots$. Among an infinity of possible choices the simplest one is $h_k(s) = \sin(\pi k(s+1))$ and it defines a discrete parameter $k = \pm1, \pm2, \pm3, \ldots$. The function $\omega_k(x)$ acquires new parameters now and is formally obtained by calculating the inverse Mellin transform

$$\omega_{1,2,k}^{(r)}(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \rho_{1,2}^{(r)}(s-1)\sin(\pi k s)x^{-s}ds, \quad k = \pm1, \pm2, \pm3, \ldots. \tag{23}$$

It turns our that for both $\rho_{1,2}^{(r)}(s)$ the integration in Eq.(23) can be performed:
a) **TM1**: for $\rho_1^{(r)}(n) = (2rn)!$, $\omega_{1,k}^{(r)}(x)$ is a special case of earlier evaluation [9] and reads:

$$\begin{aligned} \omega_{1,k}^{(r)}(x) &= \frac{1}{2rx^{\frac{2r-1}{2r}}} e^{-x^{\frac{1}{2r}}} \sin\left[k\pi\left(\tfrac{2r-1}{2r}\right) + x^{\frac{1}{2r}}\tan\left(\tfrac{k\pi}{2r}\right)\right], \quad (r > |k|) \\ &= W_1^{(r)}(x)\sin\left[k\pi\left(\tfrac{2r-1}{2r}\right) + x^{\frac{1}{2r}}\tan\left(\tfrac{k\pi}{2r}\right)\right]. \end{aligned} \tag{24}$$

In Eq.(24) we notice a pleasant factorization of $W_1^{(r)}(x)$.
b) **TM2**: for $\rho_2^{(r)}(n) = [(rn)!]^2$ the corresponding function $\omega_{2,k}^{(r)}(x)$ has to be, in the first place, represented as

$$\omega_{2,k}^{(r)}(x) = \mathcal{M}^{-1}[\underbrace{\Gamma(rs-s+1)}_{\mathrm{I}}\underbrace{\Gamma(rs-s+1)\sin(\pi ks)}_{\mathrm{II}};x] \tag{25}$$

which can be conceived as another case of Mellin convolution. A little thought gives the two partners to be convoluted as

$$\mathrm{I} \to W_1^{(r/2)}(x) = \frac{1}{rx^{\frac{r-1}{r}}}e^{-x^{\frac{1}{r}}}, \tag{26}$$

(see Eq.(24)) and

$$\mathrm{II} \to \omega_{1,k}^{(r/2)}(x) = \tfrac{1}{rx^{\frac{r-1}{r}}}e^{-x^{\frac{1}{r}}}\sin\left[k\pi\left(\tfrac{r-1}{r}\right) + x^{\frac{1}{r}}\tan\left(\tfrac{k\pi}{r}\right)\right], \tag{27}$$

(compare Eq.(24)). Thus, the integral form of $\omega_{2,k}^{(r)}(x)$ is

$$\omega_{2,k}^{(r)}(x) = \int_0^\infty W_1^{(r/2)}\left(\frac{x}{t}\right)\omega_{1,k}^{(r/2)}(t)\frac{dt}{t} \tag{28}$$

whose evaluation requires a number of changes of variables as well as the use of formula 2.5.37.2, p. 453 of vol.1 Ref.[13], but is essentially elementary. The final result is

$$\begin{aligned}\omega_{2,k}^{(r)}(x) &= \tfrac{2}{rx^{\frac{r-1}{r}}}\mathrm{Re}\left[e^{i\pi\left(\frac{1}{2}-k\frac{r-1}{r}\right)}K_0\Big(2x^{\frac{1}{2r}}\left(1+i\tan\left(\tfrac{\pi k}{r}\right)\right)^{1/2}\Big)\right]\\ &\equiv \tfrac{2}{rx^{\frac{r-1}{r}}}V_k^{(r)}(x)\end{aligned} \tag{29}$$

where we note a "near" factorization of $W_2^{(r)}(x)$.

Armed with explicit forms for $\omega_{1,k}^{(r)}(x)$ and $\omega_{2,k}^{(r)}(x)$ we are in position now to write down families of non-unique solutions. Their structure has the form: *principal solution* + const$\cdot\omega_k(x)$. More precisely:
TM1:

$$\tilde{W}_1^{(r)}(\epsilon,k,x) = W_1^{(r)}(x)\left[1+\epsilon\sin\left(k\pi\left(\frac{2r-1}{2r}\right)+x^{\frac{1}{2r}}\tan\left(\frac{k\pi}{2r}\right)\right)\right] \tag{30}$$

for real ϵ, $|\epsilon|<1$.
TM2:

$$\tilde{W}_2^{(r)}(\gamma,k,x) = W_2^{(r)}(x)\left[1+\gamma\frac{V_k^{(r)}(x)}{K_0(2x^{\frac{1}{2r}})}\right] \tag{31}$$

for $r > 2|k|$.
As $\left[1+\gamma\frac{V_k^{(r)}(x)}{K_0(2x^{1/2r})}\right]$ is an oscillating function of bounded variation, a constant $\gamma = \gamma(k,r)$ can be always found to assure the overall positivity of $\tilde{W}_2^{(r)}(\gamma,k,x)$. The above technique for obtaining non-unique solutions can be readily extended to moment sequences more general than $\rho_{1,2}^{(r)}(n)$. We shall simply mention two such extensions without entering into details.

For $\rho_3^{(r)}(n) = [(rn)!]^3$ we begin with the sequence $(n!)^3$ for which the solution is

$$\int_0^\infty x^n \text{MeijerG}\,([\,[\ ],[\ ]\,],[\,[0,0,0],[\ ]\,],x)\,dx = (n!)^3, \tag{32}$$

where we use a convenient and self-explanatory notation for Meijer's G-function borrowed from that of computer algebra systems. Observe that $\rho_3^{(r)}(n)$ corresponds to triple convolution of probability distribution function characterizing $(rn)!$, hence the use of Meijer's G-function. For applications of Meijer's G function in probability theory see Refs.[41, 42]. The extension, *via* Eq. (10), leads to the principal solution

$$\int_0^\infty x^n \left[\frac{1}{rx^{\frac{r-1}{r}}}\text{MeijerG}\left([\,[\ ],[\ ]\,],[[0,0,0],[\ ]],x^{\frac{1}{r}}\right)\right]dx = [(rn)!]^3. \tag{33}$$

The integrand in Eq. (33) is a positive function for $x > 0$ which cannot be represented by any other known special function. It possesses an infinite series representation in terms of polygamma functions, which we will not quote here. The Carleman sum S is convergent but the criterion C2 is not conclusive. Only the criterion C3 permits to ascertain the non-uniqueness. The corresponding function $\omega_{3,k}^{(r)}(x)$ is defined as

$$\omega_{3,k}^{(r)}(x) = \mathcal{M}^{-1}[\underbrace{\Gamma^2(rs-s+1)}_{\text{I}}\,\underbrace{\Gamma(rs-s+1)\sin(\pi ks)}_{\text{II}};x] \tag{34}$$

which can be calculated as Mellin convolution of I $\to W_2^{(r)}(x)$ and II $\to \omega_{1,k}^{(r/2)}(x)$, see Eqs. (24) and (29), respectively.

As a final example consider the sequence $\rho_4^{(r)}(n) = \rho_1^{(r)}(n)\rho_2^{(r)}(n)$. The corresponding Mellin convolution of two principal solutions $W_{1,2}^{(r)}(x)$, see Eqs. (30) and (31), yields directly the principal solution for $\rho_4^{(r)}(n)$:

$$\begin{aligned}\int_0^\infty x^n \left[\frac{4^{r-1}}{r\sqrt{\pi}x^{\frac{2r-1}{r}}}\text{MeijerG}\left([\,[\ ],[\ ]\,],[[r-\tfrac{1}{2},r,r,r],[\ ]],\tfrac{1}{4}x^{\frac{1}{r}}\right)\right]dx &= (2rn)![(rn)!]^2,\\ &n=0,1,2\ldots,\quad r=1,2,\ldots,\end{aligned} \tag{35}$$

which is non-unique by **C3** only, as **C2** remains inconclusive.

6 Discussion and Conclusion

We have demonstrated a methodology for obtaining unique and non-unique solutions of the Stieltjes moment problem using the Mellin convolution method. Although the initial moment sequences were simple and classical, one is rapidly forced to leave the realm of standard special functions, as the resulting solutions are special cases of Meijer G-functions, a well-known fact to specialists in probability and statistics. In most cases they resist the check for non-uniqueness *via* both the Carleman criterion **C1** and the Krein criterion **C2**, and Pakes-Gut criterion **C3** appears to be the only tool to decide this question. We have generated parametrized families of non-unique solutions exemplified here by Eqs. (24) and (29). Such functions, after Stoyanov [31], are now called *Stieltjes classes*. Their general properties and methods of explicit construction for such probability distributions like lognormal, inverse Gaussian and logistic, all leading to indeterminate Stieltjes moment problem, were investigated by Stoyanov and Tolmatz and Pakes [32, 33, 28]. In the context of work of the aforementioned authors our example Eq. (24) belongs to the class arising from Eq.(4) in [33] while the example Eq. (29) is more general. This supports our belief that the Mellin transform/convolution technique, which we have presented and advocated throughout this paper, provides a toolkit completing other methods of solving the indeterminate moment problems.

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