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Lower Bounds on the Area Requirements of Series-Parallel Graphs

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We show that there exist series-parallel graphs requiring $\Omega(n2^{\sqrt{\log n}})$ area in any straight-line or poly-line grid drawing. Such a result is achieved in two steps. First, we show that, in any straight-line or poly-line drawing of $K_{2,n}$, one side of the bounding box has length $\Omega(n)$, thus answering two questions posed by Biedl *et al.* Second, we show a family of series-parallel graphs requiring $\Omega(2^{\sqrt{\log n}})$ width and $\Omega(2^{\sqrt{\log n}})$ height in any straight-line or poly-line grid drawing. Combining the two results, the $\Omega(n2^{\sqrt{\log n}})$ area lower bound is achieved.

Keywords: planar graphs, series-parallel graphs, area requirements, graph drawing, straight-line drawings

1 Introduction

A planar graph is a graph that can be drawn in the plane so that no two edges intersect, except, possibly, at their common endpoints. Determining asymptotic bounds for the area requirements of straight-line and poly-line drawings of planar graphs is one of the classical topics in the Graph Drawing literature. Ground-breaking works at the beginning of the nineties by de Fraysseix *et al.* [dPP90] and by Schnyder [Sch90] have shown that every n-vertex planar graph admits a planar straight-line drawing on an $O(n) \times O(n)$ grid. Such a bound is worst-case optimal, even for poly-line drawings [DLT84, dPP90]. Hence, it is natural to search for interesting sub-classes of planar graphs admitting sub-quadratic area drawings.

It turns out that several important sub-classes of planar graphs contain graphs requiring quadratic area in any grid drawing.

- Every *four-connected plane graph* whose outer face has at least four vertices admits a straight-line drawing in $\lfloor \frac{n}{2} \rfloor \times \left(\lceil \frac{n}{2} \rceil - 1\right)$ area, as shown by Miura *et al.* in [MNN01], improving upon previous results by He [He97]. Miura *et al.* also observe that such a bound is tight, as shown by the graph in Fig. 1 (a).

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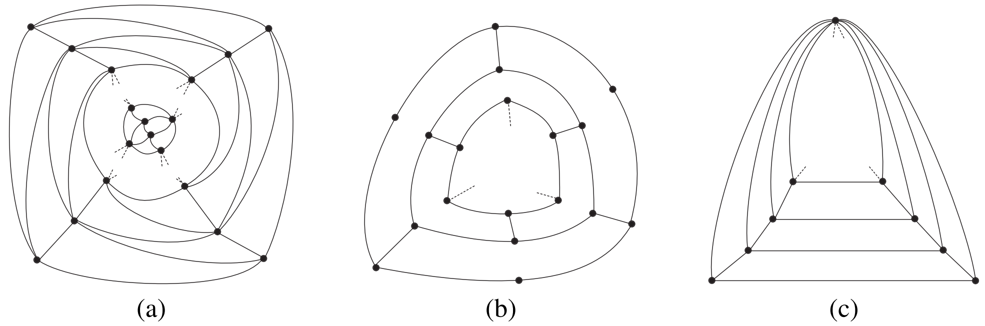

Fig. 1: (a) A four-connected plane graph requiring $\lfloor \frac{n}{2} \rfloor \times \left(\lceil \frac{n}{2} \rceil - 1\right)$ area in any poly-line drawing. (b) A plane graph with maximum degree three requiring quadratic area in any poly-line drawing. (c) A plane graph with outerplanarity two requiring quadratic area in any poly-line drawing.

- Every *bipartite plane graph* admits a straight-line drawing in $\lfloor \frac{n}{2} \rfloor \times \left(\lceil \frac{n}{2} \rceil - 1\right)$ area, as shown by Biedl and Brandenburg in [BB05]. The upper bound of Biedl and Brandenburg is tight, since bipartite plane graphs exist, very similar to the one shown by Miura *et al.* [MNN01], requiring $\lfloor \frac{n}{2} \rfloor \times \left(\lceil \frac{n}{2} \rceil - 1\right)$ area in any poly-line/straight-line drawing.

- *Planar graphs of maximum degree three* exist requiring quadratic area in any poly-line/straight-line grid drawing, as shown in Fig. 1 (b).

- Planar graphs with *outerplanarity two* exist requiring quadratic area in any poly-line/straight-line grid drawing, as shown by the graph in Fig. 1 (c), that has been presented by Biedl in [Bie05].

Planar graphs are the graphs excluding K_5 and $K_{3,3}$ as minors [Wag37]. Which are the classes of graphs excluding graphs *smaller than* K_5 and $K_{3,3}$ as minors? The answer to the previous question is a list of some of the most studied sub-classes of planar graphs. In fact, *trees* are the graphs excluding K_3 as a minor, *outerplanar graphs* are the graphs excluding K_4 and $K_{2,3}$ as minors, and *series-parallel graphs* are the graphs excluding K_4 as a minor. Such graph classes, apart from having nice characterizations in terms of excluded minors, apart from having nice alternative characterizations (a tree is a connected acyclic graph, an outerplanar graph is a graph that admits a planar embedding in which all the vertices are incident to the same face, and a series-parallel graph is a graph that can be inductively defined by series and parallel compositions of smaller series-parallel graphs), and apart from being of real interest for applications, do admit grid drawings in sub-quadratic area.

- Concerning trees, a slight modification of the *h-v drawing* algorithm by Crescenzi *et al.* [CDP92] constructs drawings in $O(n \log n)$ area. Optimal $O(n)$ area bounds are known if the degree of the tree is bounded, as proved by Garg *et al.* for poly-line drawings [GGT96] and by Garg and Rusu for straight-line drawings [GR03].

- Concerning outerplanar graphs, Biedl [Bie02] has shown how to construct poly-line drawings in $O(n \log n)$ area; Di Battista and the author [DF09] presented an algorithm for obtaining straight-line drawings in $O(n^{1.48})$ area; the author [Fra07] exhibited an algorithm for constructing straight-line drawings in $O(dn \log n)$ area, where d is the degree of the graph.

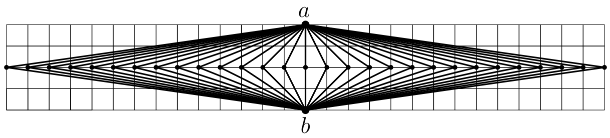

Fig. 2: A straight-line drawing of $K_{2,n}$ with linear area and linear aspect ratio.

Both for outerplanar graphs and for trees, no super-linear area lower bounds are known, neither in the case of straight-line drawings nor in the one of poly-line drawings.

In this paper we deal with series-parallel graphs, a class of planar graphs that has been widely investigated in Graph Theory and Graph Drawing (see, e.g., [VTL82, Epp92, BCD$^+$94, Di 03, DDLW06]).

The main known result on the construction of small-area grid drawings of series-parallel graphs is that every series-parallel graph admits a poly-line drawing in $O(n^{3/2})$ area. Such a bound was proved by Biedl in [Bie05, Bie10]; in these papers, the author provides a nice inductive construction of *visibility representations* of series-parallel graphs and shows how such representations can be turned into poly-line drawings with asymptotically the same area.

While poly-line drawings can be realized in $O(n^{3/2})$ area, no sub-quadratic area upper bound is known in the case of straight-line drawings. The best known quadratic upper bound for straight-line drawings is provided in [ZHN10]. In [Bie05], Biedl also proved an $\Omega(\frac{n \log n}{\log \log n})$ area lower bound for straight-line drawings of series-parallel graphs.

The $\Omega(\frac{n \log n}{\log \log n})$ area lower bound for straight-line drawings of series-parallel graphs is a direct consequence of the results in [BCLO03], where Biedl, Chan, and López-Ortiz, settling in the positive a conjecture of Felsner *et al.* [FLW03], proved that no linear-area straight-line drawing of $K_{2,n}$ can achieve constant aspect ratio. Observe that a drawing of the complete bipartite graph $K_{2,n}$ can be thought of as a drawing of n paths that start and end at the same two vertices, in the following denoted by a and b, and that do not share any other vertex. In the following we will refer such paths as the *paths of* $K_{2,n}$. Fig. 2 shows a straight-line drawing of $K_{2,n}$ with linear area and linear aspect ratio. More precisely, Biedl, Chan, and López-Ortiz proved the following:

Theorem 1 *(Biedl* et al. *[BCLO03]) Every planar straight-line grid drawing of $K_{2,n}$ on a $W \times H$ grid with $W \geq H$ satisfies $W \log H \in \Omega(n)$.*

Corollary 1 *(Biedl* et al. *[BCLO03]) Every planar straight-line grid drawing of $K_{2,n}$ on a $W \times H$ grid satisfies $\max\{W, H\} \in \Omega(n/\log n)$.*

Biedl *et al.* ask whether the $\log H$ factor in Theorem 1 can be eliminated and whether the same lower bound holds even in the case of poly-line drawings.

In this paper we answer both the questions in the affirmative. Namely, we prove the following:

Theorem 2 *Every planar straight-line or poly-line grid drawing of $K_{2,n}$ on a $W \times H$ grid satisfies $\max\{W, H\} \in \Omega(n)$.*

Such a result is achieved by first exhibiting a very simple "optimal" drawing algorithm for $K_{2,n}$. That is, if there exists a drawing of $K_{2,n}$ inside an arbitrary convex polygon P in which a and b are placed at two specified vertices of P, then our algorithm constructs one such drawing. Second, we study the drawings constructed by the mentioned algorithm inside a rectangle. Such a study reveals a surprisingly regular behavior of the drawing of the paths of $K_{2,n}$; we argue that such a behavior has a strong relationship with the generation of relatively prime numbers as expressed in the *Stern-Brocot* tree. On the base of such a relationship, we derive some arithmetical properties of the lines passing through infinite grid points in the plane, that might be interesting by their own, and we achieve the claimed lower bound.

As a consequence of Theorem 2, an $\Omega(n \log n)$ lower bound on the area requirements of poly-line and straight-line drawings of series-parallel graphs can be obtained. Namely, consider an $O(n)$-node series-parallel graph containing $K_{2,n}$ and an n-node complete ternary tree as subgraphs. Since any poly-line or straight-line drawing of an n-node complete ternary tree requires $\Omega(\log n)$ width and $\Omega(\log n)$ height (see [FLW03, Sud04]), and since the width or the height of any such a drawing has $\Omega(n)$ length (by Theorem 2), the lower bound follows. However, we can achieve a better lower bound by means of the following:

Theorem 3 *There exist series-parallel graphs requiring $\Omega(2^{\sqrt{\log n}})$ width and $\Omega(2^{\sqrt{\log n}})$ height in any straight-line or poly-line grid drawing.*

Such a result is achieved by carefully constructing a graph out of several copies of $K_{2,n}$ and by then strongly exploiting Theorem 2 and some further geometric considerations. Theorem 3, together with Theorem 2, immediately implies the following main result:

Theorem 4 *There exist series-parallel graphs requiring $\Omega(n2^{\sqrt{\log n}})$ area in any straight-line or poly-line grid drawing.*

We remark that the function $2^{\sqrt{\log n}}$ is greater than any polylogarithmic function of n and smaller than any polynomial function of n; we further remark that no super-linear area lower bound was previously known for poly-line drawings of series-parallel graphs and that $\Omega(\frac{n \log n}{\log \log n})$ was the best known area lower bound for straight-line drawings of series-parallel graphs [Bie05].

The rest of the paper is organized as follows. In Section 2 we give some preliminaries; in Section 3 we give some geometric lemmata; in Section 4 we prove Theorem 2; in Section 5 we prove Theorem 3; finally, in Section 6 we conclude and suggest some open problems.

2 Preliminaries

A *grid drawing* of a graph is a mapping of each vertex to a distinct point of the plane with integer coordinates and of each edge to a Jordan curve between the endpoints of the edge. A *planar drawing* is such that no two edges intersect except, possibly, at common endpoints. In the following we always refer to planar grid drawings. A *straight-line* drawing is such that all edges are rectilinear segments. A *poly-line* drawing is such that the edges are sequences of rectilinear segments. In a poly-line drawing a *bend* is a point in which an edge changes its slope, i.e., a point common to two consecutive segments in the sequence of segments representing the edge. In a grid drawing bends have integer coordinates. A *polygonal path* is a poly-line grid drawing of a path.

The *bounding box* of a drawing Γ is the smallest rectangle with sides parallel to the axes that covers Γ completely. The *height* (*width*) of Γ is the height (resp. width) of its bounding box. The *area* of Γ is the height of Γ times its width. The *aspect ratio* of Γ is the ratio between the maximum and minimum of its height and width.

Throughout the paper, a *grid line* is any line passing through an infinite number of grid points. Two grid lines are *consecutive* if they are parallel and no grid point is contained in the open strip delimited by

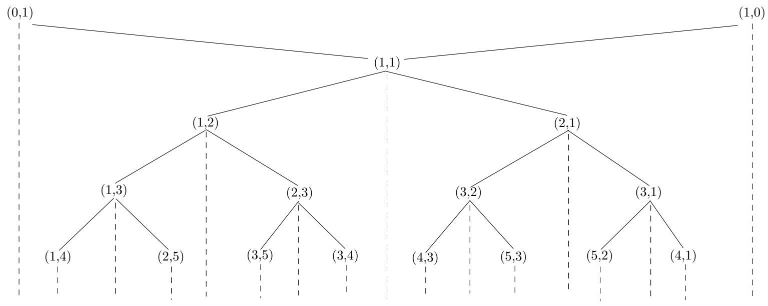

Fig. 3: The Stern-Brocot tree.

the two lines. We denote by $\overline{pq}$ the segment between two points p and q. We also denote by $\overline{a_0a_1\dots a_k}$ the polygonal line composed of segments $\overline{a_0a_1}, \overline{a_1a_2}, \dots, \overline{a_{k-1}a_k}$.

Let $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ be ordered pairs of real numbers (such pairs might represent grid points or vectors, for example) and let k be a constant. Throughout the paper, we write kp_1 to denote the pair (kx_1, ky_1), we write $p_1 + p_2$ to denote the pair $(x_1 + x_2, y_1 + y_2)$, and we write $p_1 - p_2$ to denote the pair $(x_1 - x_2, y_1 - y_2)$. We denote by $\vec{v} \cdot a$ the *scalar product* between a vector $\vec{v} = (v_x, v_y)$ and a point $a = (a_x, a_y)$ in the plane, that is, $\vec{v} \cdot a = v_x a_x + v_y a_y$.

The *Stern-Brocot tree* [Ste58, Bro60] is an infinite tree whose nodes are in bijective mapping with the irreducible positive rational numbers, or equivalently, in bijective mapping with the ordered pairs of relatively prime integers. See Fig. 3.

The Stern-Brocot tree has two nodes $(0, 1)$ and $(1, 0)$ which are both connected to the same node $(1, 1)$. Nodes $(0, 1)$ and $(1, 0)$ are the *left parent* and the *right parent* of $(1, 1)$, respectively. Further, $\frac{1}{0}$ and $\frac{0}{1}$ are the *left generating fraction* and the *right generating fraction* of $\frac{1}{1}$, respectively. An ordered binary tree is then rooted at $(1, 1)$ as follows. Consider a node (x, y) of the tree. Such a node has two children. The left child of (x, y) is the node $(x, y) + (x', y')$, where (x', y') is the ancestor of (x, y) that is closer to (x, y) (in terms of graph-theoretic distance on the tree) and that has (x, y) in its right subtree. Then, $\frac{y'}{x'}$ and $\frac{y}{x}$ are the *left generating fraction* and the *right generating fraction* of $\frac{y+y'}{x+x'}$, respectively. Analogously, the right child of (x, y) is the node $(x, y) + (x'', y'')$, where (x'', y'') is the ancestor of (x, y) that is closer to (x, y) and that has (x, y) in its left subtree. Then, $\frac{y}{x}$ and $\frac{y''}{x''}$ are the *left generating fraction* and the *right generating fraction* of $\frac{y+y''}{x+x''}$, respectively. The following properties of the Stern-Brocot tree are well-known and easy to observe:

Property 1 *Let (x, y) be a node of the Stern-Brocot tree and let $\frac{y'}{x'}$ and $\frac{y''}{x''}$ be the left and right generating fractions of $\frac{y}{x}$. Then, the subtree of the Stern-Brocot tree rooted at the left child of (x, y) contains all and only the pairs of relatively prime integers (z, w) such that $\frac{y}{x} < \frac{w}{z} < \frac{y'}{x'}$ and the subtree of the Stern-Brocot tree rooted at the right child of (x, y) contains all and only the pairs of relatively prime integers (z, w) such that $\frac{y''}{x''} < \frac{w}{z} < \frac{y}{x}$.*

Property 2 *Let (x, y) be a node of the Stern-Brocot tree. Then every node (x', y') that is a descendant of (x, y) is such that $x' \geq x$ and $y' \geq y$ and either $x' \neq x$ or $y' \neq y$ (or both).*

It is useful to visualize the Stern-Brocot tree in the following way. Nodes $(0, 1)$, $(1, 1)$, and $(1, 0)$ are

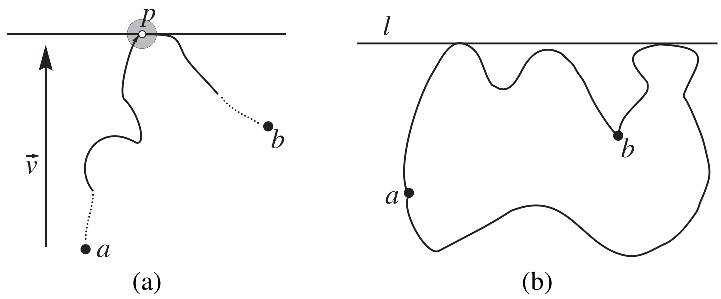

Fig. 4: (a) Illustration for the proof of Lemma 1. Disk D is the small shaded region. (b) Illustration for the proof of Lemma 2.

ordered in this way from left to right and three vertical lines are associated with these nodes. When a node (x, y) is drawn, it is placed in the strip delimited by the vertical lines associated with its left and right generating fractions, and a vertical line is associated with (x, y). In such a visualization, each node of the tree is "close" to its generating fractions and nodes (x, y) are ordered from left to right by decreasing value of $\frac{y}{x}$.

3 Geometric Lemmata

In this section we show some lemmata that will be used to prove Theorems 2 and 3. We first deal with the geometry of $K_{2,n}$ and then with the relationships between relatively prime numbers and grid lines in the plane. Recall that $K_{2,n}$ can be considered a collection of n paths (the *paths of* $K_{2,n}$) that connect two vertices a and b and that do not share any other vertex.

3.1 Lemmata on the Geometry of $K_{2,n}$

Lemma 1 *Consider any poly-line grid drawing of $K_{2,n}$, any path π of $K_{2,n}$, and any vector $\vec{v} = (v_1, v_2)$. There exists a grid point $p \in \pi$ such that $\vec{v} \cdot p \geq \vec{v} \cdot p'$, for any point $p' \in \pi$.*

Proof: If $\vec{v} \cdot a \geq \vec{v} \cdot p'$ or $\vec{v} \cdot b \geq \vec{v} \cdot p'$, for every point $p' \in \pi$, the lemma follows. Otherwise, consider the part π' of π starting at a and ending at the first point p in which $\vec{v} \cdot p \geq \vec{v} \cdot p'$, for every point $p' \in \pi$ (see Fig. 4.a). Since each point $p' \neq p$ of π' is such that $\vec{v} \cdot p' < \vec{v} \cdot p$, there exists a small disk D centered at p such that the part of π' enclosed in D is increasing in the direction determined by $\vec{v}$, when π' is oriented from a to p. Further π, when oriented from a to b, can not be increasing in the direction determined by $\vec{v}$ immediately after p, otherwise there would exist a point p'' such that $\vec{v} \cdot p'' > \vec{v} \cdot p$. It follows that π changes its slope at p and, by definition of poly-line grid drawing, p is a grid point. □

Lemma 2 *Consider any drawing of $K_{2,n}$. Let l be any line that does share any point with the open segment $\overline{ab}$. No three paths π_1, π_2, and π_3 of $K_{2,n}$ exist such that: (i) π_1, π_2, and π_3 do not share any point except for a and b; (ii) π_1, π_2, and π_3 are entirely contained in the closed half-plane delimited by l and containing a and b; and (iii) each of π_1, π_2, and π_3 shares at least one point different from a and b with l.*

Proof: Suppose, for a contradiction, that three paths π_1, π_2, and π_3 of $K_{2,n}$ with the above properties exist. Paths π_1 and π_2 form a cycle $\mathcal{C}$. Line l is external to $\mathcal{C}$ and separates a from b in the exterior of $\mathcal{C}$ (see Fig. 4.b). Consider any path π_3 between a and b. If π_3 is internal to $\mathcal{C}$, then it can not share a point different from a and b with l unless it shares points different from a and b with $\mathcal{C}$. If π_3 is external to $\mathcal{C}$, then it either shares points different from a and b with $\mathcal{C}$ or it is in part contained in the open half-plane delimited by l and not containing a and b. If π_3 is part internal and part external to $\mathcal{C}$, then it shares points different from a and b with $\mathcal{C}$. In any case we have a contradiction. $\square$

Let P be any convex polygon in the plane with vertices having integer coordinates. Let I be the set of grid points in the interior or on the border of P. Let a and b be two distinct vertices of P. Let π_1^* and π_2^* be the drawings of the two paths that connect a and b and that compose P. At least one out of π_1^* and π_2^*, say π_1^*, is different from segment $\overline{ab}$. Let M be the maximum number of paths connecting a and b that can be drawn as non-crossing polygonal paths inside or on the border of P.

Lemma 3 *There exist M non-crossing polygonal paths connecting a and b such that:*

- *Each path is inside or on the border of P; and*
- *one of such paths is π_1^*.*

Proof: Consider any drawing Γ composed of M non-crossing polygonal paths connecting a and b and contained inside or on the border of P. If a path of Γ is π_1^*, there is nothing to prove. Otherwise, observe that no two distinct paths π_i and π_j pass through points of π_1^*, as otherwise π_i and π_j cross. Hence, Γ has at most one path π passing through points of π_1^*. Remove π from Γ, if π exists, and draw a path in Γ as π_1^*. Since no path different from π passes through a point of π_1^*, the resulting drawing is planar, hence proving the lemma. $\square$

Lemma 4 *There exist M non-crossing polygonal paths connecting a and b such that:*

- *Each path is inside or on the border of P; and*
- *one of such paths is segment $\overline{ab}$.*

Proof: We prove the claim by induction on M. If $M = 1$, then drawing a path as segment $\overline{ab}$ proves the claim. Suppose $M \geq 2$. By Lemma 3, there exists a drawing Γ composed of M non-crossing polygonal paths connecting a and b such that each path is inside or on the border of P and one of such paths, say π, is π_1^*. Remove π from Γ and all the grid points π passes through, except for a and b, from I. Consider the convex polygon P' that is the convex hull of the resulting grid point-set I'. The vertices of P' have integer coordinates. Further, P' is such that $M-1$ paths can be drawn as non-crossing polygonal paths connecting a and b inside or on the border of P'. In fact Γ is a drawing having such a property. Hence, the inductive hypothesis applies and $M-1$ polygonal paths exist so that each path is inside or on the border of P' and so that one of the paths is segment $\overline{ab}$. Considering such $M-1$ paths together with the drawing of π as π_1^* proves the lemma. $\square$

Now assume that a and b are consecutive vertices of P (see Fig. 5). Let I be the set of grid points in the interior or on the border of P. As before, let π_1^* and π_2^* be the drawings of the two paths that connect

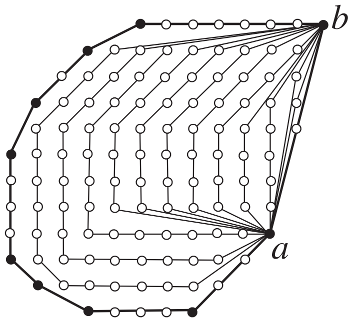

Fig. 5: Drawing the maximum number of paths in a convex polygon with vertices having integer coordinates. Black circles are vertices of P and white circles are grid points inside or on the border of P.

a and b and that compose P, where π_1^* is different from segment $\overline{ab}$. Let also M be the maximum number of paths connecting a and b that can be drawn as non-crossing polygonal paths inside or on the border of P.

We iteratively draw some paths $\pi_1, \pi_2, \cdots, \pi_N$ connecting a and b inside or on the border of P as follows. Path π_i is drawn when the current convex grid polygon is P_i containing in its interior or on its border a set I_i of grid points. At the first step $P_1 = P$ and $I_1 = I$. If P_i does not coincide with segment $\overline{ab}$, draw path π_i as the polygonal path that connects a and b, that lies on P_i, and that is different from segment $\overline{ab}$. Remove the grid points that lie on P_i, except for a and b, from I_i, obtaining a new set of grid points I_{i+1}. Then, P_{i+1} is the convex hull of I_{i+1}. If P_i coincides with segment $\overline{ab}$, draw path $\pi_i = \pi_N$ as segment $\overline{ab}$. We observe the following:

Lemma 5 *Paths $\pi_1, \pi_2, \cdots, \pi_N$ are drawn as non-crossing polygonal paths inside or on the border of P. Further, $N = M$.*

Proof: The first part of the statement is trivial. We prove that $N = M$ by induction on M. If $M = 2$, then the claim trivially holds, since π_1 is drawn as π_1^* and π_2 as $\overline{ab}$. Suppose that $M \geq 3$. By Lemma 3, there exists a drawing Γ composed of M non-crossing polygonal paths connecting a and b such that each path is inside or on the border of P and one of such paths, say π_1, is π_1^*. Remove π_1 from Γ and all the grid points π_1 passes through from I. Consider the convex polygon P' that is the convex hull of the resulting grid point-set I'. Clearly, the vertices of P' have integer coordinates. Further, P' is such that $M - 1$ non-crossing polygonal paths connecting a and b exist such that each path is inside or on the border of P'. In fact Γ is a drawing having such a property. Hence, the inductive hypothesis applies and the drawing algorithm described before the statement of the lemma draws $M - 1$ paths as non-crossing polygonal paths inside or on the border of P'. Considering such paths together with the drawing of π_1 as π_1^* proves the lemma. $\square$

3.2 A Lemma on the Arithmetics of Consecutive Grid Lines

The aim of this section is to prove the following useful lemma.

Lemma 6 *Let l_1 be a grid line with slope $\frac{y}{x}$, where $x, y > 0$ and (x, y) is a pair of relatively prime numbers. Let $\frac{y'}{x'}$ and $\frac{y''}{x''}$ be the left and right generating fractions of $\frac{y}{x}$. Consider any grid point (p_x, p_y)*

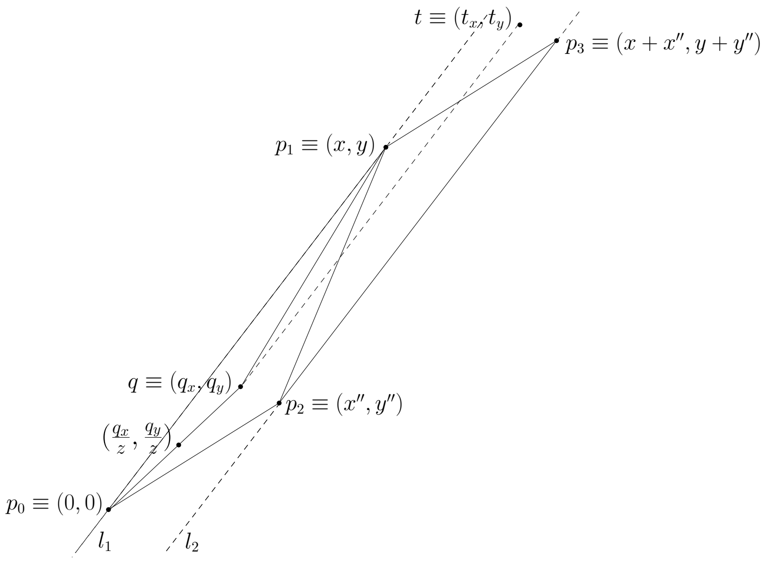

Fig. 6: Illustration for the proof of Lemma 6.

of l_1. Let l_2 (l_3) be the grid line passing through $(p_x, p_y) + (x'', y'')$ and through $(p_x, p_y) - (x', y')$ (resp. through $(p_x, p_y) - (x'', y'')$ and through $(p_x, p_y) + (x', y')$). Then, l_1 and l_2 (resp. l_1 and l_3) are consecutive grid lines.

Proof: Refer to Fig. 6. We prove the statement for l_1 and l_2, the proof for l_1 and l_3 being analogous. Suppose, for a contradiction, that l_1 and l_2 are not consecutive. First, observe that l_1 and l_2 are parallel, as l_2 has slope $\frac{p_y+y''-p_y+y'}{p_x+x''-p_x+x'} = \frac{y''+y'}{x''+x'} = \frac{y}{x}$, where the last equality holds by the definition of generating fractions.

We can assume, without loss of generality up to a simultaneous translation of l_1 and l_2, that l_1 passes through point $p_0 \equiv (0, 0)$. Denote $p_1 \equiv (x, y)$. Observe that a simultaneous translation of l_1 and l_2 does not alter whether the open strip delimited by the two lines contains a grid point, as the same translation moves any grid point between the two lines before the translation to a grid point between the two lines after the translation.

Suppose that a point $q \equiv (q_x, q_y)$ exists between l_1 and l_2. Then, we can assume that q is in the parallelogram P whose vertices are p_0, p_1, $p_2 \equiv (x'', y'')$, and $p_3 \equiv p_1 + p_2$, or on its border. Namely, if a grid point $t \equiv (t_x, t_y)$ is between l_1 and l_2, then every grid point $t \equiv (t_x + mx, t_y + my)$ is between l_1 and l_2, for all $m \in \mathbb{Z}$. Suppose that q is inside the closed triangle (p_0, p_1, p_2), the case in which it is inside (p_1, p_2, p_3) being analogous.

We can assume that (q_x, q_y) and $(x - q_x, y - q_y)$ are two pairs of relatively prime numbers. Namely, suppose that q_x and q_y have a common divisor, say z. Then, $(\frac{q_x}{z}, \frac{q_y}{z})$ is a grid point. Further, such a point is on the border of triangle (p_0, p_1, q), actually on $\overline{p_0q}$. Then, point $q \equiv (\frac{q_x}{z}, \frac{q_y}{z})$ can be considered instead of $q \equiv (q_x, q_y)$. Analogously, if $x - q_x$ and $y - q_y$ have a common divisor, say z, then point $q \equiv (x - \frac{x-q_x}{z}, y - \frac{y-q_y}{z})$ can be considered instead of $q \equiv (q_x, q_y)$. Observe that, whenever the currently

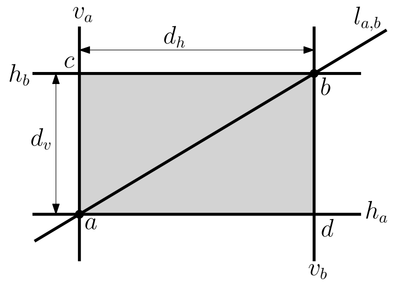

Fig. 7: Illustration of the notation for the proof of Theorem 2.

considered point $q \equiv (q_x, q_y)$ is replaced by a new grid point $q \equiv (\frac{q_x}{z}, \frac{q_y}{z})$ or $q \equiv (x - \frac{x-q_x}{z}, y - \frac{y-q_y}{z})$, the sum of the number of grid points on the border and of the number of grid points in the interior of triangle (p_0, p_1, q) decreases. Hence, eventually after a certain number of replacements, the coordinates q_x and q_y of q (and simultaneously $x - q_x$ and $y - q_y$) are relatively prime numbers.

Observe that q does not lie on $\overline{p_0 p_1}$ as it has to lie in the open strip delimited by l_1 and l_2. Further, it does not lie on $\overline{p_0 p_2}$ (on $\overline{p_1 p_2}$) as otherwise x'' and y'' (resp. x' and y') would not be relatively prime numbers.

Now consider the slope $\frac{q_y}{q_x}$. As q is inside triangle (p_0, p_1, p_2), it follows that $\frac{y''}{x''} < \frac{q_y}{q_x} < \frac{y}{x}$ and that $\frac{y}{x} < \frac{y-q_y}{x-q_x} < \frac{y'}{x'}$. By Property 1, the relatively prime pairs (q_x, q_y) and $(x - q_x, y - q_y)$ are contained in the subtree of the Stern-Brocot tree rooted at (x, y). By Property 2, $q_x \geq x$ and $q_y \geq y$ hold; further, $x - q_x \geq x$ and $y - q_y \geq y$ hold; hence, $q_x + x - q_x \geq 2x$ and $q_y + y - q_y \geq 2y$ hold. Such contradictions prove the lemma. □

4 Proof of Theorem 2

By definition, a straight-line drawing is also a poly-line drawing. Hence, it suffices to prove Theorem 2 for poly-line drawings.

Consider any poly-line grid drawing of $K_{2,n}$. Let R be the smallest axis-parallel rectangle enclosing a and b (see Fig. 7). Let $l_{a,b}$ be the line through a and b. Suppose, without loss of generality, that $y(a) \leq y(b)$. Suppose also that $x(a) < x(b)$, the case in which $x(a) \geq x(b)$ being analogous. Let c and d be the upper left corner and the lower right corner of R, respectively. Let h_a and v_a be the horizontal and vertical lines through a, respectively. Analogously, let h_b and v_b be the horizontal and vertical lines through b, respectively. Let d_h and d_v be the horizontal and vertical distance between a and b, respectively. The width W and the height H of the drawing are such that $W \geq d_h$ and $H \geq d_v$.

For any line l, denote by $H^+(l)$ (resp. by $H^-(l)$) the closed half-plane delimited by l and containing the normal vector of l increasing in the y-direction (resp. decreasing in the y-direction). If l is a vertical line, then $H^+(l)$ (resp. $H^-(l)$) denotes the closed half-plane delimited by l and containing the normal vector of l increasing in the x-direction (resp. decreasing in the x-direction). For any non-horizontal line l, we say that a point p is *to the right of* l (*to the left of* l) if p is in the open half-plane delimited by l and containing the normal vector of l increasing in the x-direction (resp. decreasing in the x-direction).

Consider the half-plane $H^+(h_b)$. By Lemma 1 with $\vec{v} = (0, 1)$, for each path π that has non-empty intersection with $H^+(h_b)$, there exists a grid point $p \in \pi$ whose y-coordinate is maximum among the

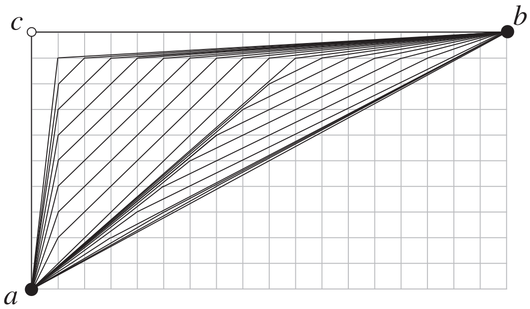

Fig. 8: Paths $\pi_1, \pi_2, \cdots, \pi_{M_1}$ in Π.

points of π. Clearly, p belongs to $H^+(h_b)$. Hence, p belongs to a horizontal grid line l that does not intersect or contain the open segment $\overline{ab}$. By Lemma 2, at most two paths of $K_{2,n}$ have their points with greatest y-coordinate belonging to l. It follows that, if a linear number of paths of $K_{2,n}$ has non-empty intersection with $H^+(h_b)$, then their points with greatest y-coordinate belong to a linear number of distinct horizontal grid lines and hence $H \in \Omega(n)$.

Similar arguments show that, if a linear number of paths of $K_{2,n}$ have non-empty intersection with $H^-(h_a)$, $H^+(v_b)$, or $H^-(v_a)$, then $H \in \Omega(n)$, $W \in \Omega(n)$, or $W \in \Omega(n)$, respectively.

If there exists no linear number of paths of $K_{2,n}$ having non-empty intersection with $H^+(h_b)$, $H^-(h_a)$, $H^+(v_b)$, or $H^-(v_a)$, then a linear number of paths of $K_{2,n}$ is completely inside or on the border of R. We show that this implies that $\max\{d_h, d_v\} \in \Omega(n)$, and hence that $\max\{W, H\} \in \Omega(n)$.

Let M be the maximum number of paths of $K_{2,n}$ that can be drawn inside or on the border of R. By Lemma 4, there exists a drawing of M paths connecting a and b, and completely lying inside or on the border of R, such that one of the paths is drawn as segment $\overline{ab}$. Since $M \in \Omega(n)$, then either a linear number of paths of $K_{2,n}$ is contained inside or on the border of the triangle T_1 having a, b, and c as vertices, or a linear number of paths of $K_{2,n}$ is contained inside or on the border of the triangle T_2 having a, b, and d as vertices. Suppose that a linear number of paths of $K_{2,n}$ is contained inside or on the border of T_1, the other case being symmetric.

Let $M_1 \in \Omega(n)$ be the maximum number of paths of $K_{2,n}$ that can be drawn inside T_1 and let I_1 be the set of grid points inside or on the border of T_1. By Lemma 5, a sequence of M_1 non-crossing polygonal paths $\Pi = (\pi_1, \pi_2, \cdots, \pi_{M_1})$ connecting a and b and completely inside or on the border of T_1 can be drawn by repeating the following two operations, for $1 \leq i < M_1$: (1) consider the current convex grid polygon P_i (when $i = 1$ then $P_1 = T_1$); let I_i be the set of grid points inside or on the border of P_i; draw path π_i as the part of P_i that connects a and b, and that is different from segment $\overline{ab}$; (2) delete from I_i the grid points π_i passes through, obtaining a set of grid points I_{i+1}. Convex polygon P_{i+1} is the convex hull of I_{i+1}. Path π_{M_1} is drawn as segment $\overline{ab}$. See Fig. 8.

In order to prove that $M_1 \in \Omega(n)$ implies $\max\{d_h, d_v\} \in \Omega(n)$, we study paths $\pi_1, \pi_2, \cdots, \pi_{M_1}$ and prove that they have a very regular behavior that is strongly related to the generation of relatively prime numbers as in the Stern-Brocot tree. In the following, we first sketch a description of the geometry of paths $\pi_1, \pi_2, \cdots, \pi_{M_1}$, we then detail such a description, we later prove the geometric claims to be correct, and we finally prove that $\max\{d_h, d_v\} \in \Omega(n)$. In the reminder of the section we assume that $d_h, d_v > 3$. Clearly, if one of d_h and d_v is $O(1)$, then the other one must be $\Omega(M_1)$, and there is nothing to prove.

4.1 Sketch of the geometry of paths $\pi_1, \pi_2, \cdots, \pi_{M_1}$

First, we observe that each path in Π is composed of two or three segments, i.e., each path has one or two bends. A sequence of paths that are consecutive in Π and that are each composed of three segments is such that all the "second segments" of the paths have the same slope.

In a sequence of paths such that the second segments of the paths have the same slope, all the bends lie on two lines, having slopes one greater and one smaller than $\frac{d_v}{d_h}$, the slope of segment $\overline{ab}$. Moreover, the two lines on which such bends lie have slope $\frac{y_1}{x_1}$ and $\frac{y_2}{x_2}$, where (x_1, y_1) and (x_2, y_2) are two pairs of relatively prime numbers; the slope of the second segments of the paths that have such bends is $\frac{y_1+y_2}{x_1+x_2}$, where (x_1+x_2, y_1+y_2) is a pair of relatively prime numbers, and $\frac{y_1}{x_1}$ and $\frac{y_2}{x_2}$ are the generating fractions of $\frac{y_1+y_2}{x_1+x_2}$.

The more sequences of three-segments paths that are consecutive in Π are considered, the more the slopes of the first, of the second, and of the third segments of the paths approach to the slope of segment $\overline{ab}$. Namely, if a sequence of paths is such that their bends lie on two lines with slopes $\frac{y_1}{x_1}$ and $\frac{y_2}{x_2}$ and their second segments have slope $\frac{y_1+y_2}{x_1+x_2}$, then the next sequence of paths whose second segments have the same slope is such that the bends of such paths lie on two lines with slopes $\frac{y_1}{x_1}$ and $\frac{y_1+y_2}{x_1+x_2}$ or with slopes $\frac{y_2}{x_2}$ and $\frac{y_1+y_2}{x_1+x_2}$, depending on whether $\frac{y_1+y_2}{x_1+x_2} < \frac{d_v}{d_h} < \frac{y_1}{x_1}$ or $\frac{y_2}{x_2} < \frac{d_v}{d_h} < \frac{y_1+y_2}{x_1+x_2}$, respectively, and the second segments of such paths have slope $\frac{2y_1+y_2}{2x_1+x_2}$ or $\frac{y_1+2y_2}{x_1+2x_2}$, respectively.

In order to analyze $\max\{d_h, d_v\}$ as a function of M_1, we subdivide Π into disjoint sub-sequences $\Pi_1, \Pi_2, \cdots, \Pi_f$ and we argue that Π_1 has at most $\max\{d_h, d_v\}$ paths and that Π_i has at most $\max\{d_h, d_v\}/2^{i-2}$ paths, for $2 \le i \le f$; such bounds lead to the conclusion that, as long as $M_1 \in \Omega(n)$, $\max\{d_h, d_v\} \in \Omega(n)$.

4.2 Details of the geometry of paths $\pi_1, \pi_2, \cdots, \pi_{M_1}$

Path π_1 is $\overline{acb}$. Let $p_1 \equiv c + (1, -1)$. Consider the following two sequences of grid points. See Fig. 9.a. Sequence $S_{0,1}$ is composed of points $p_k^{0,1} \equiv p_1 - (k-1)(0,1)$, for $1 \le k \le i_1$, where i_1 is the largest integer such that point $p_1 - (i_1 - 1)(0, 1)$ is contained inside T_1 (recall that T_1 is the triangle having a, b, and c as vertices). Sequence $S_{1,0}$ is composed of points $p_k^{1,0} \equiv p_1 + (k-1)(1, 0)$, for $1 \le k \le j_1$, where j_1 is the largest integer such that point $p_1 + (j_1 - 1)(1, 0)$ is contained inside T_1. Notice that the points of $S_{0,1}$ lie on a line with slope $\frac{1}{0} = \infty$ and the points of $S_{1,0}$ lie on a line with slope $\frac{0}{1} = 0$.

A sub-sequence Π_1 of Π, starting at π_2 and composed of paths consecutive in Π, "uses" the points in $S_{0,1}$ and in $S_{1,0}$, i.e., each path in Π_1 passes through a point in $S_{0,1}$ or a point in $S_{1,0}$. Actually, the first paths in Π_1 pass through a point in $S_{0,1}$ *and* a point in $S_{0,1}$. The paths that use the points in $S_{0,1}$ and in $S_{1,0}$ terminate when a path uses a point in $S_{0,1}$ and a point in $S_{1,0}$ that are collinear with one of a and b. Observe that when one of $S_{0,1}$ and $S_{1,0}$ is over, it is always the case that the last drawn path uses a point in $S_{0,1}$ and a point in $S_{1,0}$ that are collinear with one of a and b.

Then, path π_{k+1} is the polygonal path $\overline{ap_k^{0,1}p_k^{1,0}b}$, for $k = 1, 2, \cdots, k_1$, where k_1 is the smallest index greater than 1 such that a, $p_{k_1}^{0,1}$, and $p_{k_1}^{1,0}$ are collinear or $p_{k_1}^{0,1}$, $p_{k_1}^{1,0}$, and b are collinear. When one of $S_{0,1}$ and $S_{1,0}$ is "over", that is, there exist paths passing through all of its points, then a, $p_{k_1}^{0,1}$, and $p_{k_1}^{1,0}$ are collinear or $p_{k_1}^{0,1}$, $p_{k_1}^{1,0}$, and b are collinear. Notice that $p_1^{0,1} = p_1^{1,0} = p_1$, hence π_2 is composed of only

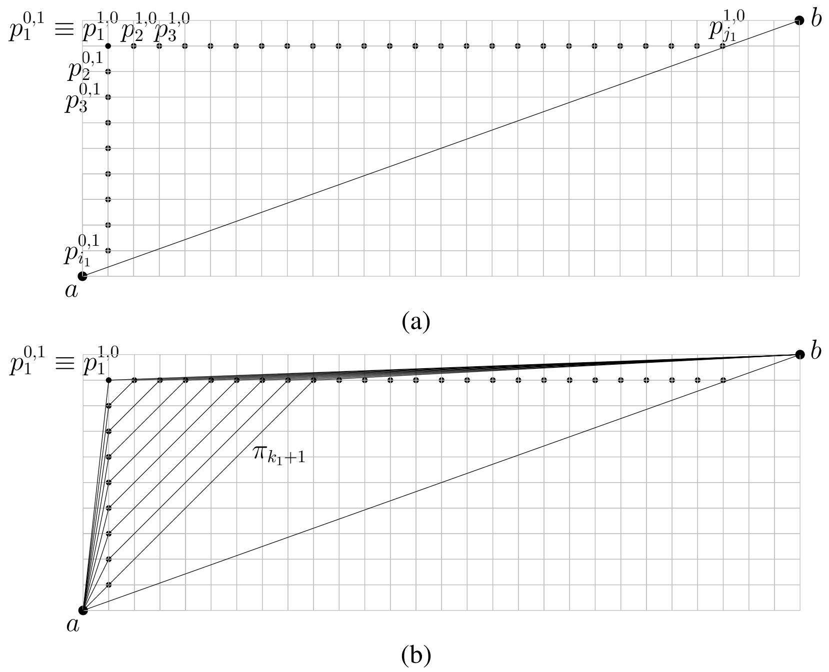

Fig. 9: (a) Sequences $S_{1,0}$ and $S_{0,1}$. (b) Paths π_{k+1}, with $1 \leq k \leq k_1$.

two segments. The second segment of path π_{k+1}, for $k = 2, 3, \cdots, k_1$, has slope $\frac{1}{1}$. Observe that $\frac{1}{0}$ and $\frac{0}{1}$ are the generating fractions of $\frac{1}{1}$. See Fig. 9.b.

Then, three cases have to be considered, namely the one in which a, $p_{k_1}^{0,1}$, $p_{k_1}^{1,0}$, and b are all collinear, the one in which a, $p_{k_1}^{0,1}$, and $p_{k_1}^{1,0}$ are collinear (and b is not), and the one in which $p_{k_1}^{0,1}$, $p_{k_1}^{1,0}$, and b are collinear (and a is not). In the first case, path π_{k_1+1} coincides with segment $\overline{ab}$, hence $\pi_{k_1+1} = \pi_{M_1}$. In the second case (the third case is analogous to the second one), sequence $S_{0,1}$ is replaced by a sequence $S_{1,1}$ defined as follows. See Fig. 10.a. The points of $S_{1,1}$ are the points $p_k^{1,1} \equiv p_{k_1+1}^{1,0} - (k-1)(1,1)$, for $1 \leq k \leq i_2$, where i_2 is the largest integer such that point $p_{k_1+1}^{1,0} - (i_2 - 1)(1,1)$ is contained inside T_1.

Some paths in Π_1 use the points in $S_{1,1}$ and the remaining points in $S_{1,0}$. The paths that use the points in $S_{1,1}$ and in $S_{1,0}$ terminate when a path uses a point in $S_{1,1}$ and a point in $S_{1,0}$ that are collinear with one of a and b. Observe that when one of $S_{1,1}$ and $S_{1,0}$ is over, it is always the case that the last drawn path uses a point in $S_{1,1}$ and a point in $S_{1,0}$ that are collinear with one of a and b.

Then, path π_{k_1+k+1} is the polygonal path $\overline{ap_k^{1,1}p_{k_1+k}^{1,0}b}$, for $k = 1, 2, \cdots, k_2$, where k_2 is the smallest index such that a, $p_{k_2}^{1,1}$, and $p_{k_1+k_2}^{1,0}$ are collinear or $p_{k_2}^{1,1}$, $p_{k_1+k_2}^{1,0}$, and b are collinear. When one of $S_{1,1}$ and $S_{1,0}$ is over, then a, $p_{k_2}^{1,1}$, and $p_{k_1+k_2}^{1,0}$ are collinear or $p_{k_2}^{1,1}$, $p_{k_1+k_2}^{1,0}$, and b are collinear. Notice that $p_1^{1,1} = p_{k_1+1}^{1,0}$, hence π_{k_1+2} is composed of only two segments. Also, observe that the bends of paths π_{k_1+k+1}, with $k = 1, 2, \cdots, k_2$, lie on two lines with slope $\frac{1}{1} = 1$ and $\frac{0}{1} = 0$, while the second segments of such paths lie on lines with slope $\frac{1+0}{1+1} = \frac{1}{2}$, where $\frac{0}{1}$ and $\frac{1}{1}$ are the generating fractions of $\frac{1}{2}$. See

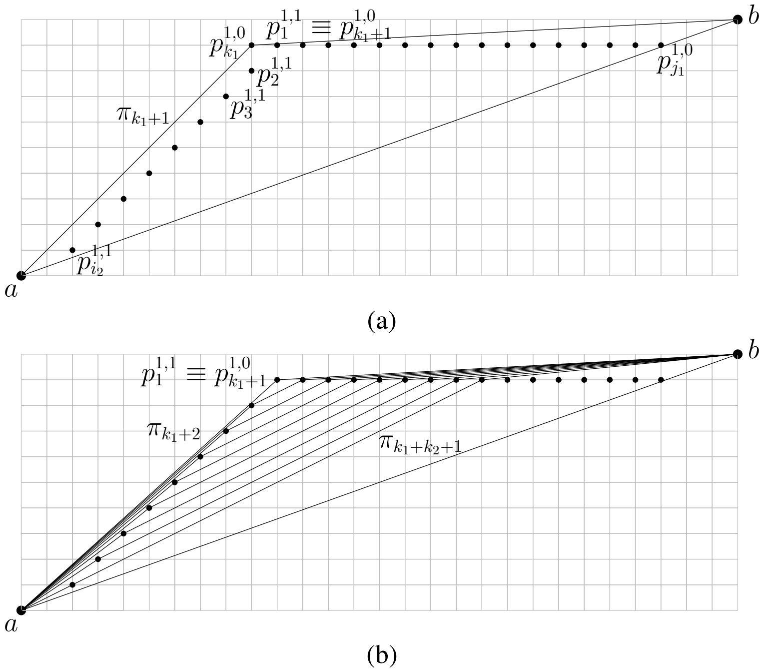

Fig. 10: (a) Sequence $S_{1,1}$. (b) Paths π_{k_1+k+1}, with $1 \leq k \leq k_2$.

Fig. 10.b.

Again, three cases have to be considered, namely the one in which a, $p^{1,1}_{k_2}$, $p^{1,0}_{k_1+k_2}$, and b are all collinear, the one in which a, $p^{1,1}_{k_2}$, and $p^{1,0}_{k_1+k_2}$ are collinear (and b is not), and the one in which $p^{1,1}_{k_2}$, $p^{1,0}_{k_1+k_2}$, and b are collinear (and a is not). In the first case, path $\pi_{k_1+k_2+1}$ coincides with segment $\overline{ab}$, hence $\pi_{k_1+k_2+1} = \pi_{M_1}$. Otherwise, a, $p^{1,1}_{k_2}$, and $p^{1,0}_{k_1+k_2}$ are collinear (and b is not), or $p^{1,1}_{k_2}$, $p^{1,0}_{k_1+k_2}$, and b are collinear (and a is not). Suppose that a, $p^{1,1}_{k_2}$, and $p^{1,0}_{k_1+k_2}$ are collinear (and b is not). Then $S_{1,1}$ is replaced by a sequence $S_{2,1}$ of points lying on a line with slope $\frac{1}{2}$. Namely, such points have coordinate $p^{2,1}_k = p^{1,0}_{k_1+k_2+1} - (k-1)(2,1)$, for $1 \leq k \leq i_3$, where i_3 is the largest integer such that point $p^{1,0}_{k_1+k_2+1} - (i_3-1)(2,1)$ is internal to T_1. See Fig. 11.a.

Some paths in Π_1 use the points in $S_{2,1}$ and the remaining points in $S_{1,0}$, that is, path $\pi_{k_1+k_2+k+1}$, with $1 \leq k \leq k_3$, passes through point $p^{2,1}_k$ and through point $p^{1,0}_{k_1+k_2+k}$, where k_3 is the smallest index such that a, $p^{2,1}_{k_3}$, and $p^{1,0}_{k_1+k_2+k_3}$ are collinear or $p^{2,1}_{k_3}$, $p^{1,0}_{k_1+k_2+k_3}$, and b are collinear. Observe that when one of $S_{2,1}$ and $S_{1,0}$ is over, it is always the case that the last drawn path uses a point in $S_{2,1}$ and a point in $S_{1,0}$ that are collinear with one of a and b. Observe also that the bends of paths $\pi_{k_1+k_2+k+1}$, with $k = 1, 2, \cdots, k_3$, lie on two lines with slope $\frac{1}{2}$ and $\frac{0}{1}$, while the second segments of such paths lie on lines with slope $\frac{1+0}{2+1} = \frac{1}{3}$, where $\frac{1}{2}$ and $\frac{0}{1}$ are the generating fractions of $\frac{1}{3}$. See Fig. 11.b.

The above argument iterates till a path is drawn that passes through a, through a point $p^{l,1}_{k_{l+1}}$ of the current sequence $S_{l,1}$, through a point $p^{1,0}_{k_1+k_2+\cdots+k_l+k_{l+1}}$ of $S_{1,0}$, and through b in such a way that

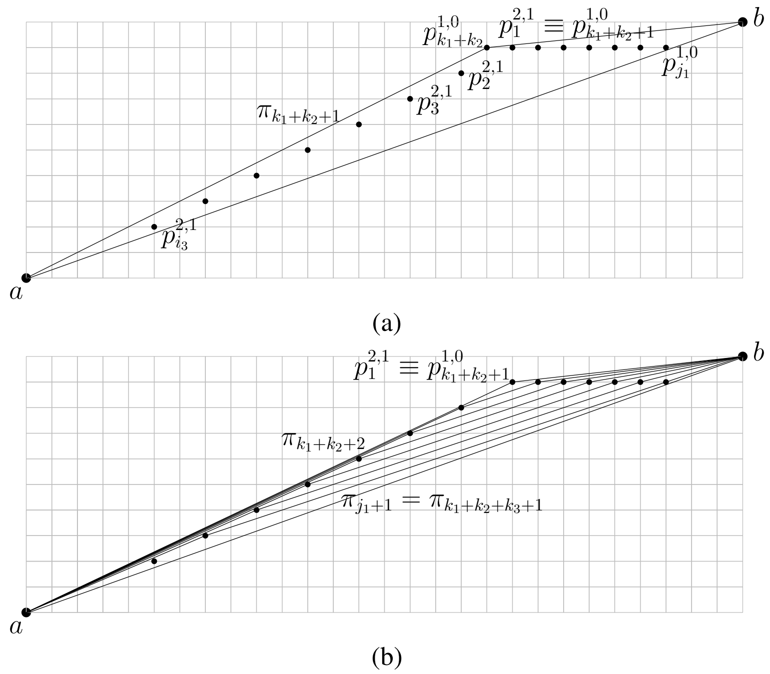

Fig. 11: (a) Sequence $S_{2,1}$. (b) Paths $\pi_{k_1+k_2+k+1}$, with $1 \leq k \leq k_3$.

$p^{l,1}_{k_{l+1}}$, $p^{1,0}_{k_1+k_2+\cdots+k_l+k_{l+1}}$, and b are collinear. If sequence $S_{1,0}$ is over, that is, all its points have been traversed by paths in Π, then the last drawn path passes through a, through a point $p^{l,1}_{k_{l+1}}$ of $S_{l,1}$, through the last point $p^{1,0}_{k_1+k_2+\cdots+k_l+k_{l+1}}$ of $S_{1,0}$, and through b, where $p^{l,1}_{k_{l+1}}$, $p^{1,0}_{k_1+k_2+\cdots+k_l+k_{l+1}}$, and b are collinear. Hence, all the paths that come after π_1 in Π_1 pass through distinct points of $S_{1,0}$, till a path is drawn that passes through a, through a point $p^{l,1}_{k_{l+1}}$ of $S_{l,1}$, through a point $p^{1,0}_{k_1+k_2+\cdots+k_l+k_{l+1}}$ of $S_{1,0}$, and through b in such a way that $p^{l,1}_{k_{l+1}}$, $p^{1,0}_{k_1+k_2+\cdots+k_l+k_{l+1}}$, and b are collinear. Thus, $\Pi_1 = (\pi_2, \pi_3, \cdots, \pi_{k_1+k_2+\cdots+k_l+k_{l+1}+1})$ is the desired sub-sequence Π_1 of Π. Further, there exists an index $l \geq 1$ such that: (1) all the points $p^{i,1}_j$ are traversed by paths in Π_1, for $0 \leq i \leq l-1$ and $1 \leq j \leq k_{i+1}$, and a, $p^{i,1}_{k_{i+1}}$, and $p^{1,0}_{k_1+k_2+\cdots+k_{i+1}}$ are collinear, for $0 \leq i \leq l-1$; (2) some points of $S_{l,1}$ are possibly traversed by paths in Π_1, and $p^{l,1}_{k_{l+1}}$, $p^{1,0}_{k_1+k_2+\cdots+k_l+k_{l+1}}$, and b are collinear. In the example in Figs. 9–11, we have $l = 2$; indeed, all the points $p^{0,1}_j$ are traversed by paths in Π_1, for $1 \leq j \leq k_1$; a, $p^{0,1}_{k_1}$, and $p^{1,0}_{k_1}$ are collinear; all the points $p^{1,1}_j$ are traversed by paths in Π_1, for $1 \leq j \leq k_2$; a, $p^{1,1}_{k_2}$, and $p^{1,0}_{k_1+k_2}$ are collinear; some points of $S_{2,1}$ are traversed by a path in Π_1; $p^{2,1}_{k_3}$, $p^{1,0}_{k_1+k_2+k_3}$, and b are collinear.

After drawing path $\pi_{k_1+k_2+\cdots+k_l+k_{l+1}+1}$ (that passes through a, $p^{l,1}_{k_{l+1}}$, $p^{1,0}_{k_1+k_2+\cdots+k_l+k_{l+1}}$, and b in such a way that $p^{l,1}_{k_{l+1}}$, $p^{1,0}_{k_1+k_2+\cdots+k_l+k_{l+1}}$, and b are collinear), either a is collinear with $p^{l,1}_{k_{l+1}}$, $p^{1,0}_{k_1+k_2+\cdots+k_l+k_{l+1}}$, and b, or not. In the former case, path $\pi_{k_1+k_2+\cdots+k_l+k_{l+1}+1}$ coincides with segment

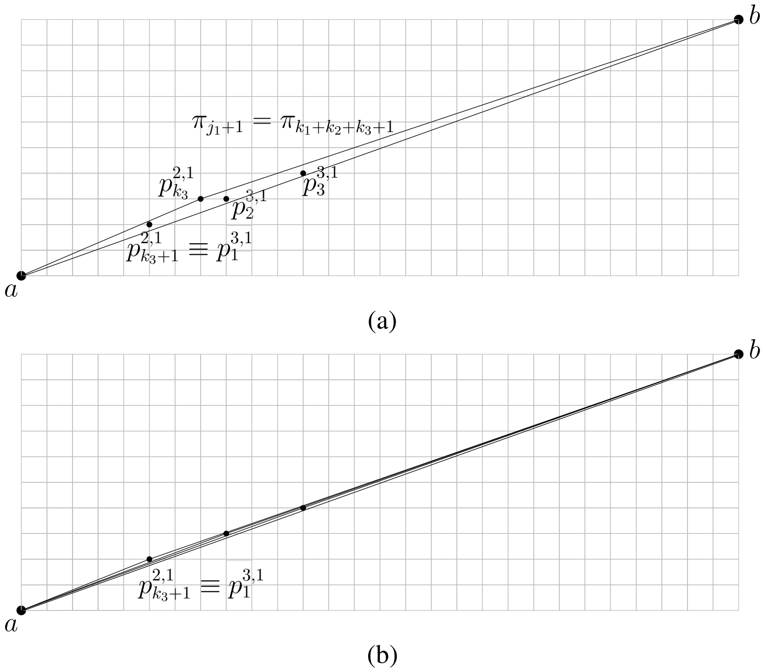

Fig. 12: (a) Sequence $S_{3,1}$. (b) The paths in Π_2.

$\overline{ab}$, hence $\pi_{k_1+k_2+\cdots+k_l+k_{l+1}+1} = \pi_{M_1}$. In the latter case, $S_{l,1}$ still contains points not traversed by any path in Π_1. Then, sequence $S_{1,0}$ is now replaced by a sequence $S_{l+1,1}$, whose points lie on a line with slope $\frac{0+1}{1+l} = \frac{1}{l+1}$ passing through the first point of $S_{l,1}$ that is not traversed by a path in Π_1, that is, point $p^{l,1}_{k_{l+1}+1}$. See Fig. 12.a, where there exists exactly one point of $S_{2,1}$ that is not traversed by a path in Π_1.

The whole argument is now repeated again. Namely, a sub-sequence Π_2 of Π uses the points in $S_{l,1}$ not traversed by paths in Π_1 and the points in $S_{l+1,1}$, i.e., each path in Π_2 passes through a point in $S_{l,1}$ or a point in $S_{l+1,1}$. Actually, the first paths in Π_2 pass through a point in $S_{l,1}$ *and* a point in $S_{l+1,1}$.

Again, Π_2 is generally found in several steps, where at each step two sequences S_{x_1,y_1} and S_{x_2,y_2} of grid points are considered, where one between S_{x_1,y_1} and S_{x_2,y_2} is $S_{l,1}$ or $S_{l+1,1}$ (at the first step $S_{x_1,y_1} = S_{l,1}$ and $S_{x_2,y_2} = S_{l+1,1}$ both hold). The points on S_{x_1,y_1} (on S_{x_2,y_2}) lie on a line with slope $\frac{y_1}{x_1}$ (resp. $\frac{y_2}{x_2}$), where (x_1, y_1) and (x_2, y_2) are pairs of relatively prime numbers. The second segments of the paths drawn when S_{x_1,y_1} and S_{x_2,y_2} are considered have slope $\frac{y_1+y_2}{x_1+x_2}$, where $(x_1 + x_2, y_1 + y_2)$ is a pair of relatively prime numbers, and $\frac{y_1}{x_1}$ and $\frac{y_2}{x_2}$ are the generating fractions of $\frac{y_1+y_2}{x_1+x_2}$. At each step, a path eventually passes through two points of S_{x_1,y_1} and S_{x_2,y_2} collinear with a or with b. Then, one between S_{x_1,y_1} and S_{x_2,y_2} (depending on whether the last path drawn in the step passes through two points of S_{x_1,y_1} and S_{x_2,y_2} collinear with a or with b) is replaced by a sequence of points lying on a line with slope $\frac{y_1+y_2}{x_1+x_2}$, hence starting a new step. After a certain number of steps, both $S_{l,1}$ and $S_{l+1,1}$ have been replaced by other sequences of points. When the last path that passes through a point of $S_{l,1}$ or of $S_{l+1,1}$ is drawn (that is, when the last path of Π_2 is drawn), it passes through a, through two points in the currently considered sequences S_{x_1,y_1} and S_{x_2,y_2}, and through b, so that either these four points are collinear, or a and two points in S_{x_1,y_1} and S_{x_2,y_2} are collinear and b is not, or b and two points in S_{x_1,y_1} and S_{x_2,y_2} are collinear and a is not. In the first case, the last drawn path is π_{M_1}, that hence coincides with segment $\overline{ab}$. In the second and the third case, either S_{x_1,y_1} or S_{x_2,y_2} is replaced by a sequence $S_{y_1+y_2,x_1+x_2}$ whose grid points lie on a line with slope $\frac{y_1+y_2}{x_1+x_2}$, depending on whether a and two points in the currently considered sequences S_{x_1,y_1} and S_{x_2,y_2} are collinear and b is not, or b and two points in the currently considered sequences S_{x_1,y_1} and S_{x_2,y_2} are collinear and a is not. The whole argument is then repeated again, searching for a sub-sequence Π_3 of Π such that Π_3 uses the points in S_{x_1,y_1} and the points in $S_{x_1+x_2,y_1+y_2}$. Clearly, there exists an index f such that $\Pi = \{\pi_1\} \cup \Pi_1 \cup \Pi_2 \cup \cdots \Pi_f \cup \{\overline{ab}\}$.

In the example considered in Figs. 9–12, the sequences considered at the first step, when determining Π_2, are $S_{2,1}$ and $S_{3,1}$. The slope of the second segments is $\frac{1+1}{2+3} = \frac{2}{5}$, although no path composed of three segments is drawn. Namely, the first path of Π_2 passes through the only point of $S_{2,1}$ not traversed by paths in Π_1. The sequences considered at the second step are $S_{2+3,1+1} = S_{5,2}$ and $S_{3,1}$. Sequence $S_{5,2}$ has only one point $p_1^{5,2} \equiv p_2^{3,1}$. The slope of the second segments is $\frac{2+1}{5+3} = \frac{3}{8}$, although no path composed of three segments is drawn. Namely, the second path of Π_2 passes through the only point of $S_{5,2}$. The sequences considered at the third step are $S_{5+3,2+1} = S_{8,3}$ and $S_{3,1}$. Sequence $S_{8,3}$ has only one point $p_1^{8,3} \equiv p_3^{3,1}$. The slope of the second segments is $\frac{3+1}{8+3} = \frac{4}{11}$, although no path composed of three segments is drawn. Namely, the third path of Π_2 passes through the only point of $S_{8,3}$ and the last point of $S_{3,1}$ (the two points actually coincide). Sequence Π_2 is over, as all the points in $S_{2,1}$ and in $S_{3,1}$ are traversed by paths in Π_1 or in Π_2. Further, since $S_{8,3}$ and $S_{3,1}$ end simultaneously, the only path of Π after Π_2 is segment $\overline{ab}$.

4.3 Proof of correctness of the geometry of paths $\pi_1, \pi_2, \cdots, \pi_{M_1}$

We now prove that paths $\pi_1, \pi_2, \cdots, \pi_{M_1}$ have the geometry described in Section 4.2.

In order to do that, we describe five possible sets of geometric features (in the following called *Conditions 1–5*) that can hold after drawing path π_i, we show that after drawing path π_2 Condition 4 is satisfied, and we prove that, if after drawing path π_i one of Conditions 1–5 is satisfied, then after drawing path π_{i+1} one of Conditions 1–5 is still satisfied (unless we are in a special case in which we can directly estimate the number of paths that come after π_i in Π).

After paths $\pi_1, \pi_2, \ldots, \pi_i$ have been drawn, a grid point is *occupied* if it has been traversed by a path π_j, with $j \leq i$, and is *free* otherwise. After a path π_i is drawn, we associate with the next path π_{i+1} to be drawn two sequences S_{x_1,y_1} and S_{x_2,y_2} of points, such that the following properties are satisfied:

- *Property S1:* x_1 and y_1 are relatively prime numbers; x_2 and y_2 are relatively prime numbers;
- *Property S2:* $\frac{y_1}{x_1} > \frac{d_v}{d_h} > \frac{y_2}{x_2}$;
- *Property S3:* $\frac{y_1}{x_1}$ and $\frac{y_2}{x_2}$ are the left and right generating fractions of $\frac{y_1+y_2}{x_1+x_2}$, respectively;
- *Property S4:* All the points in a (possibly empty) initial sub-sequence of S_{x_1,y_1} and all the points in a (possibly empty) initial sub-sequence of S_{x_2,y_2} are occupied; all the other points of S_{x_1,y_1} and S_{x_2,y_2} are free and lie inside polygon $\pi_i \cup \overline{ab}$;
- *Property S5:* The half-line $\vec{l}(x_1, y_1)$ starting at the first point $p_1^{x_1,y_1}$ of S_{x_1,y_1}, having slope $\frac{y_1}{x_1}$, and directed towards decreasing y-coordinates intersects the interior of segment $\overline{ab}$ in a point

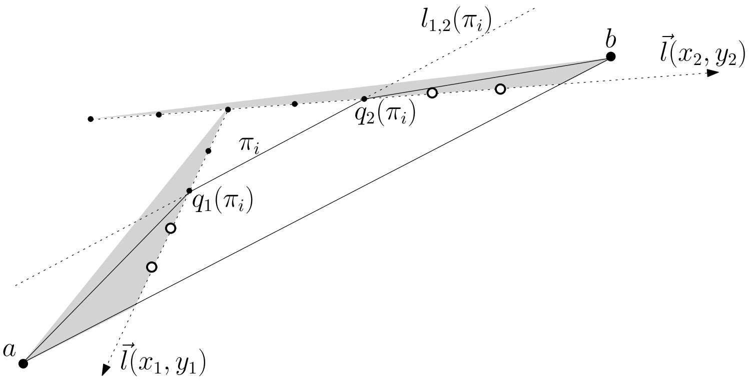

Fig. 13: After drawing π_i, Condition 1 is satisfied. In all the figures of Section 4.3, black dots represent occupied points of S_{x_1,y_1} and S_{x_2,y_2}, white dots represent free points of S_{x_1,y_1} and S_{x_2,y_2}, and the shaded triangles are $T(S_{x_1,y_1}, a)$ and $T(S_{x_2,y_2}, b)$. The slopes of the lines in the figures do not correspond to slopes of grid lines in the plane. This allows us to improve the readability of the drawings.

$q(S_{x_1,y_1}, \overline{ab})$; the half-line $\vec{l}(x_2, y_2)$ starting at the first point $p_1^{x_2,y_2}$ of S_{x_2,y_2}, having slope $\frac{y_2}{x_2}$, and directed towards increasing x-coordinates intersects the interior of segment $\overline{ab}$ in a point $q(S_{x_2,y_2}, \overline{ab})$;

- *Property S6:* There exists no grid point internal to the triangle $T(S_{x_1,y_1}, a)$ having $p_1^{x_1,y_1}$, $q(S_{x_1,y_1}, \overline{ab})$, and a as vertices; there exists no grid point internal to the triangle $T(S_{x_2,y_2}, b)$ having $p_1^{x_2,y_2}$, $q(S_{x_2,y_2}, \overline{ab})$, and b as vertices.

Conditions 1–5 are as follows:

Condition 1. Path π_i is $\overline{aq_1(\pi_i)q_2(\pi_i)b}$; $q_1(\pi_i)$ and $q_2(\pi_i)$ are the last occupied points of S_{x_1,y_1} and S_{x_2,y_2}, respectively; segment $\overline{q_1(\pi_i)q_2(\pi_i)}$ has slope $\frac{y_1+y_2}{x_1+x_2}$; the line $l_{1,2}(\pi_i)$ through $q_1(\pi_i)$ and $q_2(\pi_i)$ has a and b to its right; finally, both S_{x_1,y_1} and S_{x_2,y_2} have free points (see Fig. 13).

Condition 2. Path π_i is $\overline{aq_1(\pi_i)q_2(\pi_i)b}$; $q_1(\pi_i)$ and $q_2(\pi_i)$ are the last occupied points of S_{x_1,y_1} and S_{x_2,y_2}, respectively; segment $\overline{q_1(\pi_i)q_2(\pi_i)}$ has slope $\frac{y_1+y_2}{x_1+x_2}$; the line $l_{1,2}(\pi_i)$ through $q_1(\pi_i)$ and $q_2(\pi_i)$ has a and b to its right; finally, neither S_{x_1,y_1} nor S_{x_2,y_2} has free points (see Fig. 14).

Condition 3. Path π_i is $\overline{aq_1(\pi_i)b}$; further, either (i) $q_1(\pi_i)$ is the last occupied point of S_{x_1,y_1} and all the points of S_{x_2,y_2} are free; the first free point of S_{x_1,y_1} coincides with the first point of S_{x_2,y_2}; segment $\overline{q_1(\pi_i)b}$ has slope $\frac{y_2}{x_2}$; $\frac{y_1}{x_1}$ is a generating fraction of $\frac{y_2}{x_2}$; the line $l_{1,2}(\pi_i)$ through $q_1(\pi_i)$ with slope $\frac{y_1+y_2}{x_1+x_2}$ has a and b to its right (see Fig. 15); or (ii) $q_1(\pi_i)$ is the last occupied point of S_{x_2,y_2} and all the points of S_{x_1,y_1} are free; the first free point of S_{x_2,y_2} coincides with the first point of S_{x_1,y_1}; segment $\overline{aq_1(\pi_i)}$ has slope $\frac{y_1}{x_1}$; $\frac{y_2}{x_2}$ is a generating fraction of $\frac{y_1}{x_1}$; the line $l_{1,2}(\pi_i)$ through $q_1(\pi_i)$ with slope $\frac{y_1+y_2}{x_1+x_2}$ has a and b to its right.

Condition 4. Path π_i is $\overline{aq_1(\pi_i)b}$; $q_1(\pi_i)$ is the last occupied point of S_{x_1,y_1} and the last occupied point of S_{x_2,y_2}; the line $l_{1,2}(\pi_i)$ through $q_1(\pi_i)$ with slope $\frac{y_1+y_2}{x_1+x_2}$ has a and b to its right; both S_{x_1,y_1} and S_{x_2,y_2} have free points (see Fig. 16).

Condition 5. Path π_i is $\overline{aq_1(\pi_i)b}$; $q_1(\pi_i)$ is the last occupied point of S_{x_1,y_1} and the last occupied point of S_{x_2,y_2}; the line $l_{1,2}(\pi_i)$ through $q_1(\pi_i)$ with slope $\frac{y_1+y_2}{x_1+x_2}$ has a and b to its right; neither S_{x_1,y_1} nor S_{x_2,y_2} has free points (see Fig. 17).

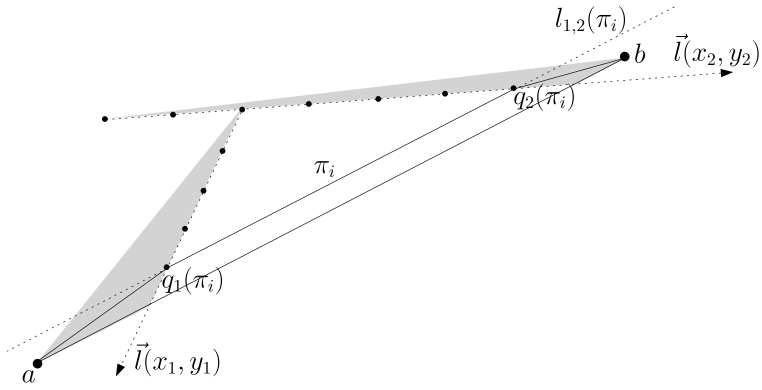

Fig. 14: After drawing π_i, Condition 2 is satisfied.

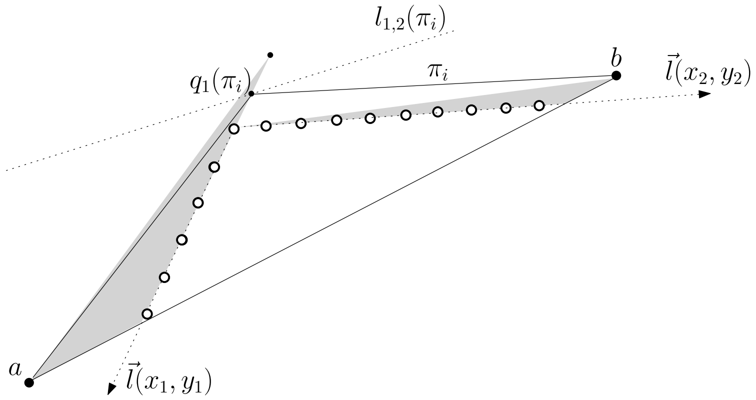

Fig. 15: After drawing π_i, Condition 3 is satisfied.

We are now ready to prove the following:

Lemma 7 *Suppose that after drawing path π_i one of Conditions 1–5 is satisfied. Then, after drawing path π_{i+1} either one of Conditions 1–5 is satisfied or all the free points inside polygon $\pi_{i+1} \cup \overline{ab}$ lie on a specific grid line.*

First we prove that after drawing path π_2 Condition 4 is satisfied. Clearly, such a path is $\overline{ap_1^{0,1}b}$, where $p_1^{0,1} = p_1^{1,0} \equiv c + (1, -1)$. Let $S_{0,1}$ and $S_{1,0}$ be defined as in Section 4.2. Then, $S_{x_1,y_1} = S_{0,1}$ and $S_{x_2,y_2} = S_{1,0}$ are associated with path π_3, clearly satisfying Properties S1–S6. Further, $p_1^{0,1}$ is the last occupied point of $S_{0,1}$ and $S_{1,0}$; moreover, as $|d_h|, |d_v| > 3$, the line through $p_1^{0,1}$ with slope $\frac{1}{1}$ has a and b to its right, and both $S_{0,1}$ and $S_{1,0}$ have free points. It follows that, after drawing π_2, Condition 4 is satisfied, with $S_{0,1}$ and $S_{1,0}$ associated with path π_3.

Next, suppose that after drawing π_i one of Conditions 1–5 is satisfied, where sequences S_{x_1,y_1} and S_{x_2,y_2} are associated with π_{i+1}; then, we argue about the drawing of path π_{i+1} and about the sequences to be associated with π_{i+2}.

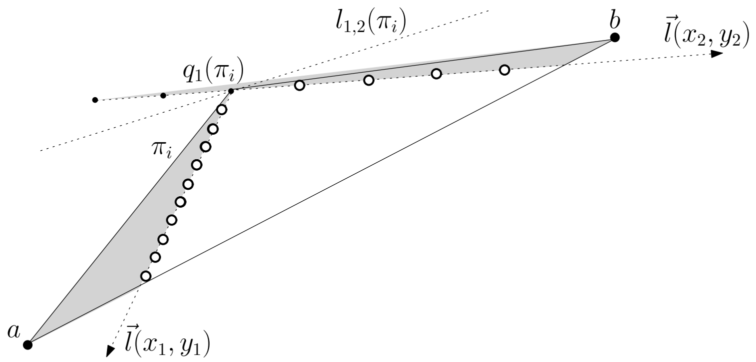

Fig. 16: After drawing π_i, Condition 4 is satisfied.

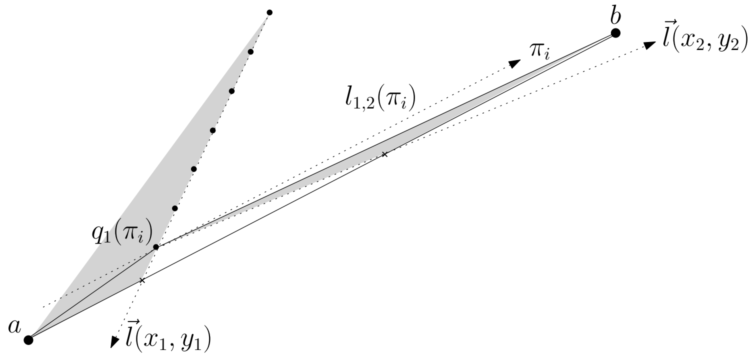

Fig. 17: After drawing π_i, Condition 5 is satisfied.

Suppose that after drawing π_i Condition 1 is satisfied. Consider the first free point of S_{x_1,y_1}, that is, point $q_1(\pi_{i+1}) \equiv q_1(\pi_i) - (x_1, y_1)$. Also consider the first free point of S_{x_2,y_2}, that is, point $q_2(\pi_{i+1}) \equiv q_2(\pi_i) + (x_2, y_2)$. Such points exist by the hypotheses of Condition 1.

We will prove that π_{i+1} passes through $q_1(\pi_{i+1})$ and $q_2(\pi_{i+1})$, that is, either π_{i+1} is $\overline{aq_1(\pi_{i+1})q_2(\pi_{i+1})b}$, or π_{i+1} is $\overline{aq_1(\pi_{i+1})b}$ with $q_2(\pi_{i+1})$ being a point of $\overline{q_1(\pi_{i+1})b}$, or π_{i+1} is $\overline{aq_2(\pi_{i+1})b}$ with $q_1(\pi_{i+1})$ being a point of $\overline{aq_2(\pi_{i+1})}$. Denote by $l_{1,2}(\pi_i)$ and $l_{1,2}(\pi_{i+1})$ the lines through $q_1(\pi_i)$ and $q_2(\pi_i)$ and through $q_1(\pi_{i+1})$ and $q_2(\pi_{i+1})$, respectively.

Claim 1 *Path π_{i+1} passes through $q_1(\pi_{i+1})$ and $q_2(\pi_{i+1})$.*

Proof: Refer to Fig. 18. Since $l_{1,2}(\pi_i)$ has slope $\frac{y_1+y_2}{x_1+x_2}$, by the hypotheses of Condition 1, the slope of $l_{1,2}(\pi_{i+1})$ is:

$$\frac{y(q_2(\pi_i)) + y_2 - (y(q_1(\pi_i)) - y_1)}{x(q_2(\pi_i)) + x_2 - (x(q_1(\pi_i)) - x_1)} = \frac{y_1 + y_2 + (y(q_2(\pi_i)) - y(q_1(\pi_i)))}{x_1 + x_2 + (x(q_2(\pi_i)) - x(q_1(\pi_i)))} =$$
$$\frac{y_1 + y_2 + m(y_1 + y_2)}{x_1 + x_2 + m(x_1 + x_2)} = \frac{(m+1)(y_1 + y_2)}{(m+1)(x_1 + x_2)} = \frac{y_1 + y_2}{x_1 + x_2},$$

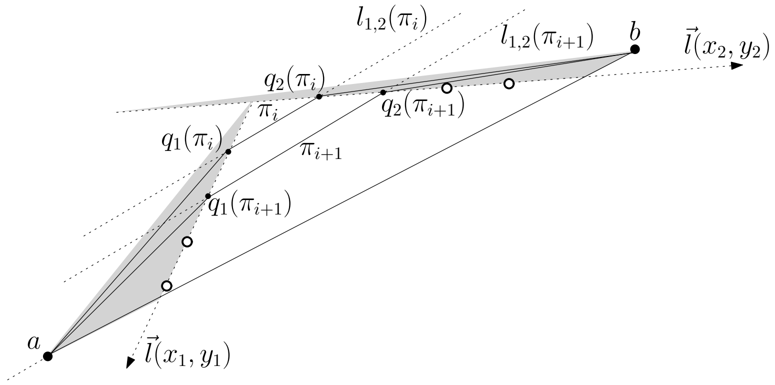

Fig. 18: Drawing of π_{i+1} when Condition 1 holds.

By Property S3, $\frac{y_1}{x_1}$ and $\frac{y_2}{x_2}$ are the generating fractions of $\frac{y_1+y_2}{x_1+x_2}$, hence, by Lemma 6, $l_{1,2}(\pi_i)$ and $l_{1,2}(\pi_{i+1})$ are consecutive grid lines. Then no grid point is internal to polygon $(q_1(\pi_i), q_2(\pi_i), q_2(\pi_{i+1}), q_1(\pi_{i+1}))$. As triangles $(a, q_1(\pi_i), q_1(\pi_{i+1}))$ and $(b, q_2(\pi_i), q_2(\pi_{i+1}))$ are enclosed in $T(S_{x_1,y_1}, a)$ and in $T(S_{x_2,y_2}, b)$, respectively, polygon $\pi_i \cup (a, q_1(\pi_{i+1}), q_2(\pi_{i+1}), b)$ contains no grid point. Hence, as long as $\overline{ab} \cup (a, q_1(\pi_{i+1}), q_2(\pi_{i+1}), b)$ is a convex polygon, we have $\pi_{i+1} = (a, q_1(\pi_{i+1}), q_2(\pi_{i+1}), b)$.

Consider the possible placements of a and b with respect to $l_{1,2}(\pi_{i+1})$. Neither a nor b is to the left of $l_{1,2}(\pi_{i+1})$, as such vertices are to the right of $l_{1,2}(\pi_i)$, by the hypotheses of Condition 1, and hence, if they were to the left of $l_{1,2}(\pi_{i+1})$, they would be in the open strip delimited by $l_{1,2}(\pi_i)$ and $l_{1,2}(\pi_{i+1})$, which are consecutive grid lines, thus contradicting Lemma 6.

Hence, either a and b are both on $l_{1,2}(\pi_{i+1})$, or one of a and b is on $l_{1,2}(\pi_{i+1})$ and the other one is to the right of such a line, or both a and b are to the right of $l_{1,2}(\pi_{i+1})$. It follows that $\overline{ab} \cup (a, q_1(\pi_{i+1}), q_2(\pi_{i+1}), b)$ is a convex polygon and hence that $\pi_{i+1} = (a, q_1(\pi_{i+1}), q_2(\pi_{i+1}), b)$. □

Now we discuss which condition is satisfied after drawing π_{i+1}, discussing the case in which a and b are both on $l_{1,2}(\pi_{i+1})$, the case in which one of a and b is on $l_{1,2}(\pi_{i+1})$ and the other one is to the right of such a line, and the case in which both a and b are to the right of $l_{1,2}(\pi_{i+1})$.

- First, we prove the following:

  Claim 2 *Vertices a and b can not be both on $l_{1,2}(\pi_{i+1})$.*

  Proof: Suppose, for a contradiction, that a and b are both on $l_{1,2}(\pi_{i+1})$. Then, we have that $q_1(\pi_{i+1})$ and $q_2(\pi_{i+1})$ are both on segment $\overline{ab}$. However, this implies that $q_1(\pi_{i+1})$ and $q_2(\pi_{i+1})$ are not inside T_1, thus violating Property S4, a contradiction. □

- Second, consider the case in which a and b are both to the right of $l_{1,2}(\pi_{i+1})$.

  - Suppose that both S_{x_1,y_1} and S_{x_2,y_2} have free points as in Fig. 18. Then we have the following:

    Claim 3 *After drawing π_{i+1} Condition 1 is satisfied with S_{x_1,y_1} and S_{x_2,y_2} associated with path π_{i+2}.*

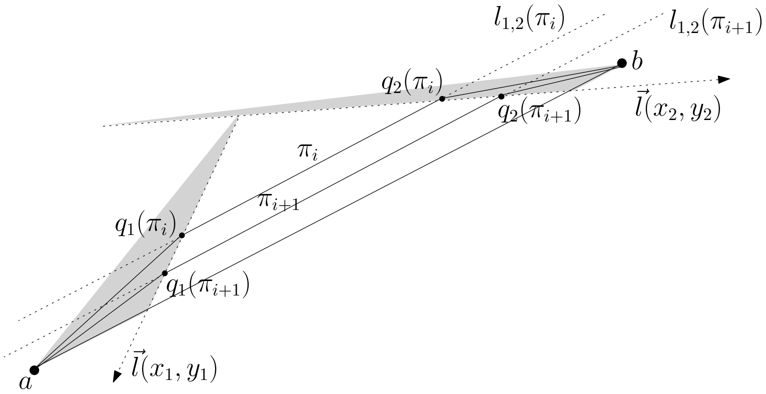

Fig. 19: Vertices a and b are both to the right of $l_{1,2}(\pi_{i+1})$ and neither S_{x_1,y_1} nor S_{x_2,y_2} has free points.

Proof: Sequences S_{x_1,y_1} and S_{x_2,y_2} satisfy Properties S1, S2, S3, S5, S6 because they satisfy them before drawing π_{i+1}; further, they satisfy Property S4, because they satisfy it before drawing π_{i+1} and because π_{i+1} traverses the first free points of S_{x_1,y_1} and S_{x_2,y_2}; as shown in the proof of Claim 1, path π_{i+1} has three segments, the second one having slope $\frac{y_1+y_2}{x_1+x_2}$ and the bends of π_{i+1} are on the last occupied points of S_{x_1,y_1} and S_{x_2,y_2}; by hypothesis, $l_{1,2}(\pi_{i+1})$ has a and b to its right and both S_{x_1,y_1} and S_{x_2,y_2} have free points. □

– Suppose that neither S_{x_1,y_1} nor S_{x_2,y_2} has free points, as in Fig. 19. Then, we have the following:

Claim 4 *After drawing π_{i+1} Condition 2 is satisfied with S_{x_1,y_1} and S_{x_2,y_2} associated with path π_{i+2}.*

Proof: Sequences S_{x_1,y_1} and S_{x_2,y_2} satisfy Properties S1, S2, S3, S5, S6 because they satisfy them before drawing π_{i+1}; further, they satisfy Property S4, because they satisfy it before drawing π_{i+1} and because π_{i+1} traverses the first free points of S_{x_1,y_1} and S_{x_2,y_2}; as shown in the proof of Claim 1, path π_{i+1} has three segments, the second one having slope $\frac{y_1+y_2}{x_1+x_2}$ and the bends of π_{i+1} are on the last occupied points of S_{x_1,y_1} and S_{x_2,y_2}; by hypothesis, $l_{1,2}(\pi_{i+1})$ has a and b to its right and neither S_{x_1,y_1} nor S_{x_2,y_2} has free points. □

– Next, we prove the following:

Claim 5 *Suppose that S_{x_1,y_1} has free points. Then S_{x_2,y_2} also has free points.*

Proof: We prove that if S_{x_1,y_1} has free points and S_{x_2,y_2} has not, then $q_2(\pi_{i+1})$ lies on segment $\overline{q_1(\pi_{i+1})b}$ and hence b is on $l_{1,2}(\pi_{i+1})$, a contradiction to the hypotheses. Refer to Fig. 20. Denote by q'_1 and q'_2 points $q'_1 \equiv q_1(\pi_{i+1}) - (x_1, y_1)$ and $q'_2 \equiv q_2(\pi_{i+1}) + (x_2, y_2)$. Consider the line $l'_{1,2}$ through q'_1 and q'_2. Such a line has slope $\frac{y_1+y_2}{x_1+x_2}$. This can be proved analogously as it was shown that line $l_{1,2}(\pi_{i+1})$ has slope $\frac{y_1+y_2}{x_1+x_2}$ in the proof of Claim 1. Since $\frac{y_1}{x_1}$ and $\frac{y_2}{x_2}$ are the generating fractions of $\frac{y_1+y_2}{x_1+x_2}$, $l_{1,2}(\pi_{i+1})$ and $l'_{1,2}$ are consecutive grid lines, hence they do not have any grid point between them, by Lemma 6. However, $l'_{1,2}$

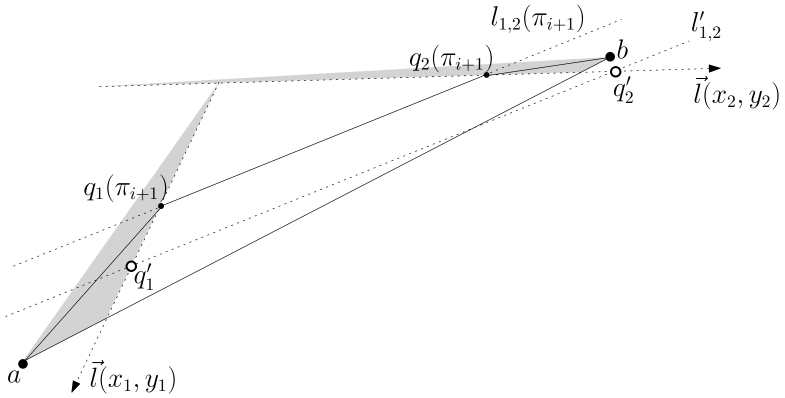

Fig. 20: The case in which vertices a and b are both to the right of $l_{1,2}(\pi_{i+1})$, S_{x_1,y_1} has free points, and S_{x_2,y_2} has no free point does not occur.

has b to the left as $l'_{1,2}$ intersects the interior of segment $\overline{ab}$, by Property S5. Thus b can not be to the right of $l_{1,2}(\pi_{i+1})$, a contradiction. $\square$

The proof of following claim is analogous to the proof of Claim 5.

Claim 6 *Suppose that S_{x_2,y_2} has free points. Then S_{x_1,y_1} also has free points.*

It follows that if a and b are both to the right of $l_{1,2}(\pi_{i+1})$, then after drawing π_{i+1} either Condition 1 or Condition 2 is satisfied with S_{x_1,y_1} and S_{x_2,y_2} associated with path π_{i+2}.

- Third, consider the case in which a is to the right of $l_{1,2}(\pi_{i+1})$ and b is on $l_{1,2}(\pi_{i+1})$. Then, observe that π_{i+1} is $\overline{aq_1(\pi_{i+1})b}$ with $q_2(\pi_{i+1})$ being a point of $\overline{q_1(\pi_{i+1})b}$.

Suppose that S_{x_1,y_1} has no free point left. We have the following:

Claim 7 $\pi_{i+2} = \pi_{M_1} = \overline{ab}$.

Proof: Refer to Fig. 21. Consider points $q'_1 \equiv q_1(\pi_{i+1}) - (x_1, y_1)$ and $q'_2 \equiv q_2(\pi_{i+1}) + (x_2, y_2)$. Consider the line $l'_{1,2}$ through q'_1 and q'_2. Such a line has slope $\frac{y_1+y_2}{x_1+x_2}$. This can be proved as it was shown that line $l_{1,2}(\pi_{i+1})$ has slope $\frac{y_1+y_2}{x_1+x_2}$ in the proof of Claim 1. Then, $l'_{1,2}$ and $l_{1,2}(\pi_{i+1})$ are consecutive grid lines and the open strip delimited by them contains the interior of triangle $(b, q_1(\pi_{i+1}), q(S_{x_1,y_1}, \overline{ab}))$, that hence has no grid point in its interior, by Lemma 6. Since $T(S_{x_1,y_1}, a)$ has no grid point in its interior, by Property S6, then polygon $\pi_{i+1} \cup \overline{ab}$ has no grid point in its interior, and hence $\pi_{i+2} = \pi_{M_1} = \overline{ab}$. $\square$

Next, suppose that S_{x_1,y_1} has free points. Refer to Fig. 22. Consider the first free point on S_{x_1,y_1}, that is, point $q'_1 \equiv q_1(\pi_{i+1}) - (x_1, y_1)$. Consider the sequence of grid points $S_{x_1+x_2,y_1+y_2}$ whose points have coordinates $q'_1 + m(x_1+x_2, y_1+y_2)$, where $0 \leq m \leq i^*$, where i^* is the largest integer such that $q'_1 + i^*(x_1 + x_2, y_1 + y_2)$ is inside T_1.

We have the following:

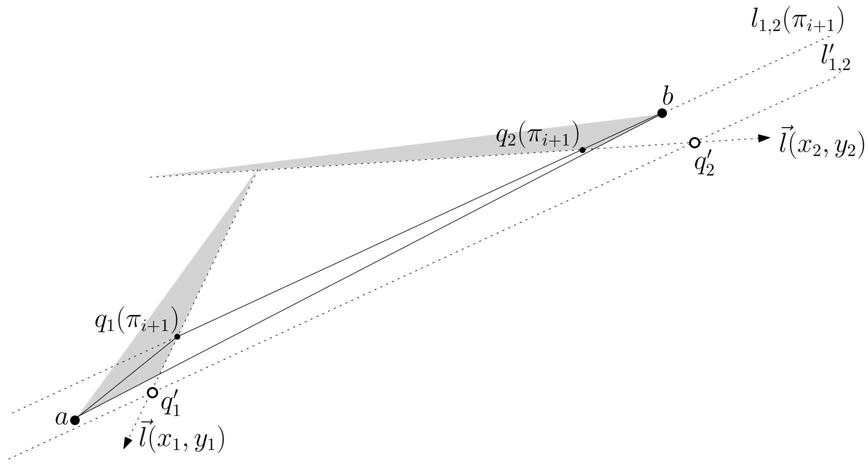

Fig. 21: If a is to the right of $l_{1,2}(\pi_{i+1})$, b is on $l_{1,2}(\pi_{i+1})$, and S_{x_1,y_1} has no free point, then $\pi_{i+2} = \pi_{M_1} = \overline{ab}$.

Claim 8 *Sequences S_{x_1,y_1} and $S_{x_1+x_2,y_1+y_2}$ satisfy Properties S1–S6.*

Proof: Sequences S_{x_1,y_1} and $S_{x_1+x_2,y_1+y_2}$ satisfy Property S1, as sequences S_{x_1,y_1} and S_{x_2,y_2} satisfy Properties S1 and S3. Since line $l_{1,2}(\pi_{i+1})$ has slope $\frac{y_1+y_2}{x_1+x_2}$, as shown in the proof of Claim 1, and since a is to the right of this line, it follows that $\frac{y_1+y_2}{x_1+x_2} < \frac{d_v}{d_h}$, and hence S_{x_1,y_1} and $S_{x_1+x_2,y_1+y_2}$ satisfy Property S2. S_{x_1,y_1} and $S_{x_1+x_2,y_1+y_2}$ satisfy Property S3; namely, since $\frac{y_1}{x_1}$ and $\frac{y_2}{x_2}$ are the generating fractions of $\frac{y_1+y_2}{x_1+x_2}$, $\frac{y_1}{x_1}$ and $\frac{y_1+y_2}{x_1+x_2}$ are the generating fractions of $\frac{2y_1+y_2}{2x_1+x_2}$. Since S_{x_1,y_1} and S_{x_2,y_2} satisfy Property S4 after drawing π_i, since π_{i+1} passes through the first free point of S_{x_1,y_1}, and since all the points of $S_{x_1+x_2,y_1+y_2}$ are free, it follows that S_{x_1,y_1} and $S_{x_1+x_2,y_1+y_2}$ satisfy Property S4. As S_{x_1,y_1} and S_{x_2,y_2} satisfy Property S5 after drawing π_i, then $\vec{l}(x_1, y_1)$ intersects the interior of segment $\overline{ab}$; since the line with slope $\frac{y_1+y_2}{x_1+x_2}$ through $q_1(\pi_{i+1})$ intersects $\overline{ab}$ in b and since q'_1 is internal to $\pi_{i+1} \cup \overline{ab}$, then $\vec{l}(x_1 + x_2, y_1 + y_2)$ intersects the interior of segment $\overline{ab}$, thus S_{x_1,y_1} and $S_{x_1+x_2,y_1+y_2}$ satisfy Property S5. Sequences S_{x_1,y_1} and

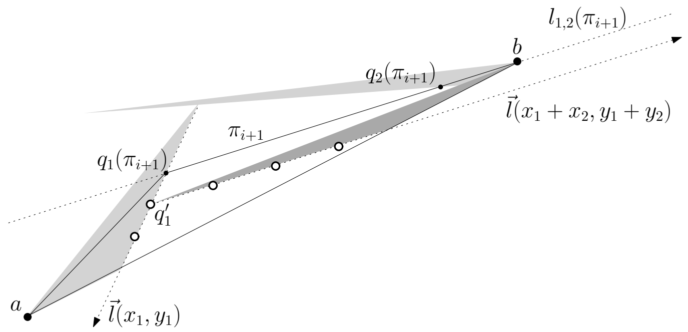

Fig. 22: Illustration for the case in which a is to the right of $l_{1,2}(\pi_{i+1})$, b is on $l_{1,2}(\pi_{i+1})$, and S_{x_1,y_1} has free points.

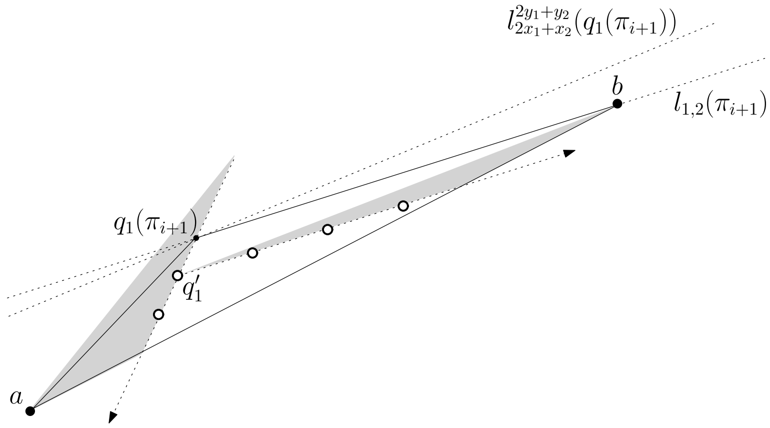

Fig. 23: If $\frac{2y_1+y_2}{2x_1+x_2} \leq \frac{d_v}{d_h}$, then $l^{2y_1+y_2}_{2x_1+x_2}(q_1(\pi_{i+1}))$ has a to its right.

S_{x_2,y_2} satisfy Property S6 after drawing π_i, hence $T(S_{x_1,y_1}, a)$ contains no grid point; further, $T(S_{x_1+x_2,y_1+y_2}, b)$ is entirely contained in the strip delimited by $l_{1,2}(\pi_{i+1})$ and by the line through q'_1 with slope $\frac{y_1+y_2}{x_1+x_2}$, hence, by Lemma 6, it contains no grid point. Thus S_{x_1,y_1} and $S_{x_1+x_2,y_1+y_2}$ satisfy Property S6. □

As already observed, π_{i+1} is $\overline{aq_1(\pi_{i+1})b}$; further, $q_1(\pi_{i+1})$ is the last occupied point of S_{x_1,y_1} and all the points of $S_{x_1+x_2,y_1+y_2}$ are free, the first free point of S_{x_1,y_1} coincides with the first point of $S_{x_1+x_2,y_1+y_2}$, segment $\overline{q_1(\pi_{i+1})b}$ has slope $\frac{y_1+y_2}{x_1+x_2}$, $\frac{y_1}{x_1}$ is a generating fraction of $\frac{y_1+y_2}{x_1+x_2}$, and both S_{x_1,y_1} and $S_{x_1+x_2,y_1+y_2}$ have free points. Hence, if the line $l^{2y_1+y_2}_{2x_1+x_2}(q_1(\pi_{i+1}))$ through $q_1(\pi_{i+1})$ with slope $\frac{2y_1+y_2}{2x_1+x_2}$ has a and b to its right, then, after drawing π_{i+1}, Condition 3 is satisfied, where sequences S_{x_1,y_1} and $S_{x_1+x_2,y_1+y_2}$ are associated with path π_{i+2}.

We prove that if $l^{2y_1+y_2}_{2x_1+x_2}(q_1(\pi_{i+1}))$ has not both a and b to its right, then none of Conditions 1–5 is satisfied and in such a special case we can directly estimate the number of paths that come after π_{i+1} in Π.

Claim 9 *If $l^{2y_1+y_2}_{2x_1+x_2}(q_1(\pi_{i+1}))$ has not both a and b to its right, then all the free points inside $\pi_{i+1} \cup \overline{ab}$ lie on a specific grid line.*

Proof: Since the line through $q_1(\pi_{i+1})$ with slope $\frac{y_1+y_2}{x_1+x_2}$, that is $l_{1,2}(\pi_{i+1})$, passes through b and since $\frac{2y_1+y_2}{2x_1+x_2} > \frac{y_1+y_2}{x_1+x_2}$ as $\frac{y_1}{x_1} > \frac{y_2}{x_2}$, it follows that $l^{2y_1+y_2}_{2x_1+x_2}(q_1(\pi_{i+1}))$ has b to its right.

Suppose, for a contradiction, that $\frac{2y_1+y_2}{2x_1+x_2} \leq \frac{d_v}{d_h}$, as in Fig. 23. Then $l^{2y_1+y_2}_{2x_1+x_2}(q_1(\pi_{i+1}))$ has a to its right, since it has the line through b with slope $\frac{2y_1+y_2}{2x_1+x_2}$ to its right, and since this line has a to its right or on it, thus contradicting the assumption that $l^{2y_1+y_2}_{2x_1+x_2}(q_1(\pi_{i+1}))$ has not both a and b to its right.

Next, suppose that $\frac{2y_1+y_2}{2x_1+x_2} > \frac{d_v}{d_h}$ and suppose that $l^{2y_1+y_2}_{2x_1+x_2}(q_1(\pi_{i+1}))$ does not have a to its right, that is, $l^{2y_1+y_2}_{2x_1+x_2}(q_1(\pi_{i+1}))$ intersects segment $\overline{ab}$.

Consider the grid point $q \equiv q_1(\pi_{i+1}) - (2x_1 + x_2, 2y_1 + y_2) \in l^{2y_1+y_2}_{2x_1+x_2}(q_1(\pi_{i+1}))$. Such a point is to the left of the line through $q_1(\pi_{i+1})$ with slope $\frac{y_1}{x_1}$, that is the line through the points of S_{x_1,y_1}, since $\frac{y_1}{x_1} > \frac{2y_1+y_2}{2x_1+x_2}$ (the last inequality holds because $\frac{y_1}{x_1} > \frac{y_2}{x_2}$). Further, q is to the left of l_{ab}, since q'_1 is to the left of l_{ab}, since $q \equiv q'_1 - (x_1 + x_2, y_1 + y_2)$, and since $\frac{y_1+y_2}{x_1+x_2} < \frac{d_v}{d_h}$. In order for $l^{2y_1+y_2}_{2x_1+x_2}(q_1(\pi_{i+1}))$ to intersect $\overline{ab}$, q has to be either on the line through $q_1(\pi_{i+1})$ and a, or to the right of such a line. Hence, q is either on segment $\overline{aq_1(\pi_{i+1})}$ or it is inside triangle $(a, q_1(\pi_{i+1}), q(S_{x_1,y_1}, \overline{ab}))$. In the latter case, shown in Fig. 24, q is inside $T(S_{x_1,y_1}, a)$, as $(a, q_1(\pi_{i+1}), q(S_{x_1,y_1}, \overline{ab}))$ is a subset of $T(S_{x_1,y_1}, a)$. However, by Property S4, $T(S_{x_1,y_1}, a)$ contains no grid point, hence such a case never occurs.

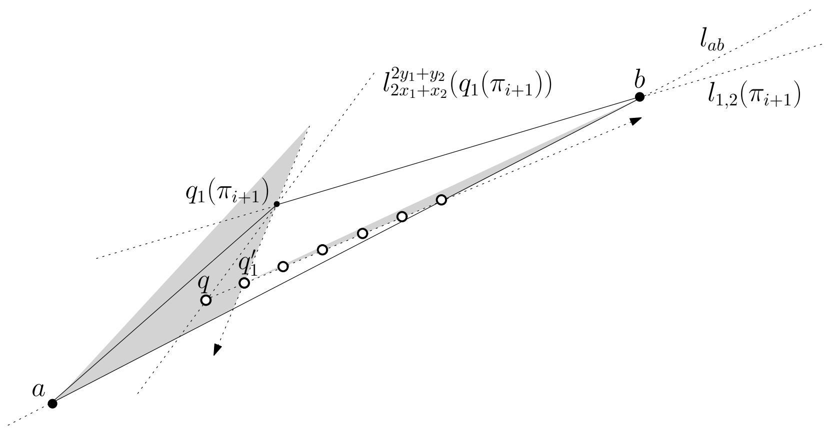

Fig. 24: Line $l^{2y_1+y_2}_{2x_1+x_2}(q_1(\pi_{i+1}))$ can not intersect the interior of segment $\overline{ab}$, as otherwise q would be inside $T(S_{x_1,y_1}, a)$.

Assume that q is on $\overline{aq_1(\pi_{i+1})}$. Suppose first that point $q''_1 \equiv q'_1 - (x_1, y_1)$ is inside T_1, as in Fig. 25. Then, since $\frac{y_1+y_2}{x_1+x_2} < \frac{d_v}{d_h} < \frac{y_1}{x_1}$ and since $\frac{y_1}{x_1} > \frac{2y_1+y_2}{2x_1+x_2}$, point $q'''_1 \equiv q''_1 - (x_1 + x_2, y_1 + y_2)$ is inside triangle $(a, q_1(\pi_{i+1}), q(S_{x_1,y_1}, \overline{ab}))$. Then, q'''_1 is inside $T(S_{x_1,y_1}, a)$, as $(a, q_1(\pi_{i+1}), q(S_{x_1,y_1}, \overline{ab}))$ is a subset of $T(S_{x_1,y_1}, a)$. However, by Property S4, $T(S_{x_1,y_1}, a)$ contains no grid point, hence such a case never occurs.

Assume that q'_1 is the only free point of S_{x_1,y_1} inside T_1. Further, $q \neq a$. Indeed, if $q = a$, then, since q'_1 is inside T_1, $\frac{d_v}{d_h} < \frac{y_1+y_2}{x_1+x_2}$, a contradiction. We prove that all the grid points inside $\pi_{i+1} \cup \overline{ab}$ lie on the line $l^{y_1+y_2}_{x_1+x_2}(q'_1)$ with slope $\frac{y_1+y_2}{x_1+x_2}$ through q'_1. Since triangle $(a, q_1(\pi_{i+1}), q(S_{x_1,y_1}, \overline{ab}))$ is a subset of $T(S_{x_1,y_1}, a)$, it contains no grid point; the only grid point on the line through $q_1(\pi_{i+1})$ with slope $\frac{y_1}{x_1}$ is q'_1, which lies on $l^{y_1+y_2}_{x_1+x_2}(q'_1)$; $l^{y_1+y_2}_{x_1+x_2}(q'_1)$ and the line $l^{y_1+y_2}_{x_1+x_2}(q''_1)$ with slope $\frac{y_1+y_2}{x_1+x_2}$ through q''_1 are consecutive grid lines, as $\frac{y_1}{x_1}$ is a generating fraction of $\frac{y_1+y_2}{x_1+x_2}$, hence they contain no grid point between them, by Lemma 6; it follows that there is no grid point in the interior of triangle $(q'_1, q(S_{x_1,y_1}, \overline{ab}), q(S_{x_1+x_2,y_1+y_2}, \overline{ab}))$. Finally, $l^{y_1+y_2}_{x_1+x_2}(q'_1)$ and the line $l^{y_1+y_2}_{x_1+x_2}(q_1(\pi_{i+1}))$ with slope $\frac{y_1+y_2}{x_1+x_2}$ through $q_1(\pi_{i+1})$ are consecutive grid lines, as $\frac{y_1}{x_1}$ is a generating fraction of $\frac{y_1+y_2}{x_1+x_2}$, hence they contain no grid point between them, by Lemma 6; it follows that there is no

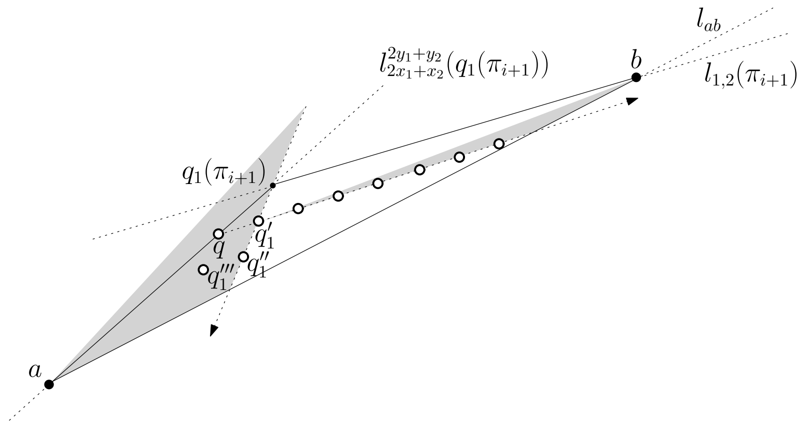

Fig. 25: If line $l^{2y_1+y_2}_{2x_1+x_2}(q_1(\pi_{i+1}))$ contains segment $\overline{aq_1(\pi_{i+1})}$, point q''_1 can not be inside T_1, as otherwise q'''_1 would be inside $T(S_{x_1,y_1}, a)$.

grid point in the interior of polygon $(q_1(\pi_{i+1}), q'_1, q(S_{x_1+x_2,y_1+y_2}, \overline{ab}), b)$. Hence, under the above assumptions, all the free points inside $\pi_{i+1} \cup \overline{ab}$ lie on $l^{y_1+y_2}_{x_1+x_2}(q'_1)$, thus proving the claim. □

Observe that the number of paths that come after π_{i+1} in Π is at most the number of free points inside $\pi_{i+1} \cup \overline{ab}$ plus one. Hence if all the free points inside $\pi_{i+1} \cup \overline{ab}$ lie on a specific grid line, the number of paths that come after π_{i+1} in Π is at most the number of points on a specific grid line.

- Fourth, the case in which b is to the right of $l_{1,2}(\pi_{i+1})$ and a is on $l_{1,2}(\pi_{i+1})$ can be discussed analogously to the previous case.

Suppose that after drawing π_i Condition 2 is satisfied. We have the following:

Claim 10 *Polygon $\pi_i \cup \overline{ab}$ has no internal grid point.*

Proof: Consider the points $q'_1 \equiv q_1(\pi_i) - (x_1, y_1)$ and $q'_2 \equiv q_2(\pi_i) + (x_2, y_2)$. Consider the lines $l_{1,2}(\pi_i)$ through $q_1(\pi_i)$ and $q_2(\pi_i)$ and $l'_{1,2}$ through q'_1 and q'_2. Line $l_{1,2}(\pi_i)$ has slope $\frac{y_1+y_2}{x_1+x_2}$ by the hypotheses of Condition 2. Line $l'_{1,2}$ has slope $\frac{y_1+y_2}{x_1+x_2}$. This can be proved analogously as it was shown that line $l_{1,2}(\pi_{i+1})$ has slope $\frac{y_1+y_2}{x_1+x_2}$ in the proof of Claim 1. By the hypotheses of Condition 2 and since $\vec{l}(x_1, y_1)$ and $\vec{l}(x_2, y_2)$ intersect segment $\overline{ab}$, points q'_1 and q'_2 lie in the closed half-plane delimited by l_{ab} and not containing c. Then, by Lemma 6, no grid point is in the open strip delimited by $l_{1,2}(\pi_i)$ and $l'_{1,2}$. If at least one of q'_1 and q'_2 is to the right of l_{ab}, then one of a and b is to the left of $l'_{1,2}$, hence it is in the open strip delimited by $l_{1,2}(\pi_i)$ and $l'_{1,2}$. It follows that both q'_1 and q'_2 are on l_{ab}. Then, no grid point is internal to polygon $(q_1(\pi_i), q_2(\pi_i), q'_2, q'_1)$. As $(a, q_1(\pi_i), q'_1)$ and $(b, q_2(\pi_i), q'_2)$ are enclosed in $T(S_{x_1,y_1}, a)$ and in $T(S_{x_2,y_2}, b)$, respectively, polygon $\pi_i \cup \overline{ab}$ contains no grid point. □

Since polygon $\pi_i \cup \overline{ab}$ has no internal point, $\pi_i = \pi_{M_1-1}$. Hence, $\pi_{i+1} = \pi_{M_1} = \overline{ab}$.

Suppose that after drawing π_i Condition 3 is satisfied. By the hypotheses of the case, π_i is $\overline{aq_1(\pi_i)b}$. Suppose that Condition 3(i) is satisfied, the case in which Condition 3(ii) is satisfied being analogous.

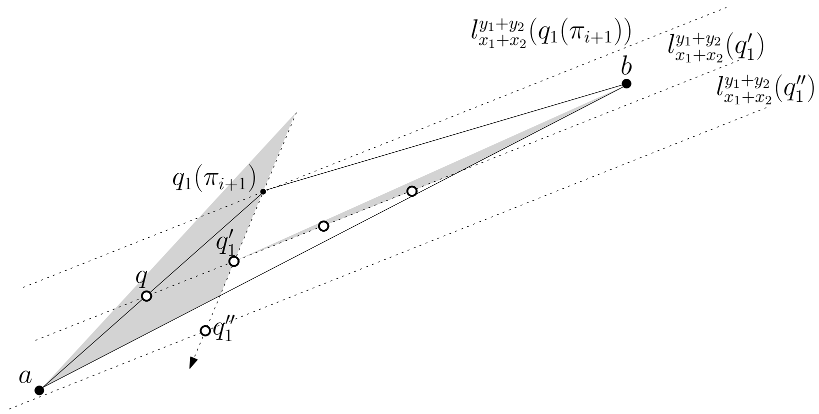

Fig. 26: If line $l^{2y_1+y_2}_{2x_1+x_2}(q_1(\pi_{i+1}))$ contains segment $\overline{aq_1(\pi_{i+1})}$ and q'_1 is the free only point of S_{x_1,y_1} inside T_1, then all the grid points inside $\pi_{i+1} \cup \overline{ab}$ lie on $l^{y_1+y_2}_{x_1+x_2}(q'_1)$.

Consider the first free point of S_{x_1,y_1}, that is, point $q_1(\pi_{i+1}) \equiv q_1(\pi_i) - (x_1, y_1)$. By the hypotheses of Condition 3, such a point exists and it is also the first point of S_{x_2,y_2}. We have the following:

Claim 11 *π_{i+1} is $\overline{aq_1(\pi_{i+1})b}$.*

Proof: Since, by the hypotheses of Condition 3, the second segment of π_i lies on a line l_1 with slope $\frac{y_2}{x_2}$, since the line l_2 passing through the points of S_{x_2,y_2} has slope $\frac{y_2}{x_2}$, and since $\frac{y_1}{x_1}$ is a generating fraction of $\frac{y_2}{x_2}$, then l_1 and l_2 are consecutive grid lines. By Lemma 6, there exists no grid point in the strip delimited by l_1 and l_2, hence there exists no grid point inside polygon $(q_1(\pi_i), q_1(\pi_{i+1}), b, q(S_{x_2,y_2}, \overline{ab}))$; thus, there exists no grid point inside triangle $(q_1(\pi_i), q_1(\pi_{i+1}), b)$. Further, there exists no grid point inside triangle $(q_1(\pi_i), q_1(\pi_{i+1}), a)$, as such a triangle is a subset of $T(S_{x_1,y_1}, a)$. Since $(a, q_1(\pi_{i+1}), b)$ is a convex polygon, π_{i+1} is $\overline{aq_1(\pi_{i+1})b}$. □

Let $l_{1,2}(\pi_{i+1})$ be the line through $q_1(\pi_{i+1})$ with slope $\frac{y_1+y_2}{x_1+x_2}$.

Now we discuss which condition is satisfied after drawing π_{i+1}, discussing the case in which a and b are both on $l_{1,2}(\pi_{i+1})$, the case in which one of a and b is on $l_{1,2}(\pi_{i+1})$ and the other one is to the right of such a line, and the case in which both a and b are to the right of $l_{1,2}(\pi_{i+1})$.

Neither a nor b is to the left of $l_{1,2}(\pi_{i+1})$. This can be shown as when Condition 1 is satisfied. Hence, either a and b are both on $l_{1,2}(\pi_{i+1})$, or one of a and b is on $l_{1,2}(\pi_{i+1})$ and the other one is to the right of this line, or both a and b are to the right of $l_{1,2}(\pi_{i+1})$.

Now we discuss which condition is satisfied after drawing π_{i+1}.

- First, it can be shown that a and b are not both on $l_{1,2}(\pi_{i+1})$ as in the proof of Claim 2.
- Second, consider the case in which a and b are both to the right of $l_{1,2}(\pi_{i+1})$.
  - If both S_{x_1,y_1} and S_{x_2,y_2} have free points, then, after drawing π_{i+1} Condition 4 is satisfied with S_{x_1,y_1} and S_{x_2,y_2} associated with path π_{i+2}. This can be shown as in the proof of Claim 3.

  - If neither S_{x_1,y_1} nor S_{x_2,y_2} has free points, then, after drawing π_{i+1} Condition 5 is satisfied with S_{x_1,y_1} and S_{x_2,y_2} associated with path π_{i+2}. This can be shown as in the proof of Claim 4.
  - The case in which exactly one of S_{x_1,y_1} and S_{x_2,y_2} has free points never occurs. This can be shown as in the proof of Claim 5.

- Third, consider the case in which a is to the right of $l_{1,2}(\pi_{i+1})$ and b is on $l_{1,2}(\pi_{i+1})$.

  If S_{x_1,y_1} has no free point left, then $\pi_{i+2} = \pi_{M_1} = \overline{ab}$. This can be shown as in the proof of Claim 7.

  If S_{x_1,y_1} has free points, consider the first free point on S_{x_1,y_1}, that is, point $q'_1 \equiv q_1(\pi_{i+1}) - (x_1, y_1)$. Consider the sequence of grid points $S_{x_1+x_2,y_1+y_2}$ whose points have coordinates $q'_1 + m(x_1 + x_2, y_1 + y_2)$, where $0 \leq m \leq i^*$, where i^* is the largest integer such that $q'_1 + i^*(x_1 + x_2, y_1 + y_2)$ is inside T_1. Then, as in the proof of Claims 8 and 9, we have that after drawing π_{i+1} either Condition 3 is satisfied, where sequences S_{x_1,y_1} and $S_{x_1+x_2,y_1+y_2}$ are associated with path π_{i+2}, or all the free points inside $\pi_{i+1} \cup \overline{ab}$ lie on a specific grid line and hence the number of paths that come after π_{i+1} in Π is at most the number of points on such a grid line.

- Fourth, the case in which b is to the right of $l_{1,2}(\pi_{i+1})$ and a is on $l_{1,2}(\pi_{i+1})$ can be discussed analogously to the previous case.

Suppose that after drawing π_i Condition 4 is satisfied. By the hypotheses of Condition 4, π_i is $\overline{aq_1(\pi_i)b}$, where $q_1(\pi)$ is the last occupied point of S_{x_1,y_1} and the last occupied point of S_{x_2,y_2}. Consider the first free point of S_{x_1,y_1}, that is, point $q_1(\pi_{i+1}) \equiv q_1(\pi_i) - (x_1, y_1)$, and consider the first free point of S_{x_2,y_2}, that is, point $q_2(\pi_{i+1}) \equiv q_1(\pi_i) + (x_2, y_2)$. Such points exist, by the hypotheses of Condition 4. Then, $\pi_{i+1} = (a, q_1(\pi_{i+1}), q_2(\pi_{i+1}), b)$. This can be shown as in the proof of Claim 1. Denote by $l_{1,2}(\pi_i)$ and $l_{1,2}(\pi_{i+1})$ the lines through $q_1(\pi_i)$ and $q_2(\pi_i)$ and through $q_1(\pi_{i+1})$ and $q_2(\pi_{i+1})$, respectively.

Now we discuss which condition is satisfied after drawing π_{i+1}, discussing the case in which a and b are both on $l_{1,2}(\pi_{i+1})$, the case in which one of a and b is on $l_{1,2}(\pi_{i+1})$ and the other one is to the right of such a line, and the case in which both a and b are to the right of $l_{1,2}(\pi_{i+1})$.

Neither a nor b is to the left of $l_{1,2}(\pi_{i+1})$. This can be shown as when Condition 1 is satisfied. Hence, either a and b are both on $l_{1,2}(\pi_{i+1})$, or one of a and b is on $l_{1,2}(\pi_{i+1})$ and the other one is to the right of this line, or both a and b are to the right of $l_{1,2}(\pi_{i+1})$.

Now we discuss which condition is satisfied after drawing π_{i+1}.

- First, it can be shown that a and b are not both on $l_{1,2}(\pi_{i+1})$ as in the proof of Claim 2.

- Second, consider the case in which a and b are both to the right of $l_{1,2}(\pi_{i+1})$.

  - If both S_{x_1,y_1} and S_{x_2,y_2} have free points, then, after drawing π_{i+1} Condition 1 is satisfied with S_{x_1,y_1} and S_{x_2,y_2} associated with path π_{i+2}. This can be shown as in the proof of Claim 3.
  - If neither S_{x_1,y_1} nor S_{x_2,y_2} has free points, then, after drawing π_{i+1} Condition 2 is satisfied with S_{x_1,y_1} and S_{x_2,y_2} associated with path π_{i+2}. This can be shown as in the proof of Claim 4.

- The case in which exactly one of S_{x_1,y_1} and S_{x_2,y_2} has free points never occurs. This can be shown as in the proof of Claim 5.

- Third, consider the case in which a is to the right of $l_{1,2}(\pi_{i+1})$ and b is on $l_{1,2}(\pi_{i+1})$.

  If S_{x_1,y_1} has no free point left, then $\pi_{i+2} = \pi_{M_1} = \overline{ab}$. This can be shown as in the proof of Claim 7.

  If S_{x_1,y_1} has free points, consider the first free point on S_{x_1,y_1}, that is, point $q'_1 \equiv q_1(\pi_{i+1}) - (x_1, y_1)$. Consider the sequence of grid points $S_{x_1+x_2,y_1+y_2}$ whose points have coordinates $q'_1 + m(x_1 + x_2, y_1 + y_2)$, where $0 \leq m \leq i^*$, where i^* is the largest integer such that $q'_1 + i^*(x_1 + x_2, y_1 + y_2)$ is inside T_1. Then, as in the proof of Claims 8 and 9, we have that after drawing π_{i+1} either Condition 3 is satisfied, where sequences S_{x_1,y_1} and $S_{x_1+x_2,y_1+y_2}$ are associated with path π_{i+2}, or all the free points inside $\pi_{i+1} \cup \overline{ab}$ lie on a specific grid line and hence the number of paths that come after π_{i+1} in Π is at most the number of points on such a grid line.

- Fourth, the case in which b is to the right of $l_{1,2}(\pi_{i+1})$ and a is on $l_{1,2}(\pi_{i+1})$ can be discussed analogously to the previous case.

Suppose that after drawing π_i Condition 5 is satisfied. We have the following claim, that can be proved analogously to Claim 10:

Claim 12 *Polygon $\pi_i \cup \overline{ab}$ has no internal grid point.*

4.4 *Proof that* $\max\{d_h, d_v\} \in \Omega(n)$.

We now compute how many paths exist in Π, as a function of d_h and d_v. Denote by $S_{x^i_1,y^i_1}$ and by $S_{x^i_2,y^i_2}$ the sequences of grid points that are used by Π_i, where the grid points in $S_{x^i_1,y^i_1}$ lie on a line with slope y^i_1/x^i_1 and the grid points in $S_{x^i_2,y^i_2}$ lie on a line with slope y^i_2/x^i_2. Notice that, following the notation of Section 4.2, $S_{x^1_1,y^1_1} = S_{0,1}$ and $S_{x^1_2,y^1_2} = S_{1,0}$. Further, if a, $p^{0,1}_{k_1}$, and $p^{1,0}_{k_1}$ are collinear (and b is not), then $S_{x^2_1,y^2_1} = S_{l,1}$, and $S_{x^2_2,y^2_2} = S_{l+1,1}$, where l is defined as in Section 4.2, while if $p^{0,1}_{k_1}$, $p^{1,0}_{k_1}$, and b are collinear (and a is not), then $S_{x^2_1,y^2_1} = S_{1,l}$, and $S_{x^2_2,y^2_2} = S_{1,l+1}$. We claim that $x^i_1, y^i_1, x^i_2, y^i_2 \geq 2^{i-2}$, for $i \geq 2$. Notice that, since $l \geq 1$, we already observed that such a claim holds when $i = 2$. From the discussion in Section 4.2 (and subsequent proof in Section 4.3), we have that y^i_1 is obtained as the sum of the numerators y^{i-1}_a and y^{i-1}_b of the slopes of two lines containing grid points traversed by paths in Π_{i-1}. Inductively, $y^{i-1}_a + y^{i-1}_b \geq y^{i-1}_1 + y^{i-1}_2 \geq 2^{i-3} + 2^{i-3} \geq 2^{i-2}$. Analogously $y^i_2, x^i_1, x^i_2 \geq 2^{i-2}$.

The number of paths in Π_i is the number of grid points in the one out of $S_{x^i_1,y^i_1}$ and $S_{x^i_2,y^i_2}$, with the greatest number of points. When $i = 1$, each of $S_{0,1}$ and $S_{1,0}$ has at most $\max\{d_h, d_v\}$ grid points. Further, for $i \geq 2$, $S_{x^i_1,y^i_1}$ and $S_{x^i_2,y^i_2}$ lie on lines with slopes whose numerators and denominators are greater than or equal to 2^{i-2}. Hence, each of such sequences has at most $\frac{\max\{d_h,d_v\}}{2^{i-2}} + 1$ grid points. In the special case in which the geometry of paths stops to satisfy Conditions 1–5, all the remaining free points lie on a specific grid line, as proved in Section 4.3. Since each remaining path uses one of such free points and no more than $\max\{d_h, d_v\}$ free points lie on a specific grid line, there are at most $\max\{d_h, d_v\}$ paths that are drawn in such a special case. Such paths are below called *final paths*. Hence, the total number of paths in Π is at most

$$\underbrace{1}_{\pi_1} + \underbrace{\max\{d_h, d_v\}}_{\text{paths in } \Pi_1} + \underbrace{\sum_{i=2}^{f} \left(\frac{\max\{d_h, d_v\}}{2^{i-2}} + 1 \right)}_{\text{paths in } \Pi_i, \text{ for } 2 \leq i \leq f} + \underbrace{\max\{d_h, d_v\}}_{\text{final paths}} + \underbrace{1}_{\overline{ab}}$$

$$\leq 2 + \max\{d_h, d_v\} + 2\max\{d_h, d_v\} + (f-1) + \max\{d_h, d_v\} \qquad = O(\max\{d_h, d_v\}),$$

where the last equality holds since $f = O(\log(\max\{d_h, d_v\})) = O(\max\{d_h, d_v\})$, because $x_1^i, y_1^i, x_2^i, y_2^i \geq 2^{i-2}$ and because both the numerator and the denominator of any line slope can not exceed $\max\{d_h, d_v\}$.

Since the number of paths in Π is $\Omega(n)$, then $\max\{d_h, d_v\} \in \Omega(n)$ and hence $\max\{W, H\} \in \Omega(n)$. Theorem 2 follows.

5 Proof of Theorem 3

In this section we prove Theorem 3, that is, we present an n-vertex series-parallel graph G_k requiring $\Omega(2^{\sqrt{\log n}})$ width and $\Omega(2^{\sqrt{\log n}})$ height in any straight-line or poly-line grid drawing. As an immediate consequence of Theorems 2 and 3 we have a proof of Theorem 4, that is, there exist series-parallel graphs requiring $\Omega(n2^{\sqrt{\log n}})$ area in any straight-line or poly-line grid drawing.

In order to prove Theorem 3, we heavily exploit Theorem 2, stating that there exists a constant $c > 0$ such that every planar straight-line or poly-line grid drawing of $K_{2,n}$ in a $W \times H$ grid satisfies $\max\{W, H\} \geq c \cdot n$. Then, let $d = \min\{\frac{c}{4}, \frac{1}{8}\}$ and let $f(n)$ be a function to be computed later.

Graph G_k is inductively defined as follows. Graph G_1 is $K_{2,f(n)-2}$. Graph G_{i+1} is defined as follows. Consider $f(n)$ copies $G_i^{1,1}, G_i^{1,2}, G_i^{2,1}, G_i^{2,2}, \dots, G_i^{j,1}, G_i^{j,2}, \dots, G_i^{\frac{f(n)}{2},1}, G_i^{\frac{f(n)}{2},2}$ of G_i; construct $\frac{f(n)}{2}$ series-parallel graphs $G_i^1, G_i^2, \dots, G_i^{\frac{f(n)}{2}}$, where G_i^j is the series composition of $G_i^{j,1}$ and $G_i^{j,2}$; then, G_{i+1} is the parallel composition of graphs $G_i^1, G_i^2, \dots, G_i^{\frac{f(n)}{2}}$. See Fig. 27.

First, we prove Theorem 3 for sufficiently large graphs, that is, for graphs having a number of vertices that is at least some constant n_0 to be determined later. From now till it is otherwise specified, assume that $n \geq n_0$.

Suppose that $f(n) \geq 8$, $\forall n \geq n_0$. Let n be the number of vertices of graph G_k. We have the following main lemma.

Lemma 8 *Let Γ_i be any poly-line grid drawing of G_i and let a_i and b_i be the poles of G_i, for each $1 \leq i \leq k$. Then, one of the following holds:*

- *Condition 1: Both the height and the width of Γ_i are greater than or equal to $d \cdot f(n)$;*
- *Condition 2: The width of Γ_i is greater than or equal to $d \cdot f(n)$ and Γ_i contains a polygonal path l_i connecting a_i to b_i that has height greater than or equal to 2^i and such that, for every point $p \in l_i$, $\min\{y(a_i), y(b_i)\} \leq y(p) \leq \max\{y(a_i), y(b_i)\}$; or the height of Γ_i is greater than or equal to $d \cdot f(n)$ and Γ_i contains a polygonal path l_i connecting a_i to b_i that has width greater than or equal to 2^i and such that, for every point $p \in l_i$, $\min\{x(a_i), x(b_i)\} \leq x(p) \leq \max\{x(a_i), x(b_i)\}$.*

Proof: We prove the statement by induction on i.

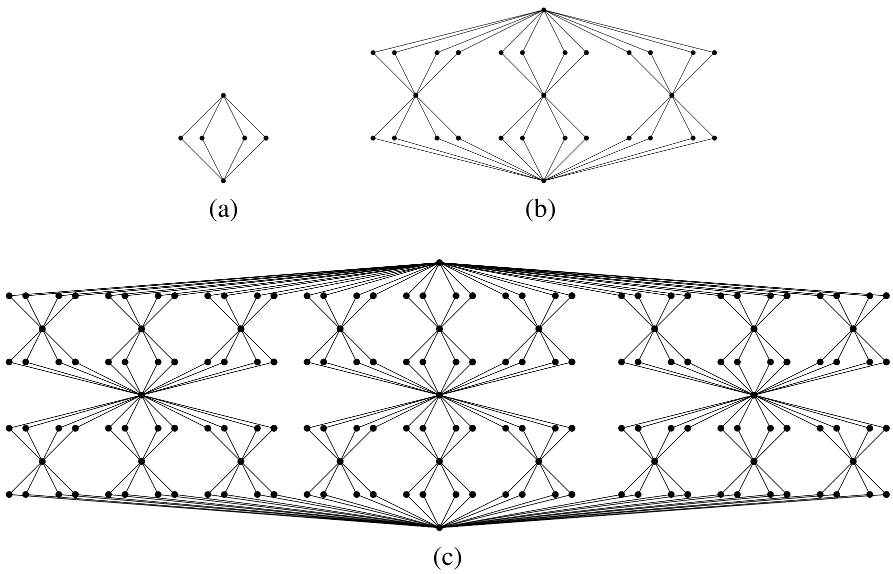

Fig. 27: Graphs G_i, with $f(n) = 6$. (a) G_1. (b) G_2. (c) G_3.

In the base case, consider any poly-line grid drawing Γ_1 of G_1. By Theorem 2, one of the height and the width of Γ_1, say the width of Γ_1, is at least $c \cdot f(n)$, hence it is at least $d \cdot f(n)$. We prove that the height of Γ_1 is at least $d \cdot f(n)$ or there exists a polygonal path l_1 connecting a_1 to b_1 that has height greater than or equal to 2 and such that, for every point $p \in l_1$, $\min\{y(a_1), y(b_1)\} \leq y(p) \leq \max\{y(a_1), y(b_1)\}$.

Denote by $l(a_1)$ and $l(b_1)$ the horizontal lines through a_1 and b_1, respectively, where we suppose, without loss of generality, that $y(a_1) \leq y(b_1)$. Suppose that at least $2d \cdot f(n)$ paths of $G_1 = K_{2,f(n)-2}$ have non-empty intersection with the open half-plane $H^-(y = y(a_1))$ (that is, the half-plane $y < y(a_1)$) or with the open half-plane $H^+(y = y(b_1))$ (that is, the half-plane $y > y(b_1)$). By Lemma 1 with $\vec{v} = (0, -1)$, for each path π of G_1 that has non-empty intersection with $H^-(y = y(a_1))$, there exists a grid point $p \in \pi$ whose y-coordinate is minimum among the points of π. Clearly, p belongs to $H^-(y = y(a_1))$. Hence, p belongs to a horizontal grid line h that does not intersect or contain the segment $\overline{a_1 b_1}$. By Lemma 2, at most two paths of G_1 have their points with smallest y-coordinate belonging to h. It can be proved analogously that at most two paths of G_1 have their points with greatest y-coordinate belonging to the same horizontal grid line in $H^+(y = y(b_1))$. Hence, as $2d \cdot f(n)$ paths of G_1 have non-empty intersection with $H^-(y = y(a_1))$ or with $H^+(y = y(b_1))$, it follows that Γ_1 has height at least $d \cdot f(n)$.

Now suppose that less then $2d \cdot f(n)$ paths of G_1 have non-empty intersection with $H^-(y = y(a_1))$ or with $H^+(y = y(b_1))$. Then, since $d \leq \frac{1}{8}$, at least $f(n) - 2 - 2d \cdot f(n) + 1 \geq \frac{3f(n)}{4} - 1$ paths of G_1 are such that, for every point p of any such a path, $y(a_1) \leq y(p) \leq y(b_1)$. By planarity of Γ_1 at most one of such paths touches $l(a_1)$ in a point whose y-coordinate is $y(a_1)$ and whose x-coordinate is smaller than $x(a_1)$. Analogously, at most one of such paths touches $l(a_1)$ in a point whose y-coordinate is $y(a_1)$ and whose x-coordinate is greater than $x(a_1)$. The last two statements hold even if b_1 replaces a_1. Hence, as long as $\frac{3f(n)}{4} - 1 \geq 5$, which is always the case whenever $f(n) \geq 8$, there is at least one path of G_1 whose every point p has $y(a_1) \leq y(p) \leq y(b_1)$ and whose only vertex $v \neq a_1, b_1$ has y-coordinate greater than

$y(a_1)$ and smaller than $y(b_1)$. It follows that the polygonal path (a_1, v, b_1) connecting the poles of G_1 has height at least two and is such that, for every point $p \in (a_1, v, b_1)$, $y(a_1) \leq y(p) \leq y(b_1)$, thus proving the base case of the induction.

Now let's consider the inductive case. Let Γ_{i+1} be any poly-line grid drawing of G_{i+1}, containing drawings $\Gamma_i^{1,1}, \Gamma_i^{1,2}, \Gamma_i^{2,1}, \Gamma_i^{2,2}, \dots, \Gamma_i^{j,1}, \Gamma_i^{j,2}, \dots, \Gamma_i^{\frac{f(n)}{2},1}, \Gamma_i^{\frac{f(n)}{2},2}$ of graphs $G_i^{1,1}, G_i^{1,2}, G_i^{2,1}, G_i^{2,2}, \dots, G_i^{j,1}, G_i^{j,2}, \dots, G_i^{\frac{f(n)}{2},1}, G_i^{\frac{f(n)}{2},2}$, respectively. By induction, for each $1 \leq j \leq \frac{f(n)}{2}$ and each $1 \leq k \leq 2$, $\Gamma_i^{j,k}$ satisfies Condition 1 or Condition 2.

If there exists a drawing $\Gamma_i^{j,k}$, for some $1 \leq j \leq \frac{f(n)}{2}$ and $1 \leq k \leq 2$, such that $\Gamma_i^{j,k}$ satisfies Condition 1, then the width and the height of $\Gamma_i^{j,k}$ are both greater than or equal to $d \cdot f(n)$, hence the width and the height of Γ_{i+1} are both greater than or equal to $d \cdot f(n)$, and there is nothing else to prove.

Hence, we can assume that, for every $1 \leq j \leq \frac{f(n)}{2}$ and $1 \leq k \leq 2$, $\Gamma_i^{j,k}$ satisfies Condition 2. If there exist two drawings $\Gamma_i^{j',k'}$ and $\Gamma_i^{j'',k''}$, for some $1 \leq j', j'' \leq \frac{f(n)}{2}$ and $1 \leq k', k'' \leq 2$, where $j' = j''$ and $k' = k''$ do not hold simultaneously, such that the width of $\Gamma_i^{j',k'}$ is greater than or equal to $d \cdot f(n)$ and such that the height of $\Gamma_i^{j'',k''}$ is greater than or equal to $d \cdot f(n)$, then the width and the height of Γ_{i+1} are both greater than or equal to $d \cdot f(n)$, and there is nothing else to prove.

Hence, we can assume (without loss of generality, up to a rotation of the Cartesian axes) that, for every $1 \leq j \leq \frac{f(n)}{2}$ and $1 \leq k \leq 2$, the width of $\Gamma_i^{j,k}$ is greater than or equal to $d \cdot f(n)$ and $\Gamma_i^{j,k}$ contains a polygonal path $l_i^{j,k}$ connecting a_i to b_i that has height greater than or equal to 2^i and such that, for every point $p \in l_i^{j,k}$, $\min\{y(a_i), y(b_i)\} \leq y(p) \leq \max\{y(a_i), y(b_i)\}$.

Denote by l_i^j the path connecting a_{i+1} and b_{i+1} composed of $l_i^{j,1}$ and $l_i^{j,2}$. Denote by $l(a_{i+1})$ and $l(b_{i+1})$ the horizontal lines through a_{i+1} and b_{i+1}, respectively, where we suppose, without loss of generality, that $y(a_{i+1}) \leq y(b_{i+1})$. Analogously to the base case, it can be proved that if at least $2d \cdot f(n)$ paths l_i^j have non-empty intersection with the open half-plane $H^-(y = y(a_{i+1}))$ (that is, the half-plane $y < y(a_{i+1})$) or with the open half-plane $H^+(y = y(b_{i+1}))$ (that is, the half-plane $y > y(b_{i+1})$), then Γ_{i+1} has height at least $d \cdot f(n)$.

Now suppose that less then $2d \cdot f(n)$ paths l_i^j have non-empty intersection with $H^-(y = y(a_{i+1}))$ or with $H^+(y = y(b_{i+1}))$. Then, since $d \leq \frac{1}{8}$, at least $f(n) - 2d \cdot f(n) + 1 \geq \frac{3f(n)}{4} + 1$ paths l_i^j are such that, for every point p of any such a path, $y(a_{i+1}) \leq y(p) \leq y(b_{i+1})$. By planarity of Γ_{i+1} at most one of such path l_i^j touches $l(a_{i+1})$ in a point whose y-coordinate is $y(a_{i+1})$ and whose x-coordinate is smaller than $x(a_{i+1})$. Analogously, at most one of such paths l_i^j touches $l(a_{i+1})$ in a point whose y-coordinate is $y(a_{i+1})$ and whose x-coordinate is greater than $x(a_{i+1})$. The last two statements hold even if b_{i+1} replaces a_{i+1}. Hence, as long as $\frac{3f(n)}{4} + 1 \geq 5$, which is always the case whenever $f(n) \geq 8$, there is at least one path l_i^j composed of path $l_i^{j,1}$, that connects the poles a_{i+1} and v of $G_i^{j,1}$, and of path $l_i^{j,2}$, that connects the poles b_{i+1} and v of $G_i^{j,2}$, such that v has y-coordinate greater than $y(a_{i+1})$ and smaller than $y(b_{i+1})$. By inductive hypothesis, $l_i^{j,1}$ has height greater than or equal to 2^i and, for every point $p \in l_i^{j,1}$, $y(a_{i+1}) \leq y(p) \leq y(v)$; further, $l_i^{j,2}$ has height greater than or equal to 2^i and, for every point $p \in l_i^{j,2}$, $y(v) \leq y(p) \leq y(b_{i+1})$; hence, l_i^j has height greater than or equal to 2^{i+1} and, for every point $p \in l_i^j$, $y(a_{i+1}) \leq y(p) \leq y(b_{i+1})$, thus completing the induction. □

Corollary 2 *Any poly-line grid drawing of G_k has height and width that are both greater than or equal to* $\min\{d \cdot f(n), 2^k\}$.

We now compute $f(n)$ as a function of k and n. By construction $|G_1| = f(n)$; since G_i is composed of $f(n)$ copies of G_{i-1}, $|G_i| \leq f(n) \cdot |G_{i-1}|$; hence, inductively, we obtain that $|G_k| \leq f^k(n)$. Assuming that $|G_k| = n$, then $f^k(n) \geq n$, that is, $f(n) \geq n^{\frac{1}{k}}$.

By Corollary 2, any poly-line grid drawing Γ_k of G_k has height and width that are both greater than or equal to $\min\{d \cdot f(n), 2^k\} \geq \min\{d \cdot n^{\frac{1}{k}}, 2^k\}$. Then, we choose k in such a way that $n^{\frac{1}{k}}$ and 2^k are equal. This is done as follows.

$$\begin{aligned} 2^k &= n^{\frac{1}{k}}; \\ \log_2(2^k) &= \log_2(n^{\frac{1}{k}}); \\ k\log_2(2) &= \frac{1}{k}\log_2(n); \\ k^2 &= \log_2(n); \\ k &= \sqrt{\log_2(n)}. \end{aligned}$$

By Corollary 2, both the height and the width of Γ_k, with $k = \sqrt{\log_2(n)}$, are greater than or equal to $\min\{d \cdot n^{\frac{1}{\sqrt{\log_2(n)}}}, 2^{\sqrt{\log_2(n)}}\} = d \cdot 2^{\sqrt{\log_2(n)}} = \Omega(2^{\sqrt{\log_2(n)}})$, and Theorem 3 follows if $n \geq n_0$.

Since we need $f(n) = 2^{\sqrt{\log_2(n)}} \geq 8$, $\forall n \geq n_0$, then $n_0 = 512$. Observe that the $d \cdot 2^{\sqrt{\log_2(n)}}$ lower bound is less than 1 for all $n < 512$, as $d \leq \frac{1}{8}$. Since every drawing of a graph that is not a collection of paths has height and width at least one, the $d \cdot 2^{\sqrt{\log_2(n)}}$ lower bound holds for graphs with any number (that is, even smaller than 512) of nodes, thus completing the proof of Theorem 3.

6 Conclusions and Open Problems

In this paper we have shown that there exist series-parallel graphs requiring $\Omega(n2^{\sqrt{\log n}})$ area in any straight-line or poly-line grid drawing. Such a result has been achieved in two steps. In the first one, we derived an $\Omega(n)$ lower bound for the maximum between the height and the width of any poly-line grid drawing of $K_{2,n}$. In the second one, we derived an $\Omega(2^{\sqrt{\log n}})$ lower bound for the minimum between the height and the width of any poly-line grid drawing of certain series-parallel graphs.

The best known upper bound for the area requirements of poly-line grid drawings of series-parallel graphs is $O(n^{3/2})$ [Bie05, Bie10], while no sub-quadratic area upper bound is known in the case of straight-line grid drawings. Hence, in both cases, the gap between the upper and the lower bound is large, thus justifying the following two questions:

Problem 1 *What are the area requirements for poly-line grid drawings of series-parallel graphs?*

Problem 2 *What are the area requirements for straight-line grid drawings of series-parallel graphs?*

David Wood [Woo08] conjectures the following: Let $p_1, \dots, p_k$ be positive integers. Let $G(p_1)$ be the graph obtained from K_3 by adding p_1 new vertices adjacent to v and w for each edge (v, w) of K_3. For $k \geq 2$, let $G(p_1, p_2, \dots, p_k)$ be the graph obtained from $G(p_1, p_2, \dots, p_{k-1})$ by adding p_k new vertices adjacent to v and w for each edge (v, w) of $G(p_1, p_2, \dots, p_{k-1})$. Observe that $G(p_1, p_2, \dots, p_k)$ is a series-parallel graph.

Conjecture 1 *(Wood [Woo08]) Every straight-line grid drawing of $G(p_1, p_2, \dots, p_k)$ requires $\Omega(n^2)$ area for some choice of k and $p_1, p_2, \dots, p_k$.*

We remark that, for outerplanar graphs and trees, no super-linear area lower bounds are known, hence the determination of the area requirements for the straight-line and poly-line grid drawings of such graph classes still requires research efforts. However, we believe (and Conjecture 1 strengthens such a conviction) that understanding whether every series-parallel graph admits a straight-line drawing in sub-quadratic area is the most interesting open problem in the topic of small-area planar graph drawing.

	Straight-line				***Poly-line***			
Graph Class	***UB.***	***Ref.***	***LB.***	***Ref.***	***UB.***	***Ref.***	***LB.***	***Ref.***
Planar Graphs	$O(n^2)$	[dPP90, Sch90]	$\Omega(n^2)$	[DLT84, dPP90]	$O(n^2)$	[dPP90, Sch90]	$\Omega(n^2)$	[DLT84, dPP90]
Series-Parallel Graphs	$O(n^2)$	[dPP90, Sch90]	$\Omega(n2^{\sqrt{\log n}})$	*this paper*	$O(n^{3/2})$	[Bie05]	$\Omega(n2^{\sqrt{\log n}})$	*this paper*
Outerplanar Graphs	$O(n^{1.48})$	[DF09]	$\Omega(n)$	*trivial*	$O(n \log n)$	[Bie02]	$\Omega(n)$	*trivial*
Trees	$O(n \log n)$	[CDP92]	$\Omega(n)$	*trivial*	$O(n \log n)$	[CDP92]	$\Omega(n)$	*trivial*

Tab. 1: Area bounds for grid drawings of planar graphs and their sub-classes.

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