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On the complexity of vertex-coloring edge-weightings

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Given a graph $G = (V, E)$ and a weight function $w : E \to \mathbb{R}$, a coloring of vertices of G, induced by w, is defined by $\chi_w(v) = \sum_{e \ni v} w(e)$ for all $v \in V$. In this paper, we show that determining whether a particular graph has a weighting of the edges from $\{1, 2\}$ that induces a proper vertex coloring is NP-complete.

Keywords: vertex-coloring, 1-2-3 conjecture, NP-completeness

1 Introduction

For a given graph $G = (V, E)$, let $w : E \to \mathbb{R}$ be a weight function. We say that w is *proper* if the coloring of the vertices $\chi_w(v) = \sum_{e \ni v} w(e)$, $v \in V$, is proper. In 2004, Karoński, Łuczak, and Thomason (2004) showed that any graph with no components isomorphic to K_2 has a proper weighting from a finite set of reals. Furthermore, they conjectured that every graph with no components isomorphic to K_2 has a proper weighting from $W = \{1, 2, 3\}$. Addario-Berry, Dalal, McDiarmid, Reed, and Thomason (2007) showed that the above holds if $W = \{1, \dots, 30\}$. This result was improved by Addario-Berry, Dalal, and Reed (2008), who showed that one can take $W = \{1, \dots, 16\}$. Subsequently, Wang and Yu (2008) proved that $W = \{1, \dots, 13\}$ suffices. A recent breakthrough by Kalkowski, Karoński, and Pfender (2010) showed that the set of weights can be as small as $W = \{1, 2, 3, 4, 5\}$.

On the other hand, Addario-Berry, Dalal, and Reed (2008) showed that almost all graphs have a proper weighting from $\{1, 2\}$. In this paper, we show that determining whether a particular graph has a proper weighting of the edges from $\{1, 2\}$ is NP-complete. Consequently, there is no simple characterization of graphs with proper weightings from $\{1, 2\}$, unless P=NP. Formally, let

$$\text{1-2WEIGHT} = \{G : G \text{ is a graph having a proper weighting from } \{1, 2\}\}.$$

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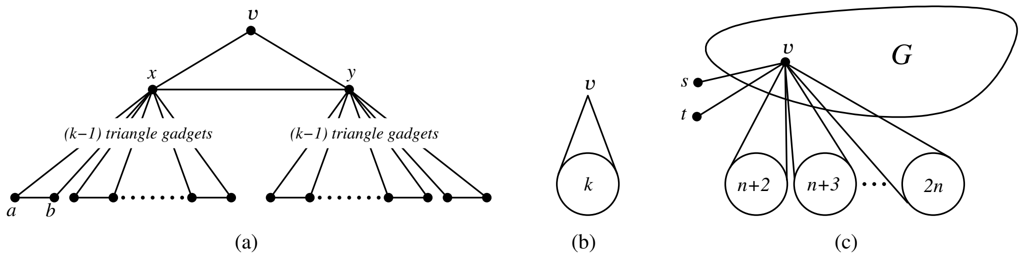

Fig. 1: A k-disallowing gadget (a) and its symbolic representation (b); a construction of $h(G)$ (c).

Theorem 1.1 *1-2WEIGHT is NP-complete.*

Before we prove this statement, we consider a similar theorem with a somewhat simpler proof, which we use as a template to prove Theorem 1.1. By analogy to 1-2WEIGHT we denote by 0-1WEIGHT the family of graphs with a proper weighting from $\{0,1\}$ and show the following:

Theorem 1.2 *0-1WEIGHT is NP-complete.*

2 0-1WEIGHT is NP-complete

Here we prove Theorem 1.2.

First note that 0-1WEIGHT is clearly in NP, since one can verify in polynomial time for a given graph whether a weighting of its edges from $\{0,1\}$ is proper.

Next we consider the well-known NP-hard problem

$$\text{3-COLOR} = \{G : G \text{ is a graph having a proper 3-vertex-coloring}\}.$$

In order to prove that 0-1WEIGHT is NP-hard (and hence NP-complete), we show a reduction from 3-COLOR to 0-1WEIGHT. To this end, we define a polynomial time reduction h, such that $G \in$ 3-COLOR if and only if $h(G) \in$ 0-1WEIGHT. To achieve this, we need two auxiliary gadgets.

We refer to the first gadget as a **triangle gadget**. This consists of a triangle xab, with x referred to as the *top* and with a and b each having no other coinciding edges. Note that any proper weighting w from $\{0,1\}$ of a graph with such a triangle must hold $w(xa) \neq w(xb)$; otherwise $\chi_w(a) = w(ab) + w(ax) = w(ba) + w(bx) = \chi_w(b)$. Hence, $\{w(xa), w(xb)\} = \{0,1\}$ and so every such triangle gadget contributes exactly 1 to $\chi_w(x)$.

The second gadget, called a k**-disallowing gadget**, consists of a *main triangle* vxy with v referred to as the *root* and with x and y each constituting the top of $k-1$ distinct triangle gadgets (see Figure 1(a)). Note that in any proper weighting w from $\{0,1\}$, $w(vx) \neq w(vy)$; otherwise, as both $\chi_w(x)$ and $\chi_w(y)$ have $k-1$ contributed by x and y's triangles, $\chi_w(x) = w(xv) + w(xy) + k - 1 = w(yv) + w(yx) + k - 1 = \chi_w(y)$. Therefore, if $w(xy) = 0$ then $\{\chi_w(x), \chi_w(y)\} = \{k-1, k\}$ and, if $w(xy) = 1$ then $\{\chi_w(x), \chi_w(y)\} = \{k, k+1\}$. In either case, v has one neighbor $z \in \{x, y\}$ with $\chi_w(z) = k$, and consequently, $\chi_w(v) \neq k$ in any proper weighting from $\{0,1\}$. Also $\{w(vx), w(vy)\} = \{0,1\}$ and hence this gadget contributes exactly 1 to $\chi_w(v)$.

Now we are ready to show a reduction from 3-COLOR to 0-1WEIGHT, h, such that $G \in$ 3-COLOR if and only if $h(G) \in$ 0-1WEIGHT. Let $G = (V, E)$ be a graph of order n. We may assume that $n \geq 3$. Otherwise, $n \leq 2$ and G is in 3-COLOR and so it suffices to take as $h(G)$ an empty graph which is trivially in 0-1WEIGHT. For $n \geq 3$ we construct the graph $h(G) = (W, F)$ as follows (see Figure 1(c)). We start with $G = (V, E)$. For each $v \in V$:

(i) connect v to two new vertices, s and t (distinct for each v);

(ii) add $n - 1$ new k-disallowing gadgets for all $k \in \{n+2, n+3, \ldots, 2n\}$ with v as their root.

Clearly, $h(G)$ can be calculated in time polynomial in the size of G.

Fact 2.1 *In $h(G)$ the following holds: any proper weighting w from $\{0, 1\}$ satisfies $\chi_w(v) \in \{n - 1, n, n+1\}$ for every $v \in V$.*

Proof: Fix $v \in V$. Since $w(vs) + w(vt) \in \{0, 1, 2\}$, v is the endpoint of $deg(v) \leq n - 1$ edges in V, and v is the root of $(n - 1)$ k-disallowing gadgets (each contributing 1 to $\chi_w(v)$), we have:

$$\chi_w(v) \in \{0, 1, 2\} + \{0, 1, \ldots, deg(v)\} + \{n - 1\} \subseteq \{n - 1, n, \ldots, 2n\},$$

where by $A + B$ we mean the set of all sums of an element from A with an element from B. Observing the above and the fact that v is the root of k-disallowing gadgets for all $k \in \{n+2, \ldots, 2n\}$, we find that any proper weighting w from $\{0, 1\}$ satisfies $\chi_w(v) \in \{n - 1, n, n+1\}$, as claimed. $\square$

It remains to show that $G \in$ 3-COLOR if and only if $h(G) \in$ 0-1WEIGHT.

First let us assume that $G \in$ 3-COLOR. That means there exists a proper 3-coloring of G, say $\chi : V \to \{n - 1, n, n+1\}$. We define a weighting of the edges of $h(G)$, $w : F \to \{0, 1\}$ as follows. For all $e \in E$ let $w(e) = 0$. For all $v \in V$, if $\chi(v) = n - 1$ then $w(vs) = w(vt) = 0$; otherwise, if $\chi(v) = n$ then $w(vs) = 1$ and $w(vt) = 0$; and finally, if $\chi(v) = n+1$ then $w(vs) = w(vt) = 1$. All other edges (parts of gadgets) are weighted as follows: For a triangle gadget xab with root x, $w(xa) = 1, w(xb) = w(ab) = 0$. For a k-disallowing gadget with root v, and main triangle vxy, $w(vx) = w(xy) = 1, w(vy) = 0$, and the weighting of all other triangle gadgets as described above. Note that w is a proper weighting of $h(G)$ (satisfying $\chi_w(v) = \chi(v)$ for all $v \in V$), as required.

Now let us assume that $G \notin$ 3-COLOR. Therefore, for all $\chi : V \to \{n - 1, n, n+1\}$, χ is not proper. But, from Fact 2.1, any proper weighting from $\{0, 1\}$ of $h(G)$ satisfies $\chi_w(v) \in \{n - 1, n, n+1\}$ for all $v \in V$. Thus, there is no such proper weighting and hence $h(G) \notin$ 0-1WEIGHT.

This completes the proof of Theorem 1.2.

3 1-2WEIGHT is NP-complete

The proof of Theorem 1.1 extends the ideas introduced in the proof of Theorem 1.2. Since clearly 1-2WEIGHT is in NP, it remains to show that 1-2WEIGHT is NP-hard. As before, we show a reduction from 3-COLOR to 1-2WEIGHT. To this end, we define a polynomial time reduction f, such that $G \in$ 3-COLOR if and only if $f(G) \in$ 1-2WEIGHT. Below we define auxiliary gadgets.

As in Section 2, we will use a **triangle gadget**. Now note that every triangle xab, with only x having other adjacent edges (x is referred to as the *top*), contributes exactly 3 to $\chi_w(x)$ in any proper weighting w from $\{1, 2\}$.

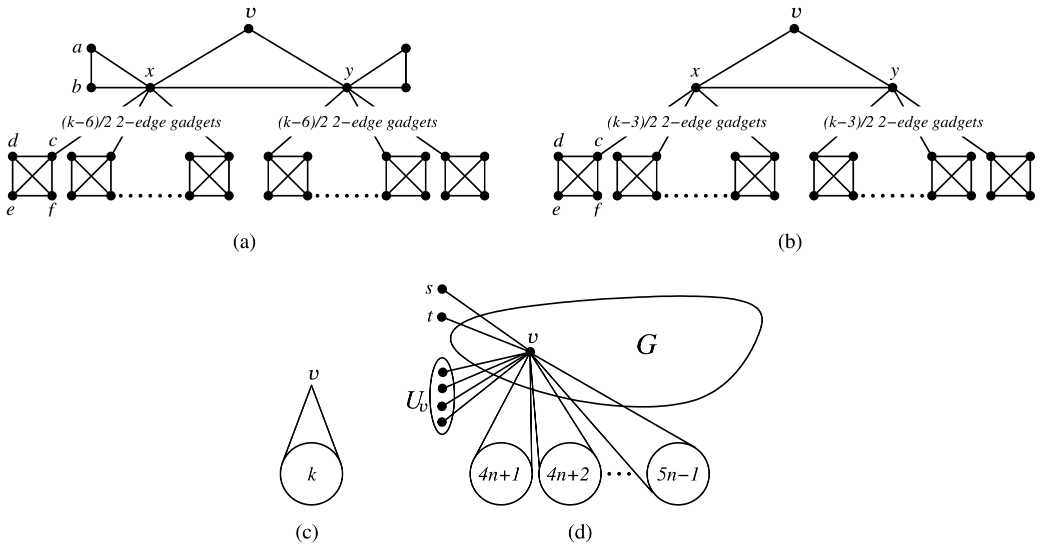

Fig. 2: A k-disallowing gadget (a) for even k, (b) for odd k and their symbolic representation (c); a construction of $f(G)$, (d).

Now we define a **2-edge gadget** consisting of the set of vertices $\{x, c, d, e, f\}$ with x and c adjacent and $\{c, d, e, f\}$ spanning a complete graph K_4. One can check that every proper weighting w from $\{1, 2\}$ of a graph adjacent to such a gadget only at x requires $w(xc) = 2$. We refer to x as the *endpoint*.

We use the above gadgets to construct another gadget, called a **k-disallowing gadget**. As we will see, this gadget has similar properties as its namesake in Section 2. We therefore allow ourselves the re-use of the name for this new, slightly different, gadget. We assume that $k \geq 8$. The k-disallowing gadget contains a *main triangle* vxy with v referred to as the *root*. Moreover, if k is even, x and y each form the endpoint of $(k-6)/2$ edge disjoint 2-edge gadgets and x and y are each tops of distinct triangle gadgets (see Figure 2(a)). If k is odd, x and y each form the endpoint of $(k-3)/2$ edge disjoint 2-edge gadgets (see Figure 2(b)). Note that in any proper weighting w from $\{1, 2\}$, $w(vx) \neq w(vy)$; otherwise, since the weight contributed by gadgets to $\chi_w(x)$ and $\chi_w(y)$ is $k-3$, then $\chi_w(x) = w(xv) + w(xy) + k - 3 = w(yv) + w(yx) + k - 3 = \chi_w(y)$. Therefore, for any k, if $w(xy) = 1$ then $\{\chi_w(x), \chi_w(y)\} = \{k-1, k\}$ and, if $w(xy) = 2$ then $\{\chi_w(x), \chi_w(y)\} = \{k, k+1\}$. In either case, v has one neighbor $z \in \{x, y\}$ with $\chi_w(z) = k$, and consequently, $\chi_w(v) \neq k$ in any proper weighting from $\{1, 2\}$. Also $\{w(vx), w(vy)\} = \{1, 2\}$, and hence this gadget contributes exactly 3 to $\chi_w(v)$.

Now we are ready to show a polynomial time reduction from 3-COLOR to 1-2WEIGHT, f, such that $G \in$ 3-COLOR if and only if $f(G) \in$ 1-2WEIGHT. Let $G = (V, E)$ be a graph of order n. As in Section 2, we may assume that $n \geq 3$. We construct the graph $f(G) = (W, F)$ as follows (see Figure 2(d)). We start with $G = (V, E)$. For each $v \in V$:

(i) connect v to two new vertices s and t (distinct for each v);

(ii) connect v to all vertices from a new set U_v (distinct for each v) with $|U_v| = n - 1 - deg(v)$;

(iii) add $n-1$ new k-disallowing gadgets for all $k \in \{4n+1, 4n+2, \dots, 5n-1\}$ with v as their root.

Clearly, $f(G)$ can be calculated in time polynomial in the size of G.

Fact 3.1 *In $f(G)$ the following holds: any proper weighting w from $\{1,2\}$ satisfies $\chi_w(v) \in \{4n-2, 4n-1, 4n\}$ for every $v \in V$.*

Proof: Fix $v \in V$. Since $w(vs) + w(vt) \in \{2,3,4\}$, v is the endpoint of $n-1$ edges with endpoints in $V \cup U_v$ and v is the root of $(n-1)$ k-disallowing gadgets (each contributing 3 to $\chi_w(v)$), we have:

$$\chi_w(v) \in \{2,3,4\} + \{n-1, \dots, 2n-2\} + \{3n-3\} = \{4n-2, \dots, 5n-1\}.$$

Observing the above and the fact that v is the root of k-disallowing gadgets for all $k \in \{4n+1, 4n+2, \dots, 5n-1\}$, we find that any proper weighting w from $\{1,2\}$ satisfies $\chi_w(v) \in \{4n-2, 4n-1, 4n\}$, as claimed. ☐

Now we show that $G \in$ 3-COLOR if and only if $f(G) \in$ 1-2WEIGHT.

First let us assume that $G \in$ 3-COLOR. That means there exists a proper 3-coloring of G, say $\chi : V \to \{4n-2, 4n-1, 4n\}$. We define a weighting of the edges of $f(G)$, $w : F \to \{1,2\}$ as follows. For all $e \in E$ let $w(e) = 1$. For all edges $e = vu$ with $v \in V$ and $u \in U_v$ we set $w(e) = 1$. For all $v \in V$, if $\chi(v) = 4n-2$ then $w(vs) = w(vt) = 1$; otherwise, if $\chi(v) = 4n-1$ then $w(vs) = 1$ and $w(vt) = 2$; finally, if $\chi(v) = 4n$ then $w(vs) = w(vt) = 2$. All other edges (parts of gadgets) are weighted as follows: For a triangle gadget xab with root x, $w(xa) = 2, w(xb) = w(ab) = 1$. For a 2-gadget defined by $\{x, c, d, e, f\}$ with x adjacent to c, we have $w(xc) = w(cd) = w(ce) = w(de) = w(df) = 2$ and $w(cf) = w(ef) = 1$. For a k-disallowing gadget with root v and main triangle vxy, $w(vx) = w(xy) = 2, w(vy) = 1$, and the weighting of all other gadgets as described above. Note that w is a proper weighting of $f(G)$ (satisfying $\chi_w(v) = \chi(v)$ for all $v \in V$), as required.

Next let us assume that $G \notin$ 3-COLOR. Therefore, for all $\chi : V \to \{4n-2, 4n-1, 4n\}$, χ is not a proper vertex coloring. But, from Fact 3.1, any proper weighting from $\{1,2\}$ of $f(G)$ satisfies $\chi_w(v) \in \{4n-2, 4n-1, 4n\}$ for all $v \in V$. Thus, there is no such proper weighting and hence $f(G) \notin$ 1-2WEIGHT.

This concludes the proof of Theorem 1.1.

4 Concluding remarks

In this paper we showed that determining whether a graph has a proper weighting from either $\{0,1\}$ or $\{1,2\}$ is NP-complete. As a matter of fact, these two problems are not the same, in the sense that the corresponding families of graphs 0-1WEIGHT and 1-2WEIGHT are not equal. For example, the graph consisting only of one 2-edge gadget is in 1-2WEIGHT, as seen before, but it is easy to check that it is not in 0-1WEIGHT. Furthermore, we believe that our approach can be generalized to show that determining whether a graph has a proper weighting from $\{a, b\}$ is NP-complete for any different rational numbers a and b. It is not clear if the same would hold for any two distinct irrational numbers.

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