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Convex Partitions of Graphs induced by Paths of Order Three

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A set C of vertices of a graph G is P_3-convex if $v \in C$ for every path uvw in G with $u, w \in C$. We prove that it is NP-complete to decide for a given graph G and a given integer p whether the vertex set of G can be partitioned into p non-empty disjoint P_3-convex sets. Furthermore, we study such partitions for a variety of graph classes.

Keywords: graph convexity; convex partition

1 Introduction

In [4] Bollobás considers "The spread of an infection on a square grid". Initially some squares of a square grid are infected. The infection spreads to an uninfected square if at least two of the neighbouring squares are infected. A natural combinatorial problem — solved using an elegant perimeter argument in [4] — is to determine the minimum number of initially infected squares such that eventually all squares become infected. This kind of spreading process has received considerable attention in a variety of contexts such as social influence [15], percolation [3], marketing strategies [10], and distributed computing [25]. In the present paper we study the above process under the perspective of graph convexity.

All graphs will be finite, simple, and undirected and we use common terminology and notation. Given a graph G and a collection $\mathcal{C}$ of subsets of the vertex set $V(G)$ of G, the pair $(G, \mathcal{C})$ is a *graph convexity* if $\emptyset \in \mathcal{C}$, $V(G) \in \mathcal{C}$, and $\mathcal{C}$ is closed under intersection. The sets in $\mathcal{C}$ are called *convex*. The *convex hull* of some set S of vertices of some graph G with respect to some graph convexity $(G, \mathcal{C})$ is the unique smallest set $H_{\mathcal{C}}(S)$ in $\mathcal{C}$ containing S. Several natural graph convexities are defined in terms of paths. If $\mathcal{P}$ is a set of paths in some graph G and $\mathcal{C}$ is the collection of all subsets C of $V(G)$ such that, for every path P in $\mathcal{P}$ between two vertices in C, the set C contains all vertices of P, then $(G, \mathcal{C})$ is a graph convexity.

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Well-studied examples are *geodetic convexity* [18, 20], where $\mathcal{P}$ consists of all shortest paths, *monophonic convexity* [16, 19], where $\mathcal{P}$ consists of all induced paths, *triangle-path convexity* [8], and m_3-*convexity* [14]. In order to capture the above spreading process we choose $\mathcal{P}_3$ as the set of all paths of order 3 in some graph G. Specifically, we call a set C of vertices in a graph G P_3-*convex* if, for every path uvw in G with $u, w \in C$, the vertex v belongs to C. (Note that all vertices of a path are distinct.) Equivalently, C is P_3-convex in G if and only if no vertex in $V(G) \setminus C$ has at least two neighbours in C. Let $(G, \mathcal{C}_3)$ denote the graph convexity where $\mathcal{C}_3$ is the collection of all P_3-convex sets of G. Convexities induced by paths of order 3 have been studied before in directed graphs and especially tournaments [17, 24, 26]. In [7] some results concerning P_3-convexity were obtained for undirected graphs.

There are some natural parameters associated with a graph convexity $(G, \mathcal{C})$. The *convexity number* of $(G, \mathcal{C})$ is the maximum cardinality of a proper convex subset of $V(G)$, and the *hull number* [6, 11, 13, 15, 18] of $(G, \mathcal{C})$ is the minimum cardinality of a set S of vertices such that $H_{\mathcal{C}}(S) = V(G)$. Furthermore, if $(G, \mathcal{C})$ is defined in terms of a set $\mathcal{P}$ of paths in G as explained above, then the *geodetic number* [9, 12] of $(G, \mathcal{C})$ is the minimum cardinality of a set S of vertices such that the union of the vertex sets of all paths in $\mathcal{P}$ between vertices in S equals $V(G)$. The geodetic number of the graph convexity $(G, \mathcal{C}_3)$ coincides with the well-studied 2-*domination number* [21, 22]. The problem considered by Bollobás in [4] concerns the hull number of $(G, \mathcal{C}_3)$ where G is the square grid $P_n \times P_n$. For further recent results concerning the hull number of $(G, \mathcal{C}_3)$, we refer the reader to [15, 6].

In the present paper we study the problem of partitioning the vertex set of a graph into P_3-convex sets. We establish the computational hardness of the corresponding decision problem in general and obtain results for several well-known graph classes [5]. Furthermore, we establish the hardness of the convexity number of $(G, \mathcal{C}_3)$. For analogous results concerning geodetic convexity, we refer to [2].

Before we proceed to the results, we recall some basic definitions. An *independent set* in a graph is a set of pairwise non-adjacent vertices. A *clique* in a graph is a set of pairwise adjacent vertices. A *colouring* of a graph is a partition of its vertex set into independent sets. The *chromatic number* of a graph G is the minimum number $\chi(G)$ of independent sets into which its vertex set can be partitioned. A graph is 2-*connected* if it is connected and the removal of a single vertex does not disconnect it. The k-*th power* of a graph G is the graph G^k with the same vertices as G such that two distinct vertices are adjacent in G^k if and only if their distance in G is at most k. The *union* of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. For two graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$, the *join* of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \wedge v \in V(G_2)\}$. A vertex u in a graph G is *universal* if it is adjacent to all vertices in $V(G) \setminus \{u\}$. A graph whose vertex set can be partitioned into an independent set and a clique is a *split graph*. A graph that does not contain an induced cycle of length at least 4 is *chordal*. A graph G is an *interval graph*, if there is a collection of intervals of the real line indexed by the elements of $V(G)$ such that two distinct vertices are adjacent in G if and only if the two corresponding intervals intersect.

2 Results

A P_3*-convex* p*-partition* of a graph is a partition of its vertex set into p disjoint non-empty P_3-convex sets. Clearly, the two trivial partitions into just one set containing all vertices of some graph G and into $|V(G)|$ distinct sets each containing exactly one vertex from $V(G)$ are both P_3-convex partitions. Hence every graph G has a P_3-convex p-partition for $p \in \{1, |V(G)|\}$. Not excluding these two trivial partitions it follows that — unlike as for many other graph partition notions such as for instance graph colourings — there is no natural minimization or maximization problem related to the existence of P_3-convex partitions. In fact, the most natural problem in this context seems to be to characterize/determine those values of p for which some given graph G has a P_3-convex p-partition. This leads to the following decision problem.

P_3-CONVEX PARTITION
Instance: A graph G and a positive integer p.
Question: Does G have a P_3-convex p-partition?

Our first result is the hardness of this problem.

Theorem 2.1 P_3-CONVEX PARTITION *is NP-complete even when restricted to split graphs.*

Since we derive this result from Theorem 2.6, we postpone its proof for after that result.

As mentioned in the Introduction, the geodetic number and the hull number of the graph convexity $(G, \mathcal{C}_3)$ have already been studied in different contexts. In [7] it was proved that the following problem is NP-complete for general graphs.

P_3-CONVEXITY NUMBER
Instance: A graph G and a positive integer k.
Question: Does G have a P_3-convex set C with $k \leq |C| < V(G)$?

It is not difficult to see that the construction given in [7] can be modified such that one obtains NP-completeness already restricted to split graphs. For the sake of completeness, we give a direct proof.

Theorem 2.2 P_3-CONVEXITY NUMBER *is NP-complete even when restricted to split graphs.*

Proof: P_3-CONVEXITY NUMBER is clearly in NP. In order to complete the proof, we describe a reduction from 3-DIMENSIONAL MATCHING ([SP1] in [23]). An instance of 3-DIMENSIONAL MATCHING is a 3-uniform hypergraph $\mathcal{H} = (X, \mathcal{E})$ where X has a partition $X_1 \cup X_2 \cup X_3$ such that every hyperedge in $\mathcal{E}$ contains exactly one vertex from each X_i for $1 \leq i \leq 3$, and the question is whether $\mathcal{H}$ has a perfect matching, i.e. whether there is a set $\mathcal{M} \subseteq \mathcal{E}$ of hyperedges such that every vertex u in X is contained in exactly one of the hyperedges in $\mathcal{M}$.

Let $(X, \mathcal{E})$ be an instance of 3-DIMENSIONAL MATCHING. Clearly, we may assume that $|X| \equiv 0 \bmod 3$ and that $|X| > 3$. We construct a split graph G and specify an integer k such that the encoding length of (G, k) is polynomially bounded in terms of the encoding length of $\mathcal{H}$ and $\mathcal{H}$ has a perfect matching if and only if G has a P_3-convex set C with $k \leq |C| < V(G)$. Let G have vertex set $V(G) = X \cup \mathcal{E}$ such that X is a clique in G, $\mathcal{E}$ is an independent set in G, and $x \in X$ and $e \in \mathcal{E}$ are adjacent in G if and only if $x \in e$. Let $k = \frac{|X|}{3}$.

If $\mathcal{M}$ is a perfect matching of $\mathcal{H}$, then $\mathcal{M}$ is a P_3-convex set in G of order k. Conversely, let U be a P_3-convex set in G with $k \leq |U| < |V(G)|$. If U contains two elements of X, then, by construction,

$U = V(G)$, which is a contradiction. If U contains a vertex in X and a vertex in $\mathcal{E}$, then, by construction, U contains two vertices in X, which is a contradiction. Since $k \geq 2$, this implies that $U \subseteq \mathcal{E}$. Since no vertex in X is contained in two of the hyperedges in U, we have $|U| \leq \frac{|X|}{3} = k$. Hence $|U| = k$ and U is a perfect matching of $\mathcal{H}$. $\square$

Our next result relates the existence of P_3-convex p-partitions of some graph to properties of suitable subgraphs. This result allows the design of efficient algorithms solving P_3-CONVEX PARTITION for classes of graphs, which have suitable decompositions. The statements (ii), (iii), and (iv) of Theorem 2.3 for instance can be used to efficiently determine the existence of P_3-convex p-partitions of cographs.

Theorem 2.3 *Let G be a graph of order n.*

(i) If H is a subgraph of G, $V_1 \cup V_2 \cup \cdots \cup V_p$ is a P_3-convex p-partition of G, $U_i = V_i \cap V(H) \neq \emptyset$ for $1 \leq i \leq p'$, and $V_i \cap V(H) = \emptyset$ for $p' + 1 \leq i \leq p$, then $U_1 \cup U_2 \cup \cdots \cup U_{p'}$ is a P_3-convex p'-partition of H.

(ii) If G is the disjoint union of the two graphs G_1 and G_2, then G has a P_3-convex p-partition if and only if there is a P_3-convex p_1-partition of G_1 and a P_3-convex p_2-partition of G_2 such that $\max\{p_1, p_2\} \leq p \leq p_1 + p_2$.

(iii) If G arises by adding a universal vertex u to some graph G', then G has a P_3-convex p-partition if and only if $p = n$ or there is a set U that is the union of the vertex sets of some components of G' such that $p = n - |U|$.

(iv) If G is the join of two disjoint graphs G_1 and G_2 such that the orders of G_1 and G_2 are at least 2, G_1 has n_1 isolated vertices, and G_2 has n_2 isolated vertices, then G has a P_3-convex p-partition if and only if $p = 1$ or $n - \min\{n_1, n_2\} \leq p \leq n$.

(v) If G is the union of two graphs G_1 and G_2 and $\{u\} = V(G_1) \cap V(G_2)$, then G has a P_3-convex p-partition if and only if there is a P_3-convex p_1-partition of G_1 in which exactly p_1' sets contain a neighbour of u but not u and a P_3-convex p_2-partition of G_2 in which exactly p_2' sets contain a neighbour of u but not u such that

$$1 + p_1' + p_2' + \max\{0, p_1 - p_1' - p_2' - 1, p_2 - p_1' - p_2' - 1\} \leq p \leq p_1 + p_2 - 1.$$

Proof: (i) Since for every $1 \leq i \leq p'$, no vertex $u \in V(G) \setminus V_i$ has two neighbours in V_i and $U_i \subseteq V_i$, the partition $U_1 \cup U_2 \cup \cdots \cup U_{p'}$ of $V(H)$ is a P_3-convex p'-partition of H.

(ii) Let $\mathcal{P} : V_1 \cup V_2 \cup \cdots \cup V_p$ be a P_3-convex p-partition of G. Let $V_1^1 \cup V_2^1 \cup \cdots \cup V_{p_1}^1$ and $V_1^2 \cup V_2^2 \cup \cdots \cup V_{p_2}^2$ denote the two P_3-convex partitions of G_1 and G_2, respectively, that are induced by $\mathcal{P}$ exactly as in part (i). Clearly, $\max\{p_1, p_2\} \leq p \leq p_1 + p_2$.

Conversely, let $V_1^1 \cup V_2^1 \cup \cdots \cup V_{p_1}^1$ and $V_1^2 \cup V_2^2 \cup \cdots \cup V_{p_2}^2$ be two P_3-convex partitions of G_1 and G_2, respectively. Let p be such that $\max\{p_1, p_2\} \leq p \leq p_1 + p_2$. Let

$$V_i = \begin{cases} V_i^1 \cup V_i^2 & , 1 \leq i \leq (p_1 + p_2) - p, \\ V_i^1 & , (p_1 + p_2 - p) + 1 \leq i \leq p_1, \\ V_{(p_2 - p) + i}^2 & , p_1 + 1 \leq i \leq p. \end{cases}$$

Since no vertex of G has a neighbour in $V(G_1)$ and a neighbour in $V(G_2)$, $V_1 \cup V_2 \cup \cdots \cup V_p$ is a P_3-convex p-partition of G.

(iii) Let $V_1 \cup V_2 \cup \cdots \cup V_p$ be a P_3-convex p-partition of G with $p < n$. Let $u \in V_1$. Clearly, $|V_i| = 1$ for $2 \leq i \leq p$. If $vw \in E(G')$ is such that $v \in V_1$, then $w \in V_1$. This implies that $U = V_1 \setminus \{u\}$ is the union of the vertex sets of some components of G', and hence $p = n - |U|$.

Conversely, if U is the union of the vertex sets of some components of G', then $U \cup \{u\}$ together with the $n - 1 - |U|$ sets each containing exactly one vertex from $V(G') \setminus U$ yields a P_3-convex $(n - |U|)$-partition of G.

(iv) Let U be a proper subset of $V(G)$. It is easy to see that U is P_3-convex if and only if U does not contain two vertices from $V(G_1)$ or two vertices from $V(G_2)$, and does not contain a vertex $u_1 \in V(G_1)$ and a vertex $u_2 \in V(G_2)$ such that u_1 is not isolated in G_1 or u_2 is not isolated in G_2. Therefore the P_3-convex sets of G are either of cardinality 1 or consist of one isolated vertex from G_1 and one isolated vertex from G_2, which easily implies the desired statement.

(v) Let $\mathcal{P} : V_1 \cup V_2 \cup \cdots \cup V_p$ be a P_3-convex p-partition of G. Let $\mathcal{P}_1 : V_1^1 \cup V_2^1 \cup \cdots \cup V_{p_1}^1$ and $\mathcal{P}_2 : V_1^2 \cup V_2^2 \cup \cdots \cup V_{p_2}^2$ denote the two P_3-convex partitions of G_1 and G_2, respectively, that are induced by $\mathcal{P}$ exactly as in (i). Since the partite set in $\mathcal{P}$ containing u contributes a non-empty set to both induced partitions $\mathcal{P}_1$ and $\mathcal{P}_2$, we have $p \leq p_1 + p_2 - 1$. Let $\mathcal{P}_1$ contain exactly p_1' sets that contain a neighbour of u but not u, and let $\mathcal{P}_2$ contain exactly p_2' sets that contain a neighbour of u but not u. If $V_i = V_r^1 \cup V_s^2$ for some indices i, r, and s, then either (a) $u \in V_r^1 \cap V_s^2$, or (b) $u \notin V_r^1 \cup V_s^2$ and either V_r^1 or V_s^2 does not contain a neighbour of u.

Case (a) occurs exactly once. Case (b) occurs at least $p_1' + p_2'$ times for the partite sets in $\mathcal{P}_1 \cup \mathcal{P}_2$ that contain a neighbour of u but not u and at least $\max\{0, p_1 - p_1' - p_2' - 1, p_2 - p_1' - p_2' - 1\}$ times for the remaining partite sets in $\mathcal{P}_1 \cup \mathcal{P}_2$. Altogether, this implies that p is at least $1 + p_1' + p_2' + \max\{0, p_1 - p_1' - p_2' - 1, p_2 - p_1' - p_2' - 1\}$.

Conversely, let $\mathcal{P}_1 : V_1^1 \cup V_2^1 \cup \cdots \cup V_{p_1}^1$ and $\mathcal{P}_2 : V_1^2 \cup V_2^2 \cup \cdots \cup V_{p_2}^2$ be two P_3-convex partitions of G_1 and G_2, respectively. We may assume that $u \in V_1^1 \cap V_1^2$. Let p_1' and p_2' be as in the statement of (v). Let $\mathcal{P}_1'$ contain the partite sets in $\mathcal{P}_1$ that contain a neighbour of u but not u and let $\mathcal{P}_2'$ contain the partite sets in $\mathcal{P}_2$ that contain a neighbour of u but not u. Note that $p_1' = |\mathcal{P}_1'|$ and $p_2' = |\mathcal{P}_2'|$. Let $q = 1 + p_1' + p_2' + \max\{0, p_1 - p_1' - p_2' - 1, p_2 - p_1' - p_2' - 1\}$ and let p be such that $q \leq p \leq p_1 + p_2 - 1$. Let $V_1 \cup V_2 \cup \cdots \cup V_q$ be a partition of $V(G)$ such that one partite set is $V_1^1 \cup V_1^2$, p_1' partite sets are the union of a partite set in $\mathcal{P}_1'$ with one or zero partite set in $\mathcal{P}_2 \setminus \mathcal{P}_2'$, p_2' partite sets are the union of a partite set in $\mathcal{P}_2'$ with one or zero partite set in $\mathcal{P}_1 \setminus \mathcal{P}_1'$, and $\max\{0, p_1 - p_1' - p_2' - 1, p_2 - p_1' - p_2' - 1\}$ partite sets are the union of one or zero partite set in $\mathcal{P}_1 \setminus \mathcal{P}_1'$ with one or zero partite set in $\mathcal{P}_2 \setminus \mathcal{P}_2'$. Such a partition clearly exists and is a P_3-convex q-partition of G. Now, replacing $p - q$ of the partite sets V_i with $i \geq 2$ that are the union of two sets V_r^1 and V_s^2 with these two sets results in a P_3-convex p-partition of G. This completes the proof. $\square$

2.1 Powers of Cycles

In this subsection, we consider P_3-convex partitions of the k-th power C_n^k of the cycle C_n of order n. Since the complete graph K_n of order n has a P_3-convex p-partition if and only if p is 1 or n, we focus on non-complete cycle powers, i.e. $k < \left\lfloor \frac{n}{2} \right\rfloor$.

We denote the *chromatic number of a graph G* by $\chi(G)$.

Theorem 2.4 *Let $n, k \in \mathbb{N}$ be such that $n \geq 3$ and $1 \leq k < \left\lfloor \frac{n}{2} \right\rfloor$. C_n^k has a P_3-convex p-partition if and only if*

either $p = 1$,

or $k = 1$ and $3 \leq p \leq n$,

or $k = 1$, $n \geq 4$, and $p = 2$,

or $k \geq 2$ and $p \geq \chi(C_n^{2k})$.

Proof: Clearly, C_n^k has the two trivial P_3-convex p-partitions for $p = 1$ and $p = n$. If $k = 1$, then every set of at most $n - 2$ consecutive vertices of C_n is P_3-convex in C_n^1. This clearly implies all statements concerning $k = 1$. Now let $k \geq 2$. Let U be a proper subset of $V(C_n^k)$ that is P_3-convex in C_n^k. If U contains two consecutive vertices of C_n, then $U = V(C_n)$, which is a contradiction. If U contains two distinct vertices u and v of distance at most $2k$ in C_n, then U contains all vertices on shortest paths in C_n between u and v and hence $U = V(C_n)$, which is a contradiction. This implies that U is an independent set of C_n^{2k}. Conversely, every independent set U of C_n^{2k} is a proper subset of $V(C_n^k)$ that is P_3-convex in C_n^k, because no vertex in $V(C_n) \setminus U$ has two neighbours in U. Therefore, C_n^k has a P_3-convex p-partition if and only if C_n^{2k} has a p-colouring, which implies all statements concerning $k \geq 2$. $\square$

Theorem 2.4 refered to the chromatic number of C_n^{2k}. We believe that the following result should be known. Since we have not been able to find a reference, we include the simple proof for completeness.

Lemma 2.5 *If $n, k \in \mathbb{N}$ are such that $2 \leq k < \left\lfloor \frac{n}{2} \right\rfloor$, then $\chi(C_n^k) = \left\lceil \frac{n}{\left\lfloor \frac{n}{k+1} \right\rfloor} \right\rceil$.*

Proof: A set of vertices I that is independent in C_n^k does not contain two vertices whose distance in C_n is at most k. This implies that every independent set of C_n^k contains at most $\left\lfloor \frac{n}{k+1} \right\rfloor$ vertices and hence $\chi(C_n^k)$ is at least the expression stated in the lemma. Let $n = (k+1)r + s$ for integers r and s with $0 \leq s \leq k$. We choose r disjoint sets $V_1, V_2, \ldots, V_r$ each containing exactly $k+1$ consecutive vertices on C_n such that each of the components of $C_n - (V_1 \cup V_2 \cup \cdots \cup V_r)$ contains either $\left\lfloor \frac{s}{r} \right\rfloor$ or $\left\lceil \frac{s}{r} \right\rceil$ many vertices. By construction, there is a colouring of C_n^k using $k+1$ distinct colours for the vertices in $V_1 \cup V_2 \cup \cdots \cup V_r$ and $\left\lceil \frac{s}{r} \right\rceil$ distinct colours for the remaining vertices. This implies

$$\chi(C_n^k) \leq (k+1) + \left\lceil \frac{s}{r} \right\rceil = \left\lceil (k+1) + \frac{s}{r} \right\rceil = \left\lceil \frac{(k+1)r + s}{r} \right\rceil = \left\lceil \frac{n}{r} \right\rceil = \left\lceil \frac{n}{\left\lfloor \frac{n}{k+1} \right\rfloor} \right\rceil$$

and hence $\chi(C_n^k)$ is at most the expression stated in the lemma, which completes the proof. $\square$

2.2 Split Graphs

The next result captures the possibilities for the P_3-convex partitions of split graphs.

Theorem 2.6 *Let G be a split graph such that $V(G)$ is the disjoint union of an independent set I and a clique K. Let $I_{\leq 1} = \{u \in I \mid d_G(u) \leq 1\}$ and let $I_{\geq 2} = I \setminus I_{\leq 1}$. Let $H = G^2[I_{\geq 2}]$.*
G has a P_3-convex p-partition if and only if $1 \leq p \leq 1 + |I_{\leq 1}|$ or $\chi(H) + |K| \leq p \leq n$.

Proof: Let $\mathcal{P} : V_1 \cup V_2 \cup \cdots \cup V_p$ be a P_3-convex p-partition of G. Clearly, either K is a subset of one partite set or every partite set contains at most one vertex from K. If K is the subset of one partite set, say V_1, then $I_{\geq 2} \subseteq V_1$ and hence $1 \leq p \leq 1 + |I_{\leq 1}|$. If every partite set contains at most one vertex from K, then no partite set contains a vertex from $I_{\geq 2}$ as well as a vertex from K. Furthermore, no partite set contains two vertices from $I_{\geq 2}$ at distance (at most) 2. Therefore, the partition of I induced by $\mathcal{P}$ yields a colouring of H and the partition of K induced by $\mathcal{P}$ consists of $|K|$ sets. Altogether we obtain $\chi(H) + |K| \leq p \leq n$.

For the converse, let p be such that $1 \leq p \leq 1 + |I_{\leq 1}|$ or $\chi(H) + |K| \leq p \leq n$. If $1 \leq p \leq 1 + |I_{\leq 1}|$, then G has a P_3-convex p-partition in which one partite set consists of $I_{\geq 2}$, K, and a subset of $I_{\leq 1}$ of cardinality $1 + |I_{\leq 1}| - p$, and all remaining $p - 1$ partite sets consist of exactly one vertex from $I_{\leq 1}$. If $p = \chi(H) + |K|$, then G has a P_3-convex p-partition, which, for every $u \in K$, contains one partite set that contains $\{u\} \cup (N_G(u) \cap I_{\leq 1})$ together with some isolated vertices of G, and contains the partite sets of a partition of $I_{\geq 2} = V(H)$ into $\chi(H)$ sets that are independent in H. Clearly, replacing a partite set V_i of this P_3-convex partition that satisfies $V_i \subseteq I_{\geq 2}$ with two non-empty subsets of V_i results in a P_3-convex partition of G. Similarly, replacing a partite set V_i of this P_3-convex partition that contains one vertex of K and one vertex u of $I_{\leq 1}$ with the two sets $V_i \setminus \{u\}$ and $\{u\}$ results in a P_3-convex partition of G. Iteratively applying these two operations, we obtain P_3-convex p-partitions for all p with $\chi(H) + |K| \leq p \leq n$, which completes the proof. $\Box$

We can now give the

Proof of Theorem 2.1: P_3-CONVEX PARTITION is clearly in NP. In order to complete the proof, we describe a reduction from CHROMATIC NUMBER ([GT4] in [23]). Given an instance (H, k) of CHROMATIC NUMBER, we construct a split graph G and specify an integer p such that the encoding length of (G, p) is polynomially bounded in terms of the encoding length of (H, k) and H has a vertex colouring with k colours if and only if G has a P_3-convex p-partition. The vertex set of G is the union of an independent set I of order $|V(H)|$ and a clique C of order $2|V(H)| + |E(H)|$. Let $f : V(H) \to I$ be bijective. For every vertex $u \in V(H)$, the vertex $f(u)$ has two neighbours $g(u)$ and $g'(u)$ in C. Furthermore, for every edge $uv \in E(H)$, the vertices $f(u)$ and $f(v)$ have a common neighbour $g(uv)$ in C. In view of the order of C, we may assume that all specified vertices in C are distinct, i.e. $C = \{g(u) \mid u \in V(G)\} \cup \{g'(u) \mid u \in V(G)\} \cup \{g(uv) \mid uv \in E(G)\}$. Let $p = k + |C|$.

Now it follows easily from Theorem 2.6 that H has a vertex colouring with k colours if and only if G has a P_3-convex p-partition, which completes the proof. $\Box$

2.3 Chordal Graphs

In this subsection, we study chordal graphs.

Theorem 2.7 *Let G be a 2-connected chordal graph.*

(i) If U is a proper subset of $V(G)$, then U is a P_3-convex set of G if and only if U is an independent set of the square G^2 of G.

(ii) G has a P_3-convex p-partition if and only if $p = 1$ or $p \geq \chi(G^2)$.

Proof: (i) Since the 'if'-part is obvious, we proceed to the proof of the 'only if'-part. For contradiction, we assume that U contains two vertices x and y with distance at most 2. We may assume that U is the smallest P_3-convex set containing x and y. Clearly, U contains all vertices on shortest paths between x and y and $G[U]$ is connected. Since $U \neq V(G)$, there is an edge $uv \in E(G)$ with $u \in V(G) \setminus U$ and $v \in U$. Since G is 2-connected, there are two internally vertex-disjoint paths between u and a vertex in $U \setminus \{v\}$. This implies the existence of a path $P : v_1 v_2 \ldots v_l$ in G with $l \geq 3$, $v_1, v_l \in U$, and $v_2, \ldots, v_{l-1} \in V(G) \setminus U$. Choosing P such that l is minimum possible, the chordality of G implies that $l = 3$, which implies the contradiction $v_2 \in U$.

(ii) By (i), G has a P_3-convex p-partition if and only if G^2 has a p-colouring. □

Note that if G is a split graph and the set $I_{\leq 1}$ as defined in Theorem 2.6 is empty, then G is a 2-connected chordal graph and the statement of Theorem 2.6 follows from Theorem 2.7 (ii).

By Theorem 2.7, P_3-convex partitions of a chordal graph G are intimately linked to colourings of its square G^2. Unfortunately, it is even hard to approximate the chromatic number of squares of chordal graphs of order n within a factor of $O\left(n^{\frac{1}{2}-\epsilon}\right)$ for any $\epsilon > 0$ [1]. Agnarsson et al. [1] establish such a hardness result already for squares of split graphs and their construction is quite similar to the construction used in the proof of our Theorem 2.1. Since interval graphs and squares of interval graphs are chordal, Theorem 2.7 allows to solve P_3-CONVEX PARTITION efficiently for 2-connected interval graphs.

Note that applying (v) of Theorem 2.3 to chordal graphs leads to some simplifications. In fact, if G is a 2-connected chordal graph, $u \in V(G)$, and $V_1 \cup V_2 \cup \cdots \cup V_p$ is a P_3-convex p-partition of G with $p \geq 2$, then, by Theorem 2.7, the partite set containing u does not contain any neighbour of u and every partite set contains at most one neighbour of u.

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