On hamiltonian chain saturated uniform hypergraphs

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received 15th April 2010, revised 5th September 2011, accepted 22nd November 2011.

We say that a hypergraph H is hamiltonian chain saturated if H does not contain a hamiltonian chain but by adding any new edge we create a hamiltonian chain in H. In this paper we ask about the smallest size of a k-uniform hamiltonian chain saturated hypergraph. We present a construction of a family of k-uniform hamiltonian chain saturated hypergraphs with $O(n^{k-1/2})$ edges.

Keywords: saturated hypergraph, hamiltonian path, hamiltonian cycle, hamiltonian chain.

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1 Introduction

Let H be a k-uniform hypergraph on the vertex set $V(H) = \{v_1, ..., v_n\}$ with $n \geq k$. The set of the edges — k-element subsets of $V(H)$ — is denoted by $\mathcal{E}(H)$. For simplicity of notation v_{n+x} with $x \geq 0$ denotes the same vertex as v_x. In [19] the authors defined the notion of a hamiltonian chain.

Definition 1 A cyclic ordering $(v_1, v_2, \ldots, v_n)$ of the vertex set is called a *hamiltonian chain* and denoted $C_n^{(k)}$, if and only if $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\} \in \mathcal{E}(H)$ whenever $1 \leq i \leq n$. An ordering $(v_1, v_2, \ldots, v_l)$ of a subset of the vertex set is called a *path* and denoted $P_l^{(k)}$, if and only if $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\} \in \mathcal{E}(H)$ whenever $1 \leq i \leq l-k+1$. A path $P_n^{(k)}$ is a *hamiltonian path.*

For $v \in V(H)$, let $H - v$ be the hypergraph obtained by deleting v and all edges incident to v. We refer to this operation as *removing* v from H.

Definition 2 We say that a hypergraph H is *hamiltonian chain saturated* if H does not contain a hamiltonian chain but by adding any new edge we create a hamiltonian chain in H.

The question investigated in this paper is the following: *What is the minimum number $sat(n, C_n^{(k)})$ of edges in a hamiltonian chain saturated k-uniform hypergraphs on n vertices?*

In fact, the above problem belongs to the much wider theory of saturated graphs and hypergraphs. Given a hypergraph F, we say that the hypergraph H is F-saturated if H has no F as a subhypergraph,

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but does contain F after the addition of any new edge. The minimum number of edges in an F-saturated hypergraph on n vertices is denoted by $sat(n, F)$ (the maximum number is denoted by $ex(n, F)$ and the problem of determining $ex(n, F)$ is the Turán's problem). There are many results on $sat(n, F)$ for graphs, including complete graphs [14], cycles [1, 2, 6, 16, 17, 20, 23], complete s-partite graphs [3, 7, 8, 21]. In particular, the problem considered in the paper is solved for graphs. Namely, $sat(n, C_n) = \lceil \frac{3n}{2} \rceil$ (apart from a few small values of n) which follows from [5, 9, 10]. Furthermore, $sat(n, P_n) = \lfloor \frac{3n-1}{2} \rfloor$ for $n \geq 54$ [11, 15]. Much less is known for $k \geq 3$. Bollobás [4] generalized Erdős, Hajnal and Moons result [14] for complete k-uniform hypergraphs. Erdős, Füredi and Tuza [13] obtained $sat(n, F)$ for some particular hypergraphs F with few edges. Pikhurko [22] proved that $sat(n, F) = O(n^{k-1})$ for any fixed hypergraph F (generalizing previous result for graphs by Erdős, Füredi and Tuza [13]). The same order of magnitude was conjectured for $sat(n, C_n^{(k)})$ [18] and $sat(n, P_n^{(k)})$ [12]. It is known that both $sat(n, C_n^{(k)})$ and $sat(n, P_n^{(k)})$ are at least $\binom{n}{k}/(k(n-k)+1)$, see [12]. Hence the lower bound is of order n^{k-1}. On the other hand in [19] a construction is given of n-vertex hamiltonian path saturated k-uniform hypergraphs with

$$\sim \left(\frac{1}{k!} - \frac{1}{2^k \lceil k/2 \rceil! \lfloor k/2 \rfloor!}\right) n^k$$

edges. In case when $k = 3$, better constructions presented in [12] give hamiltonian path saturated 3-uniform hypergraphs and hamiltonian chain saturated 3-uniform hypergraphs with $O(n^{5/2})$ edges. In this paper we present a non-obvious generalization of these construction for all $k \geq 4$. As a result we obtain that

$$sat(n, C_n^{(k)}) \leq \left(\frac{1}{k!} + \frac{1}{(k-1)!}\right) n^{k-1/2} + o\left(n^{k-1/2}\right).$$

We notice also that the same bound holds for $sat(n, P_n^{(k)})$.

2 Construction

In the sequel we assume that k is a fixed integer greater than or equal to 4.

Definition 3 Let p and q be non-negative integers and $U_0, U_1, \ldots, U_q$ be pairwise disjoint sets of vertices such that $|U_0| = p$ and $|U_i| \geq 2$ for $i = 1, 2, \ldots, q$. Define the vertex set of the hypergraph H to be $V(H) = \bigcup_{i=0}^{q} U_i$. Let $u_1 \in U_{i_1}, ..., u_k \in U_{i_k}$ and $j = \min\{i_1, ..., i_k\}$. Then $\{u_1, ..., u_k\} =: E \in \mathcal{E}(H)$ if and only if $j = 0$ or $|E \cap U_j| \geq 2$. The family of all hypergraphs obtained by this construction is denoted by $\mathcal{H}_k(p, q)$.

Notation The above number j that is determined by the edge E is denoted by j_E.

Theorem 1 *Let $H \in \mathcal{H}_k(p, q)$ where p, q are non-negative integers such that $p \geq 2$ and $q \geq \frac{k}{2}(p-1)+2$. Let $|U_i| = \alpha$ for $i = 1, ..., q-1$ and $\frac{\alpha}{2}(k-1) < |U_q| \leq k\alpha$, where α is an integer satisfying $\alpha \geq 2p(k-2)$. Then H does not have a hamiltonian chain.*

Proof. Suppose for a contradiction that H contains a hamiltonian chain $(v_1, v_2, ..., v_n)$. By removing all vertices from U_0 the chain $(v_1, v_2, ..., v_n)$ falls to m parts, $m \leq p$. Each part induces a path in $H - U_0$ or consists of at most $k-1$ vertices. Note that for any two adjacent edges E and E' belonging to one part $j_E = j_{E'}$. Indeed, E' has only one vertex which does not belong to E. Thus, since $|E \cap U_{j_E}| \geq 2$, E' contains a vertex from U_{j_E}. Hence, $j_{E'} \leq j_E$. Due to symmetry we have also $j_E \leq j_{E'}$. Therefore, every edge in a part has at least two vertices from some fixed set U_j. We say that the set U_j is a *dominating set* for this part. Let x_i denote the number of vertices of the i-th part which belong to its dominating set. Let y_i denote the number of remaining vertices in the i-th part. In this part consider now consecutive disjoint k-tuples and an r-tuple at the end, $0 \le r \le k-1$. Let x_r denote the number of vertices from the dominating set that belong to the r-tuple. Each k-tuple contains at least two vertices from the dominating set. Hence, the number of considered k-tuples does not exceed $\left\lfloor \frac{x_i - x_r}{2} \right\rfloor \le \frac{\alpha}{2}$. Thus, if $r \le k-2$ then the number of vertices in the i-th part satisfies

$$x_i + y_i \le \frac{\alpha}{2} k + (k-2). \tag{1}$$

If $r = k-1$ then $x_r \ge 1$ (because the last k vertices form an edge, too). Hence, $x_i + y_i \le \left\lfloor \frac{\alpha-1}{2} \right\rfloor k + (k-1)$, so (1) is satisfied, too. Hence, $x_i + y_i \le \frac{\alpha}{2} k + (k-2)$ or, if a part contains vertices from only one set U_j, $x_i + y_i \le |U_j|$. In particular, $x_i + y_i \le \frac{\alpha}{2} k + (k-2)$ for all parts except possibly one which contains only vertices from U_q (this is because $|U_q| < 2(\frac{k}{2}\alpha + (k-2))$). Hence, if some part contains only vertices from U_q then

$$\begin{aligned}
(q-1)\alpha + |U_q| = |U_1| + ... + |U_q| &= \sum_{i=1}^{m} (x_i + y_i) \le \sum_{i=1}^{m-1} \left(\frac{k}{2}\alpha + (k-2) \right) + |U_q| \\
&\le (p-1) \left(\frac{k}{2}\alpha + (k-2) \right) + |U_q|, \text{ hence} \\
q &\le (p-1)\frac{k}{2} + \frac{k-2}{\alpha}(p-1) + 1 < (p-1)\frac{k}{2} + 2, \text{ a contradiction.}
\end{aligned}$$

Otherwise,

$$\begin{aligned}
(q-1)\alpha + |U_q| = |U_1| + ... + |U_q| &= \sum_{i=1}^{m} (x_i + y_i) \le \sum_{i=1}^{m} \left(\frac{k}{2}\alpha + (k-2) \right) \\
&\le p \left(\frac{k}{2}\alpha + (k-2) \right), \text{ hence} \\
(q-1)\alpha &< p \left(\frac{k}{2}\alpha + (k-2) \right) - \frac{\alpha}{2}(k-1), \text{ whence} \\
q &< (p-1)\frac{k}{2} + 2, \text{ a contradiction again.}
\end{aligned}$$

□

Theorem 2 *Let t be a positive integer and let $H \in \mathcal{H}_k(2t+1, kt+2)$. Let $|U_i| = \alpha$ for $i = 1, ..., kt+1$ and $\frac{\alpha}{2}(k-1) < |U_{kt+2}| \le (\alpha - k)(k-2)$, where α is an integer satisfying $\alpha \ge (4t+2)(k-2)$. Then H is hamiltonian chain saturated.*

Proof. Let $p = 2t+1$ and $q = kt+2$. Hence, $q = \frac{k}{2}(p-1) + 2$. Thus, by Theorem 1, H does not have any hamiltonian chain. We will prove that adding any new edge E_0 produces a hamiltonian chain in H. Let E_0 have the form $\{i_1, i_2, ..., i_k\}$ with $1 \le i_1 < i_2 \le i_3, ..., \le i_k \le q$, which means that E_0 contains a vertex from set U_{i_1}, a vertex from set U_{i_2}, and so on. Note that E_0 contains only one vertex from U_{i_1}

since $E_0 \notin \mathcal{E}(H)$.
Suppose first that $i_2 \neq q$. Thus, E_0 contains at most $k-2$ vertices from U_q. Let $j_1 < j_2 < ... < j_{q-3}$ be consecutive integers from $\{1, ..., q-1\} \setminus \{i_1, i_2\}$. Let $\mathcal{C}_{j_i}$, $i = 1, ..., p-1$, be a path of the form

$$\underbrace{*, ..., *}_{k-2}, j_i, j_i, \underbrace{*, ..., *}_{k-2}, j_i, j_i, \quad \cdots \quad , \underbrace{*, ..., *}_{k-2}, j_i, j_i, \underbrace{*, ..., *}_{k-2}, j_i, j_i, \underbrace{*, ..., *}_{k-2}, (j_i)$$

which means that $\mathcal{C}_{j_i}$ contains

- all vertices from $U_{j_i} \setminus E_0$ – in the positions denoted by j_i (the symbol (j_i) means that $\mathcal{C}_{j_i}$ may or may not contain a vertex from $U_{j_i} \setminus E_0$ in this position depending on the parity of $|U_{j_i} \setminus E_0|$)
- some vertices from the set $\left(U_{j_p} \cup ... \cup U_{j_{q-3}}\right) \setminus E_0$ – in the positions denoted by $*$.

Clearly, U_{j_i} is the dominating set for $\mathcal{C}_{j_i}$ (since $i < p$). Moreover, among every k consecutive vertices there are two vertices from U_{j_i}. Thus, $\mathcal{C}_{j_i}$ is indeed a path. Let $z_{j_i} = |U_{j_i} \cap E_0|$. Note that a single $\mathcal{C}_{j_i}$ may contain $\left\lfloor \frac{\alpha - z_{j_i}}{2} \right\rfloor k + (k-2)$ vertices. Thus, the paths $\mathcal{C}_{j_1}, ..., \mathcal{C}_{j_{p-1}}$ together may contain $(q-3)\alpha$ vertices. Indeed,

$$\begin{aligned}
\sum_{i=1}^{p-1} \left(\left\lfloor \frac{\alpha - z_{j_i}}{2} \right\rfloor k + (k-2) \right) &\geq \sum_{i=1}^{p-1} \left(\frac{\alpha - z_{j_i} - 1}{2} k + (k-2) \right) \qquad (2)\\
&= \frac{\alpha}{2} k(p-1) + (p-1)(k-2) - \sum_{i=1}^{p-1} \frac{z_{j_i} + 1}{2} k \\
&= \frac{\alpha}{2} k(p-1) + (p-1)(k-2) - \frac{k}{2} \left(\sum_{i=1}^{p-1} z_{j_i} + \sum_{i=1}^{p-1} 1 \right) \\
&\geq \frac{\alpha}{2} k(p-1) + (p-1)(k-2) - \frac{k}{2}(k-2+p-1) \\
&\geq \frac{\alpha}{2} k(p-1) - \alpha = (q-3)\alpha, \text{ because } \alpha \geq \frac{k}{2}(k-2).
\end{aligned}$$

On the other hand, after removing any number of vertices outside the set U_{j_i} from a $\mathcal{C}_{j_i}$ we still have a path. Indeed, by removing some vertices from positions denoted by $*$ we do not spoil the property, that among every k consecutive vertices there are at least two vertices from U_{j_i}. Therefore we can construct $p-1$ paths $\mathcal{C}'_{j_1}, ..., \mathcal{C}'_{j_{p-1}}$ which together contain all vertices from $\left(U_{j_1} \cup ... \cup U_{j_{q-3}}\right) \setminus E_0$ (and do not contain any other vertex). Finally, let $\mathcal{C}_{i_1}$ have the form

$$\underbrace{q, ..., q}_{k-2}, i_1, i_1, \underbrace{q, ..., q}_{k-2}, i_1, i_1, \quad \cdots \quad , \underbrace{q, ..., q}_{k-2}, i_1, \underbrace{i_1, \overbrace{i_3, ..., i_k, i_2}^{k-2}}_{E_0} i_2 \underbrace{q, ..., q}_{k-2}, \quad \cdots \quad , i_2, i_2, \underbrace{q, ..., q}_{k-2}.$$

Note that

$$\begin{aligned}
\left\lfloor \frac{|U_{i_1} \cup U_{i_2}| - (k-2)}{2} \right\rfloor &\geq \frac{2\alpha - k + 1}{2} = \alpha - k + k/2 + 1/2 > \alpha - k + 2 \\
&= \frac{(\alpha - k)(k-2)}{k-2} + 2 > \left\lceil \frac{|U_q|}{k-2} \right\rceil.
\end{aligned}$$

Hence, we are able to insert all vertices from U_q to $\mathcal{C}_{i_1}$. Moreover, by adding at the beginning (or at the end) of $\mathcal{C}_{i_1}$ any number of vertices from $U_{i_1} \setminus E_0$ (from $U_{i_2} \setminus E_0$) we still have a path. Hence, we can construct a path $\mathcal{C}'{}_{i_1}$ which contains all vertices from $U_{i_1} \cup U_{i_2} \cup U_q \cup E_0$. It is easy to see now that

$$\left(\mathcal{C}'_{j_1}, 0, \mathcal{C}'_{j_2}, 0, \ \cdots\ , 0, \mathcal{C}'_{j_{p-1}}, 0, \mathcal{C}'_{i_1}, 0\right)$$

(here symbols 0 denote distinct vertices from U_0) is a hamiltonian chain in H.

Suppose now that $i_2 = q$, whence E_0 is of the form $\{i_1, q, ..., q\}$. Let $j_1 < j_2 ... < j_{q-2}$ be consecutive integers from $\{1, ..., q-1\} \setminus \{i_1\}$. We construct paths $\mathcal{C}_{j_1}$, ..., $\mathcal{C}_{j_{p-1}}$ in a similar way as previously. This time the paths $\mathcal{C}_{j_i}$ may contain $(q-2)\alpha$ vertices. This can be verified by similar computations as previously, see (2), with one difference that now $z_{j_i} = 0$ for each $i = 1, ..., p-1$. Recall, that after removing any number of vertices outside the set U_{j_i} from a $\mathcal{C}_{j_i}$ we still have a path. Therefore, we can construct $p-1$ paths $\mathcal{C}''_{j_1}$, ...,$\mathcal{C}''_{j_{p-1}}$ which together contain all vertices from $U_{j_1} \cup ... \cup U_{j_{q-2}}$ (and do not contain any other vertex). Now let $\mathcal{C}''_{i_1}$ have the form

$$i_1, i_1, ..., i_1, \underbrace{i_1, q, q, ..., q}_{E_0}, q, ..., q, q$$

and contain all vertices from $U_{i_1} \cup U_q$. Therefore,

$$\left(\mathcal{C}''_{j_1}, 0, \mathcal{C}''_{j_2}, 0, \ \cdots\ , 0, \mathcal{C}''_{j_{p-1}}, 0, \mathcal{C}''_{i_1}, 0\right)$$

is a hamiltonian chain in H. □

3 Main result

Theorem 3 *For each $n \geq k^4 + 8k^3 + 16k^2$, there exists a k-uniform, hamiltonian chain saturated hypergraph with fewer than $\left(\frac{1}{k!} + \frac{1}{(k-1)!}\right) n^{k-1/2} + o\left(n^{k-1/2}\right)$ edges.*

Proof. Let $t_0 := \left\lfloor \frac{\sqrt{n}}{2k} \right\rfloor - 1$ and $\alpha := \left\lfloor \frac{n-2t_0-2}{kt_0+1/2+k/2} \right\rfloor$. Thus, it is easy to check that $\alpha \geq (4t_0+2)k > (4t_0+2)(k-2)$. Let $H \in \mathcal{H}_k(2t_0+1, kt_0+2)$ with $|U_i| = \alpha$ for $i = 1, ..., kt_0+1$. Then

$$\frac{n-2t_0-1}{kt_0+1/2+k/2} - 1 \leq \alpha < \frac{n-2t_0-1}{kt_0+1/2+k/2}, \text{ whence}$$
$$(kt_0+1)\alpha + \frac{\alpha}{2}(k-1) < n - 2t_0 - 1 \leq (kt_0+1)\alpha + \frac{\alpha}{2}(k-1) + (kt_0 + k/2 + 1/2).$$

Thus, since $n - 2t_0 - 1 = (kt_0+1)\alpha + |U_{kt_0+2}|$, we have

$$\frac{\alpha}{2}(k-1) < |U_{kt_0+2}| \leq \frac{\alpha}{2}(k-1) + kt_0 + k/2 + 1/2 < (\alpha - k)(k-2)$$

for $k \geq 4$, $\alpha \geq (4t_0+2)k$ and $t_0 \geq k/2$. Therefore, by Theorem 2, H is a hamiltonian chain saturated hypergraph.

Moreover,

$$
\begin{aligned}
|\mathcal{E}(H)| &\le |U_0|\binom{n-1}{k-1} + \sum_{i=1}^{kt_0+1}\binom{\alpha}{2}\binom{n-2t_0-1-(i-1)\alpha}{k-2} + \binom{|U_{kt_0+2}|}{k} \\
&\le (2t_0+1)\frac{n^{k-1}}{(k-1)!} + \frac{\alpha^2}{2}\sum_{i=1}^{kt_0+1}\frac{(n-(i-1)\alpha)^{k-2}}{(k-2)!} + \frac{k^k\alpha^k}{k!}.
\end{aligned}
$$

Comparing the sum of the areas of rectangles R_i, $i = 1, ..., kt_0 + 1$, with sides $a_i = \alpha$ and $b_i = (n-(i-1)\alpha)^{k-2}$ with the area of the figure limited by the x-axis and the graph of the function $y = x^{k-2}$, $0 \le x \le n + \alpha$, we obtain that

$$
\begin{aligned}
&\sum_{i=1}^{kt_0+1}\alpha\,(n-(i-1)\alpha)^{k-2} \le \int_0^{n+\alpha} x^{k-2}dx, \text{hence,} \\
&\sum_{i=1}^{kt_0+1}(n-(i-1)\alpha)^{k-2} \le \frac{(n+\alpha)^{k-1}}{\alpha(k-1)} \qquad (3)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|\mathcal{E}(H)| &\le (2t_0+1)\frac{n^{k-1}}{(k-1)!} + \frac{\alpha(n+\alpha)^{k-1}}{2(k-1)!} + \frac{k^k\alpha^k}{k!} \qquad (4)\\
&= \left(\frac{1}{k!} + \frac{1}{(k-1)!}\right)n^{k-1/2} + o\left(n^{k-1/2}\right),
\end{aligned}
$$

because $t_0 \approx \frac{\sqrt{n}}{2k}$ and $\alpha \approx 2\sqrt{n}$. □

4 Concluding remarks

We have constructed a family of k-uniform hamiltonian chain saturated hypergraphs. The main result is Theorem 3, which gives the hypergraphs with the smallest known number of edges. Unfortunately, since the lower bound in [12] is smaller than the number of edges in our construction by a factor $n^{1/2}$ the order of magnitude for $sat(C_n^{(k)})$ remains unknown.

Note that our construction cannot be significantly improved by taking another t_0. Indeed, if we take t_0 of order greater than $n^{1/2}$ then, the first component of (4) is of order greater than $n^{k-1/2}$. Otherwise, if t_0 is of order less than $n^{1/2}$, then α is of order greater than $n^{1/2}$, whence the second component of (4) is of order greater than $n^{k-1/2}$. On the other hand, the coefficient $\frac{1}{2k}$ is chosen in order to minimize (4) and satisfy all necessary conditions that the numbers p, q, α must have (some slight improvements are still possible, however).

We observe that the same bounds can be obtained in case when we consider hamiltonian paths. The upper bound can be realized by a hypergraph $H \in \mathcal{H}_k(2t_0, kt_0 + 2)$ with t_0 and α being the same as in Theorem 3.

Theorem 4 *For each $n \ge k^4 + 8k^3 + 16k^2$, there exists a k-uniform, hamiltonian path saturated hypergraph with less than $\left(\frac{1}{k!} + \frac{1}{(k-1)!}\right)n^{k-1/2} + o\left(n^{k-1/2}\right)$ edges.*

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