Discrete Mathematics and Theoretical Computer Science DMTCS vol. **13:1**, 2011, 45–66

Cycle transversals in bounded degree graphs[†]

Marina Groshaus[1‡] Pavol Hell[2§] Sulamita Klein[3¶]
Loana Tito Nogueira[4‖] Fábio Protti[4**]

[1] *Universidad de Buenos Aires, Argentina*
[2] *Simon Fraser University, Canada*
[3] *Universidade Federal do Rio de Janeiro, Brazil*
[4] *Universidade Federal Fluminense, Brazil*

received 12th July 2010, revised 31st January 2011, accepted 9th February 2011.

In this work we investigate the algorithmic complexity of computing a minimum C_k-transversal, i.e., a subset of vertices that intersects all the chordless cycles with k vertices of the input graph, for a fixed $k \geq 3$. For graphs of maximum degree at most three, we prove that this problem is polynomial-time solvable when $k \leq 4$, and NP-hard otherwise. For graphs of maximum degree at most four, we prove that this problem is NP-hard for any fixed $k \geq 3$. We also discuss polynomial-time approximation algorithms for computing C_3-transversals in graphs of maximum degree at most four, based on a new decomposition theorem for such graphs that leads to useful reduction rules.

Keywords: approximation algorithms, cycle-transversals, transversals

1 Introduction

The graphs considered in this work are finite and with no loops and multiple edges. Let $\mathcal{H}$ be a fixed family of graphs. An $\mathcal{H}$*-subgraph* of a graph G is an *induced* subgraph of G isomorphic to a member of $\mathcal{H}$. A graph is $\mathcal{H}$*-free* if it contains no $\mathcal{H}$-subgraph. An $\mathcal{H}$*-transversal* of a graph G is a subset $T \subseteq V(G)$ such that T intersects all the $\mathcal{H}$-subgraphs of G. Clearly, if T is an $\mathcal{H}$-transversal of G then $G - T$ is $\mathcal{H}$-free. Moreover, if T is small (minimum) then $G - T$ is a large (maximum) induced $\mathcal{H}$-free subgraph of G. We remark that the term "covering" sometimes appears in the literature with the same meaning as "transversal", see for instance [8].

For a fixed family $\mathcal{H}$, the general decision problem called $\mathcal{H}$-TRANSVERSAL can be formulated as follows: given a graph G and an integer ℓ, decide whether G contains an $\mathcal{H}$-transversal T such that $|T| \leq \ell$. Yannakakis proved that this problem is NP-complete [13]; in fact, he proved a more general result which says that the problem of finding the minimum number of vertices of a graph G whose deletion results in a subgraph satisfying a property π which is hereditary on induced subgraphs is NP-hard. The following table summarizes some interesting special cases of the $\mathcal{H}$-TRANSVERSAL problem.

[†] A short version of this paper appeared in *Electronic Notes in Discrete Mathematics*, Volume 35, pp 189 – 195
[‡] E-mail: groshaus@dc.uba.ar. Partially supported by UBACyT and ANPCyT
[§] E-mail: pavol@cs.sfu.ca.
[¶] E-mail: sula@cos.ufrj.br. Partially supported by CNPq and FAPERJ.
[‖] E-mail: loana@ic.uff.br. Partially supported by CNPq and FAPERJ.
[**] E-mail: fabio@ic.uff..br. Partially supported by CNPq and FAPERJ.

Problem	G	$\mathcal{H}$	$G - T$
1	general	odd cycles	bipartite
2	general	$\{K_2\}$	stable set
3	general	$\{K_3\}$	triangle-free
4	general	$\{P_3\}$	disjoint union of cliques
5	general	$\{P_4\}$	cograph
6	chordal	$\{K_3\}$	forest
7	interval	$\{K_{1,3}\}$	indifference
8	bipartite	$\{P_4\}$	disjoint union of bicliques
9	chordal bipartite	$\{C_4\}$	forest
10	perfect	$\{K_\ell\}$	$(\ell - 1)$-colorable

Problems 1 to 5 are NP-complete due to Yannakakis' result [13]. Problem 1 is precisely the MAXIMUM INDUCED BIPARTITE SUBGRAPH problem. An edge version of this problem (whose goal is to find a maximum induced bipartite subgraph with maximum number of edges) is considered in [3]; still, an $O(mn)$ algorithm is developed in [11] to find odd cycle transversals with bounded size. Problem 2 is the well known VERTEX COVER problem [8]. Problem 3 has an interesting edge version: find the minimum number of edges whose deletion leaves a triangle-free subgraph (see [12]). Problems 4 and 5 are interesting due to their connections to parameterized edge editing theory (see [4, 9]). To the best of our knowledge, the complexities of Problems 6 to 10 remain as open questions; in particular, Problem 7 is cited in [6] as an important question in order theory (the maximum indifference order contained in an interval order).

Let k denote a fixed integer, $k \geq 3$. In this work we investigate the case $\mathcal{H} = \{C_k\}$. (C_k denotes a chordless cycle with k vertices.) We consider the following problem:

C_k-TRANSVERSAL
INPUT: a graph G, an integer ℓ
QUESTION: does G contain a C_k-transversal of size at most ℓ?

The C_k-TRANSVERSAL problem is NP-complete for general graphs as a consequence of Yannakakis' result [13]. A way of dealing with this intractability is to assume some restrictions on the input graph. Restricting its maximum degree turns out to be a widely employed and natural strategy. In particular, we can seek dichotomy results as follows: for a fixed value of Δ (the maximum degree of the input graph), find an integer p such that C_k-TRANSVERSAL is polynomial-time solvable if $k \leq p$, and NP-complete otherwise. Alternatively, we can fix k and determine p such that C_k-TRANSVERSAL is polynomial-time solvable if $\Delta \leq p$, and NP-complete otherwise. The latter approach is well studied in algorithmic graph theory. For example, coloring a graph is polynomial-time solvable for graphs of maximum degree at most three [1], and NP-complete otherwise [5]. Still in this context, Gallucio, Hell and Nešetřil conjecture that the H-colouring problem for graphs of maximum degree at most three is NP-complete, even when H is a triangle-free graph with chromatic number three [7]. (The H-colouring problem asks, for a fixed graph H, whether an input graph G admits a mapping $c : V(G) \to V(H)$ such that $xy \in E(G)$ implies $c(x)c(y) \in E(H)$.)

The following table summarizes our results on the boundary between tractability and NP-completeness for the C_k-TRANSVERSAL problem, with respect to the values of k and Δ.

	$\Delta = 2$	$\Delta = 3$	$\Delta = 4, 5 \dots$
$k = 3$	P	P	NP-c
$k = 4$	P	P	NP-c
$k = 5, 6 \dots$	P	NP-c	NP-c

If $\Delta = 2$, minimum C_k-transversals are trivially obtained in polynomial time for any k, since in this case the input graph is a disjoint union of paths and cycles. In Section 2, we show that C_k-TRANSVERSAL for graphs of maximum degree at most three is polynomial-time solvable for $k \leq 4$ and NP-complete otherwise. For graphs of maximum degree at most four, we show in Section 3 that C_k-TRANSVERSAL is NP-complete for any fixed $k \geq 3$. This NP-completeness result trivially extends to $\Delta \geq 5$.

In view of the hardness of finding minimum C_3-transversals (or *triangle-transversals*) when $\Delta = 4$, a polynomial-time approximation algorithm for this case is presented in Section 3, based on a new decomposition theorem for graphs of maximum degree at most four and certain reduction rules. Some interesting polynomial cases are also discussed.

As we shall see, the NP-completeness results above are still valid for bipartite graphs and even values of k.

Unless otherwise stated, n stands for the number of vertices of a graph. For a subset S of vertices, define $N(S) = (\cup_{v \in S} N(v)) \setminus S$. We use the notation $N(u, v)$ instead of $N(\{u, v\})$.

2 Graphs of maximum degree at most three

The following definition is useful and will be used throughout this paper:

Definition 1 *An edge $e \in E(G)$ is called a k-*free edge *if e is contained in no induced C_k of G.*

The observation below deals with the case $k = 3$:

Observation 2 *Let G be a graph such that $\Delta = 3$. If e is a 3-free edge then T is a triangle-transversal of G if and only if T is a triangle-transversal of $G - e$.*

Observation 2 leads to a simple polynomial-time algorithm for finding a minimum triangle-transversal of G in a graph of maximum degree at most three.

Theorem 3 TRIANGLE-TRANSVERSAL *is polynomial time solvable for graphs of maximum degree at most three.*

Proof: Let G be a graph with $\Delta = 3$. In order to find a minimum triangle-transversal of G, first remove 3-free edges; next, observe that each connected component of the remaining graph can be a triangle, a K_4 or a diamond (K_4 minus one edge). Hence a minimum triangle-transversal consists of one vertex per component (if it is a diamond or a triangle) or two vertices (if it is a K_4). □

We remark that Observation 2 is not true for C_k-transversals with $k > 3$. For instance, consider $k = 4$ and the graph $K_4 - e$ (a complete graph with four vertices minus one edge). Every edge of this graph is 4-free. In addition, since $K_4 - e$ contains no induced C_4, the empty set is a C_4-transversal of it. By removing the edge linking the vertices with degree two in $K_4 - e$, we obtain a new graph for which a C_4-transversal must contain at least one vertex.

Now we deal with C_4-transversals in graphs of maximum degree at most three. A C_4-transversal of G will also be called a *square-transversal*.

Definition 4 *A* circular ladder *(also called* prism*) is a cubic graph with vertex set* $V = \{a_1, \ldots, a_j\} \cup \{b_1, \ldots, b_j\}$*, for* $j \geq 3$*, and edge set*

$$E = \{a_i b_i \mid 1 \leq i \leq j\} \cup \{a_i a_{i+1}, b_i b_{i+1} \mid 1 \leq i \leq j-1\} \cup \{a_1 a_j, b_1 b_j\}.$$

A Möbius ladder *is defined similarly, with edges* $a_1 b_j, b_1 a_j$ *instead of* $a_1 a_j, b_1 b_j$*. (See Figure 1.)*

A minimum square-transversal of a circular (Möbius) ladder is $\{a_1, a_3, \ldots, a_{j-1}\}$ if j is even, or $\{a_1, a_3, \ldots, a_j\}$ if j is odd.

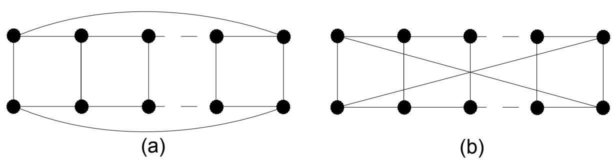

Fig. 1: (a) Circular ladder; (b) Möbius ladder.

Theorem 5 *Let* G *be a connected cubic graph graph with* $n \geq 6$*. Then* G *contains no 4-free edges if and only if* G *is a circular ladder or a Möbius ladder.*

Proof: If G is a circular (Möbius) ladder then G contains no 4-free edges. Conversely, assume that G contains no 4-free edges. Clearly, G has an even number of vertices. We need the following claims:

Claim 1*:* G *is either triangle-free or a circular ladder with* $n = 6$*.*

Proof of Claim 1: Suppose G contains a triangle abc. Since edge ab lies in an induced C_4, there must exist two new vertices d and e and edges ad, be, de in G. Edge bc must also lie in an induced C_4. If there exists an induced cycle $cbed$, by the degree restriction e has a new neighbor f and we get a contradiction, since edge ef cannot lie in an induced C_4. Thus, $cd \notin E(G)$ and there exists a new vertex f and edges cf, ef. If $df \notin E(G)$ then edge ac does not lie in an induced C_4, which is impossible. Therefore, $df \in E(G)$ and G is a circular ladder with six vertices. This completes the proof of Claim 1.

Claim 2*: If* $n \geq 8$ *then there exists an edge* $uv \in E(G)$ *such that the subgraph induced by* $N(u, v)$ *is a* $2K_2$ *(a graph formed by two isolated copies of* K_2*).*

Proof of Claim 2: Since $n \geq 8$, by Claim 1 G is triangle-free. Therefore, for any edge $uv \in E(G)$ we have $|N(u, v)| = 4$. Let $N(u) \backslash \{v\} = \{u_1, u_2\}$ and $N(v) \backslash \{u\} = \{v_1, v_2\}$, and let $u_1 u v v_1$ be an induced C_4. Clearly, $u_1 u_2, v_1 v_2 \notin E(G)$. We have three cases:

Case 1: both edges $u_2 v_1$, $u_1 v_2$ exist. If $u_2 v_2 \in E(G)$ then G is cubic with six vertices (in fact, G is a Möbius ladder with six vertices). This is a contradiction. Hence, $u_2 v_2 \notin E(G)$. But then u_2 must have a new neighbor u_3, and edge $u_2 u_3$ cannot lie in an induced C_4. Impossible. Thus Case 1 does not occur.

Case 2: $u_2 v_1 \in E(G)$ but $u_1 v_2 \notin E(G)$. Edge vv_2 must lie in an induced C_4, which is either $v v_2 u_2 u$ or $v v_2 u_2 v_1$, implying $u_2 v_2 \in E(G)$. Now, u_1 must have a new neighbor u_3, and edge $u_1 u_3$ cannot lie in an induced C_4. Again, this is impossible. Thus Case 2 does not occur as well.

Case 3: neither u_2v_1 nor u_1v_2 is an edge of G. If $u_2v_2 \in E(G)$, nothing remains to prove. Otherwise, u_2 must have a new neighbor u_3 such that $uu_1u_3u_2$ is an induced C_4, implying $u_3v_1 \notin E(G)$ (otherwise there would be a triangle). Hence, the subgraph induced by $N(u, u_1)$ is a $2K_2$. This completes the proof of Claim 2.

We conclude the proof of the theorem by induction. The result is valid for $n = 6$. Now assume $n \geq 8$. By Claim 2, choose $uv \in E(G)$ such that $N(u)\backslash\{v\} = \{u_1, u_2\}$, $N(v)\backslash\{u\} = \{v_1, v_2\}$, $u_1v_1, u_2v_2 \in E(G)$ and $u_1u_2, v_1v_2, u_1v_2, v_1u_2 \notin E(G)$. Construct $G' = (G - \{u, v\}) \cup \{u_1u_2, v_1v_2\}$. Observe that G' is cubic with $n-2$ vertices. Moreover, in G', $N(u, v)$ induces a connected subgraph; thus G' is connected. We show that G' contains no 4-free edges. It is clear that edges $u_1u_2, v_1v_2, u_1v_1, u_2v_2 \in E(G')$ are not 4-free edges. In addition, any edge $xy \in E(G')$ such that $x, y \notin \{u_1, u_2, v_1, v_2\}$ is not a 4-free edge as well. It remains to show that any edge of the form $xy \in E(G')$ with $x \in \{u_1, u_2, v_1, v_2\}$ and $y \notin \{u_1, u_2, v_1, v_2\}$ is not a 4-free edge. Suppose without loss of generality that $x = u_1$. If y and v_1 have no common neighbor in G', the only C_4 containing edge yu_1 in G must be yu_1uu_2, that is, $yu_2 \in E(G')$. Let z be the third neighbor of y. Since yz is not a 4-free edge in G', let $yzz'z''$ be a C_4 in G' containing it, where $z', z'' \notin \{u_1, u_2, v_1, v_2\}$. Then $N(y) = \{u_1, u_2, z, z''\}$, contradicting the fact that G' is cubic. Hence, there must exist a common neighbor z of y and v_1, i.e., u_1yzv_1 is an induced C_4 both in G and G'. Therefore, G' contains no 4-free edges. By the induction hypothesis, G' is a circular (Möbius) ladder. To conclude the proof of the theorem, observe that G, in this case, is also a circular ladder or Möbius ladder. □

Theorem 6 *Let G be a connected graph with $n \geq 6$, $\Delta = 3$, and containing no 4-free edges. Then G is a subgraph (not necessarily induced) of a circular ladder or Möbius ladder.*

Proof: Since G contains no 4-free edges, there are no vertices with degree one (*pendant vertices*) in G. Let $n_2(G)$ be the number of vertices with degree two in G. We use induction on $n_2(G)$. If $n_2(G) = 0$ then the result is true by Theorem 5. Assume then $n_2(G) > 0$.

Claim: There exist two nonadjacent vertices with degree two in G.

Proof of the Claim: Suppose the claim is false. Then the subset of vertices with degree two in G is a clique Q. By the degree restriction, Q induces a K_j, for some $j \in \{1, 2, 3\}$. If $j = 3$, G itself would be a K_j, which is impossible. For the case $j = 2$, assume that G is a graph with minimum number of vertices containing two adjacent vertices x, y with degree two and $n - 2$ vertices with degree three. Let $x' \neq y, y' \neq x$ be neighbors of x, y, respectively. Clearly, $x' \neq y'$. Moreover, vertices x', y' must be adjacent, otherwise edge xy would be 4-free, and edge $x'y'$ is contained in another induced C_4 which misses x, y. By degree arguments, it is easy to see that G must contain at least 8 vertices. Let $G' = G - \{x, y\}$. Note that G' contains two adjacent vertices with degree two (x' and y') and $n-4$ vertices with degree three, and satisfies $6 \leq |V(G')| < |V(G)|$, a contradiction. Thus the existence of a connected graph G satisfying the hypotheses of the statement and containing two adjacent vertices with degree two and $n - 2$ vertices with degree three is impossible. Finally, if $j = 1$, let w be the only vertex in G with degree two, and let x, y be its neighbors. Vertices x, y must have a common neighbor z other than w, otherwise edges wx, wy would be 4-free. Suppose that x, y have another common neighbor $z_1 \notin \{z, w\}$. In this case z_1 is not adjacent to z, otherwise edge zz_1 is 4-free. Let z_2 be the third neighbor of z; the only way to have edge zz_2 contained in a C_4 is to assume the existence of cycles zz_2z_1x and zz_2z_1y. Again, z_2 must have a third neighbor z_3; however, no C_4 can contain edge z_2z_3 because of the degree restrictions. Thus, the only common neighbor of x, y is z. This means that edges xz, yz are contained in distinct induced C_4's other than $wxzy$, say C_1 and C_2. Consider $G' = G - w + \{xy, ax, by, ab\}$, where a, b are new vertices. In G', edge xy is not a chord of C_1, C_2. Then G' is a graph satisfying the hypotheses of the statement and containing two adjacent vertices (a and b) with degree two and $|V(G')| - 2$ vertices with degree three. But this is impossible, as said above. This completes the proof of the claim.

Now, let x, y be two nonadjacent vertices in G with degree two. Assume without loss of generality that the distance d between x and y is minimum. Consider a path P linking x and y, isomorphic to P_j (the chordless path with j vertices), for $j = d+1 \geq 3$. If $j = 3$, define a supergraph G' of G by creating a new vertex a and new edges ax, ay; and, if $j = 4$, define G' by creating a new edge xy. In both cases, the new edges created are not 4-free, and since $n_2(G') < n_2(G)$, by the induction hypothesis G' is a subgraph of a circular (Möbius) ladder, and so is G.

It remains to consider the case $j \geq 5$. Write $P_j = a_1 a_2 \ldots a_j$. We show by induction on j that, in this case, G is isomorphic to a *ladder:* a circular ladder (or Möbius ladder) minus edges $a_1 a_j, b_1 b_j$ (or $a_1 b_j, b_1 a_j$) (recall Definition 4). The case $j = 5$ can be proved by inspection (each a_i must have a distinct neighbor b_i, and b_i cannot be adjacent to a_{i+1}, for $i = 1, 2, 3, 4$; this implies that vertices $b_1, b_2, \ldots, b_5$ induce P_5). For $j > 5$, let b_j be a neighbor of a_j such that $b_j \neq a_{j-1}$, and let $G' = G - \{a_j, b_j\}$. In G', a_1 and a_{j-1} are linked by a path isomorphic to P_{j-1}. In addition, a_{j-1} and b_j must have a common neighbor $b_{j-1} \neq a_j$. Note that edges $a_{j-2} a_{j-1}$ and $a_{j-1} b_{j-1}$ must be contained in a same induced C_4 in G. This implies that edge $a_{j-1} b_{j-1}$ is contained in a C_4 in G', that is, G' contains no 4-free edges. Thus, by the induction hypothesis, G' is a ladder formed by vertices $a_1, \ldots, a_{j-1}, b_1, \ldots, b_{j-1}$. Then G is a ladder as well. This completes the proof. □

Theorem 7 *The* SQUARE-TRANSVERSAL *problem is polynomial time solvable for graphs of maximum degree at most three.*

Proof: Remove 4-free edges from G. By Theorem 6, each connected component with at least 6 vertices of the remaining graph is a subgraph of a circular (Möbius) ladder; for such components, square-transversals can be easily obtained. For components with less than 6 vertices, square-transversals can be directly obtained by inspection. □

To conclude this section, we analyze the C_5-TRANSVERSAL problem in graphs of maximum degree at most three.

Theorem 8 C_5-TRANSVERSAL *is NP-complete for graphs of maximum degree at most three.*

Proof: The problem is clearly in NP. The hardness proof is a reduction from a special version of SAT, denoted here 3SAT$_3$: each clause contains at most three literals, and each variable occurs exactly three times, twice positively and once negatively. The NP-completeness of this problem is a consequence of the results in [2, 10].

Let $\mathcal{F}$ be a boolean formula of 3SAT$_3$ with n variables $x_1, x_2, \ldots, x_n$ and m clauses $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$. Construct a graph G of maximum degree at most three as follows. Create a subgraph G_i for each variable x_i, $1 \leq i \leq n$, as in Figure 2. Edges $a_i b_i$ and $a'_i b'_i$ represent the two positive occurrences of x_i, whilst edge $a''_i b''_i$ represents its negative occurrence.

Next, create a subgraph H_j for each clause $\mathcal{C}_j$, $1 \leq j \leq m$, as in Figure 3. Denote $W_j = \{u_j, v_j, u'_j, v'_j, u''_j, v''_j\}$.

Finally, for each clause $\mathcal{C}_j$ containing variables x_h, x_i, x_k with $h \leq i \leq k$, identify pairs of adjacent vertices as follows:

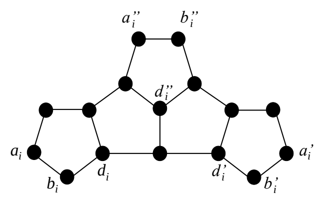

Fig. 2: Subgraph G_i corresponding to variable x_i.

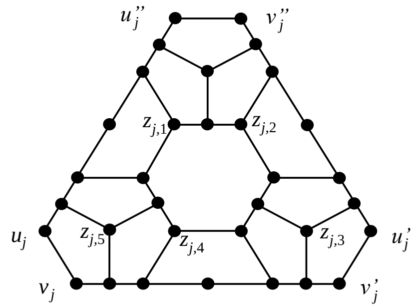

Fig. 3: Subgraph H_j corresponding to clause $\mathcal{C}_j$.

(a) if x_h occurs positively in $\mathcal{C}_j$ then:

(a.1) set $a_h = u_j, b_h = v_j$, if x_h does not occur positively in some $\mathcal{C}_{j'}$ with $j' < j$;

(a.2) set $a'_h = u_j, b'_h = v_j$, otherwise.

(b) if x_h occurs negatively in $\mathcal{C}_j$ then set $a''_h = u_j, b''_h = v_j$.

(c) if x_i occurs positively in $\mathcal{C}_j$ then:

(c.1) set $a_i = u'_j, b_i = v'_j$, if x_i does not occur positively in some $\mathcal{C}_{j'}$ with $j' < j$;

(c.2) set $a'_i = u'_j, b'_i = v'_j$, otherwise.

(d) if x_i occurs negatively in $\mathcal{C}_j$ then set $a''_i = u'_j, b''_i = v'_j$.

(e) if x_k occurs positively in $\mathcal{C}_j$ then:

(e.1) set $a_k = u''_j, b_k = v''_j$, if x_k does not occur positively in some $\mathcal{C}_{j'}$ with $j' < j$;

(e.2) set $a'_k = u''_j, b'_k = v''_j$, otherwise.

(f) if x_k occurs negatively in $\mathcal{C}_j$ then set $a''_k = u''_j, b''_k = v''_j$.

Proceed similarly for clauses containing two variables x_h, x_i (using rules (a) to (d)) and clauses containing only one variable x_h (using rules (a) and (b)).

See an example of construction of G in Figure 4. It is important to note that every induced C_5 of G is entirely contained either in a variable subgraph G_i or in a clause subgraph H_j.

The following observations are useful:

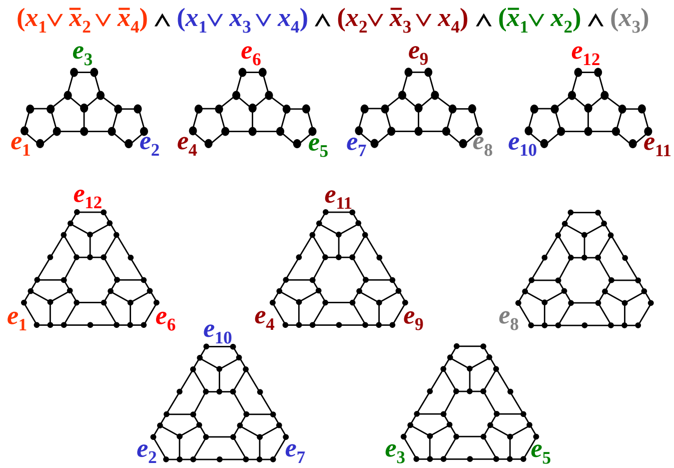

Fig. 4: Proof of Theorem 8. Graph G constructed from an instance of 3SAT$_3$. Two edges with the same label e_j are identical.

Observation 1. Let T_i be a C_5-transversal of G_i. Then $|T_i| \geq 3$. Moreover, if $|T_i| = 3$ then T_i cannot contain two vertices t_1, t_2 such that $t_1 \in \{a_i, b_i, a'_i, b'_i\}$ and $t_2 \in \{a''_i, b''_i\}$. (Note that selecting a_i, a''_i, for example, still leaves two disjoint C_5's uncovered. Suitable choices are $T_i = \{x, x', d''_i\}$, with $x \in \{a_i, b_i\}$ and $x' \in \{a'_i, b'_i\}$; or $T_i = \{x, d_i, d'_i\}$, with $x \in \{a''_i, b''_i\}$.)

Observation 2. Let T be a C_5-transversal of G and $T'_j = T \cap V(H_j)$. Then $|T'_j \backslash W_j| \geq 5$, since the subgraph induced by $T'_j \backslash W_j$ contains five disjoint C_5's. Moreover, if $T \cap W_j \neq \emptyset$ for some j then one can easily see that five vertices are sufficient in $T'_j \backslash W_j$; for instance, if $T \cap W_j = \{u''_j\}$ then $T'_j \backslash W_j$ may contain vertices $z_{j,k}$ for $k = \{1, 2, 3, 4, 5\}$ (see Figure 3).

In what follows, we prove that $\mathcal{F}$ is satisfiable if and only if G admits a C_5-transversal T of size $3n + 5m$. Assume first that $\mathcal{F}$ is satisfiable. For each $1 \leq i \leq n$, include in T vertices a_i, a'_i, d''_i if x_i is set to "true", otherwise include in T vertices a''_i, d_i, d'_i. See Figure 2. Since each clause $\mathcal{C}_j$ contains at least one true literal, $T \cap W_j \neq \emptyset$, for every $j \in \{1, \ldots, m\}$. By Observation 2, five additional vertices for each H_j suffice to complete the construction of T, yielding a C_5-transversal of G of size $3n + 5m$.

Conversely, assume that G admits a C_5-transversal T of size $3n + 5m$. Let $T_i = T \cap V(G_i)$ and $T'_j = T \cap V(H_j)$. By Observations 1 and 2, $|T_i| \geq 3$ and $|T'_j \backslash W_j| \geq 5$. As G contains m disjoint copies of H_j and n disjoint copies of G_i, it follows that $W_j \neq \emptyset, 1 \leq j \leq m$, and $|T_i| = 3,\ 1 \leq i \leq n$. By Observation 1 again, T_i cannot contain vertices t_1, t_2 such that $t_1 \in \{a_i, b_i, a'_i, b'_i\}$ and $t_2 \in \{a''_i, b''_i\}$. We can then assume $T_i = \{a_i, a'_i, d''_i\}$ or $T_i = \{a''_i, d_i, d'_i\}$. In the former case we set x_i to "true", otherwise to "false". This shows that $\mathcal{F}$ is satisfiable. □

We can transform the C_5's of Figure into C_k's, for any $k \geq 6$, by conveniently splitting some edges. See an example for $k = 6$ in Figure 5. This illustrates how one can easily prove the following theorem:

Theorem 9 C_k-TRANSVERSAL *is NP-complete for graphs of maximum degree at most three, for any fixed* $k \geq 5$.

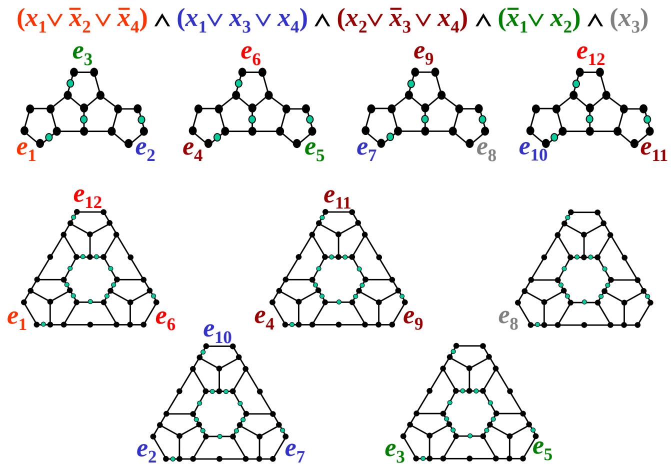

Fig. 5: Transforming C_5's into C_6's.

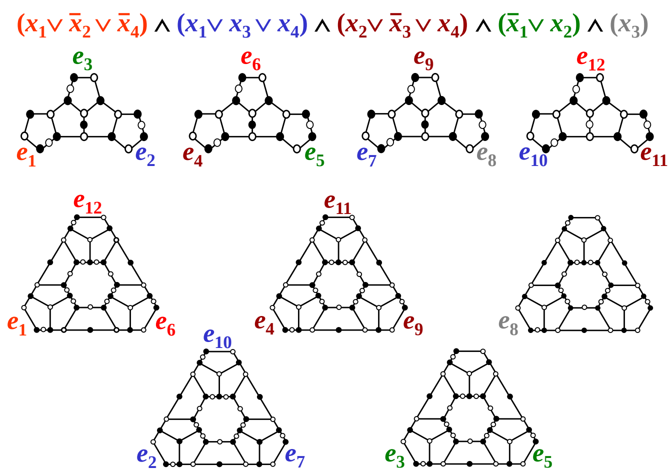

Fig. 6: Bipartite graph G for the case $k = 6$; $V(G)$ is divided into white and black vertices.

Moreover, the graph G used in the hardness proof is bipartite when k is even. See Figure 6, where $k = 6$ and $V(G)$ is divided into white and black vertices. Therefore:

Theorem 10 *C_{2k}-TRANSVERSAL is NP-complete for bipartite graphs of maximum degree at most three, for any $k \geq 3$.*

3 Graphs of maximum degree at most four

3.1 NP-completeness results

In this subsection we first prove the following NP-completeness result.

Theorem 11 TRIANGLE-TRANSVERSAL *is NP-complete for graphs of maximum degree at most four.*

Proof: The problem is clearly in NP. The hardness proof is similar to that of Theorem 8. Given an instance $\mathcal{F}$ of 3SAT$_3$ with n variables $x_1, x_2, \ldots, x_n$ and m clauses $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$, we construct G by creating a subgraph G_i for each variable x_i, as in Figure 7. Vertices a_i and a'_i represent the two positive occurrences of x_i, and vertex a''_i represents its negative occurrence.

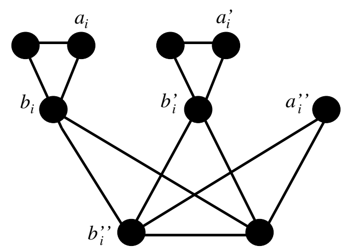

Fig. 7: Subgraph G_i corresponding to variable x_i.

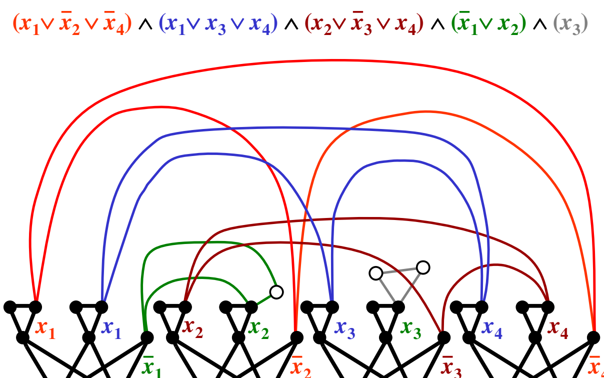

Fig. 8: Example of construction of G.

Let T_i denote a minimum triangle-transversal of G_i. Clearly, $|T_i| = 3$. We will assume that $T_i = \{a_i, a'_i, b''_i\}$ or $T_i = \{b_i, b'_i, a''_i\}$. As T_i is minimum, it cannot contain $\{a_i, a''_i\}$ or $\{a'_i, a''_i\}$.

Now, for each clause $\mathcal{C}_j$, $1 \leq j \leq m$, we create a triangle Z_j in G as follows:

- if x_i occurs in $\mathcal{C}_j$ and $\mathcal{C}_{j'}$ $(j \leq j')$ then $a_i \in V(Z_j)$ and $a'_i \in V(Z_{j'})$;
- if $\overline{x_i}$ occurs in $\mathcal{C}_j$ then $a''_i \in V(Z_j)$;
- if $|\mathcal{C}_j| = 2$ then include a new vertex c_j in $V(Z_j)$;
- if $|\mathcal{C}_j| = 1$ then include two new vertices c_j, c'_j in $V(Z_j)$.

See an example of construction of G in Figure 8.

We prove that $\mathcal{F}$ is satisfiable if and only if G admits a triangle-transversal T of size $3n$. If $\mathcal{F}$ is satisfiable, for each $1 \leq i \leq n$ include in T vertices a_i, a'_i, b''_i if x_i is set to "true", otherwise include in T vertices b_i, b'_i, a''_i. Observe that $|T| = 3n$ and every triangle in G is intersected by T.

Conversely, assume that G admits a triangle-transversal T of size $3n$. Let $T_i = T \cap V(G_i)$. Since each G_i contains three disjoint triangles and G contains n disjoint copies of G_i, we have $|T_i| = 3$, for every $1 \leq i \leq n$. Thus T_i is a minimum triangle-transversal of G_i, and as observed above it cannot contain $\{a_i, a''_i\}$ or $\{a'_i, a''_i\}$. We can then assume $T_i = \{a_i, a'_i, b''_i\}$ or $T_i = \{b_i, b'_i, a''_i\}$. In the former case we set x_i to "true", otherwise to "false". Hence, $\mathcal{F}$ is satisfiable. □

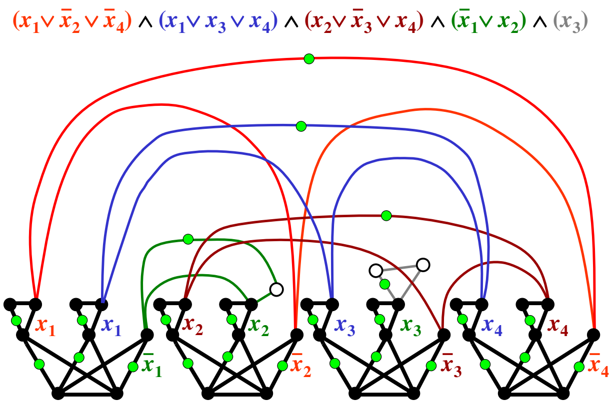

Fig. 9: Transforming triangles into C_4's.

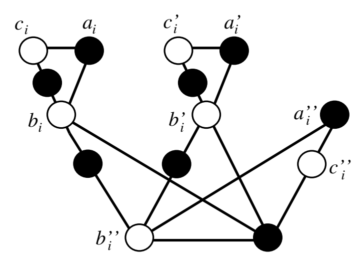

Fig. 10: Subgraph G_i corresponding to variable x_i for the case $k = 4$.

Using the same idea as in the previous section, we can transform the triangles of Figure 8 into C_k's, for any $k \geq 4$. See an example for $k = 4$ in Figure 9. Hence we again claim the following result, without

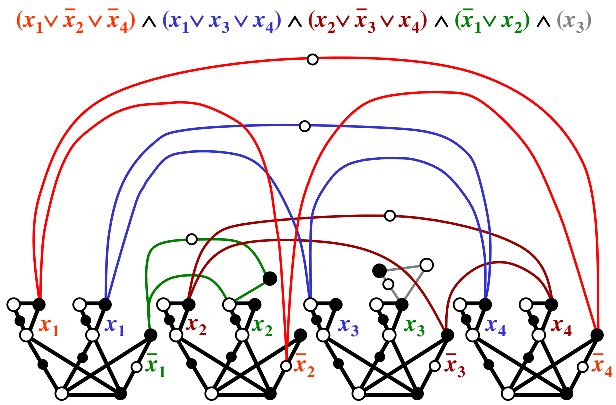

Fig. 11: Bipartite graph G for the case $k = 4$; $V(G)$ is divided into white and black vertices.

giving the formal details:

Theorem 12 C_k-TRANSVERSAL *is NP-complete for graphs of maximum degree at most four, for any fixed* $k \geq 3$*.*

Also, the subgraph G_i corresponding to variable x_i is bipartite when k is even. See an example in Figure 10. By slightly modifying the graph G of the reduction, we can make it bipartite: vertex c_i (resp., c'_i, c''_i) can be used instead of a_i (resp., a'_i, a''_i) to create the C_k's representing the clauses. See Figure 11. Therefore:

Theorem 13 C_{2k}-TRANSVERSAL *is NP-complete for bipartite graphs of maximum degree at most four, for any* $k \geq 2$*.*

3.2 A decomposition theorem and an approximation algorithm for triangle-transversals in graphs of maximum degree at most four

Consider the following naive k-approximation algorithm for finding C_k-transversals in general graphs, for a fixed $k \geq 3$. Given a graph G, initially set $T = \emptyset$ and $\mathscr{C} := \emptyset$. At each step: (i) locate an induced C_k, say C (which can be found in polynomial time, since k is fixed); (ii) set $T := T \cup V(C)$ and $\mathscr{C} := \mathscr{C} \cup \{C\}$; (iii) remove the vertices in $V(C)$ from G. Repeat (i)–(iii) until there are no more C_k's. Observe that the collection $\mathscr{C}$ is a C_k-*packing*, that is, a collection of vertex-disjoint C_k's. Also, T is clearly a C_k-transversal. If T^* is a minimum C_k-transversal, we have $|T^*| \geq |\mathscr{C}|$. Since $|T| = k|\mathscr{C}|$, it follows that $|T|/|T^*| \leq k$.

The above naive algorithm produces triangle-transversals with size at most three times the optimum. Nonetheless, a better behavior can be achieved by restricting the maximum degree of the input graph. In view of Theorem 11, we describe in this section a polynomial-time approximation algorithm for obtaining a triangle-transversal of a graph G with $\Delta = 4$.

We need the following definitions. A *tie* is a graph formed by five vertices a, b, c, d, z where $d(z) = 4$ and a, b, c, d induce $2K_2$. The vertex z is called a *bond*. A *piece* is a connected graph of maximum degree at most four containing no 3-free edges and no bonds. As we shall see, bonds and pieces play a crucial role in the algorithm. The following theorem characterizes pieces.

Theorem 14 *Let H be a piece. Then H is one of following graphs: $H_n (n \geq 3), H'_n (n \geq 7), H''_n (n \geq 8), G_4, G_{5i}$ $(1 \leq i \leq 5), G_{6j}$ $(1 \leq j \leq 5), G_7$.* (See Figure 12).

In Figure 12, the graph H_n $(n \geq 3)$ is formed by two paths $u_1 u_2 \ldots u_{\lfloor n/2 \rfloor}$ and $v_1 v_2 \ldots v_{\lceil n/2 \rceil}$, plus the following edges: $u_i v_i$ and $u_i v_{i+1}$, $1 \leq i \leq \lfloor n/2 \rfloor - 1$; $u_{\lfloor n/2 \rfloor} v_{\lfloor n/2 \rfloor}$; and, if n is odd, $u_{\lfloor n/2 \rfloor} v_{\lceil n/2 \rceil}$. The graph H'_n $(n \geq 7)$ is formed by a copy of H_n plus the following edges: $v_1 u_{\lfloor n/2 \rfloor}$; $v_1 v_{\lceil n/2 \rceil}$; and $u_1 v_{\lfloor n/2 \rfloor}$ (if n is even) or $u_1 v_{\lceil n/2 \rceil}$ (if n is odd). The graph H''_n is formed by a copy of H_n plus the edge $v_1 v_{\lceil n/2 \rceil}$ and a vertex w adjacent to $u_1, v_1, u_{\lfloor n/2 \rfloor}, v_{\lceil n/2 \rceil}$.

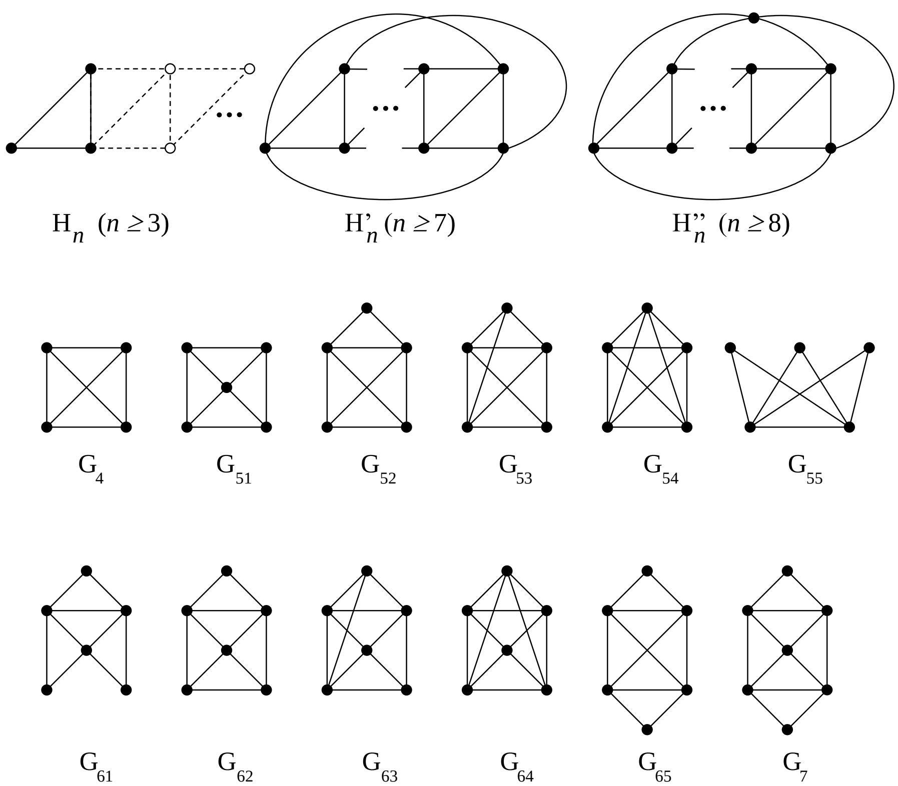

Fig. 12: All the possible pieces.

The proof of Theorem 14 is a consequence of the following two lemmas. A piece H is said to be *minimal* if $H - z$ is not a piece for any $z \in V(H)$.

Lemma 15 *If H is a minimal piece then $H = H_3$, $H = H'_n$ $(n \geq 7)$ or $H = H''_n$ $(n \geq 8)$.* (See Figure 12).

Proof: Observe first that H contains no cut vertex (since there are no 3-free edges in H, such a vertex would necessarily be a bond, a contradiction). Therefore, for an arbitrary vertex $z \in V(H)$, $H - z$ is connected and contains no bonds.

Denote by $t_H(e)$ the number of distinct triangles of H containing edge e. Since H contains no 3-free edges, $t_H(e) > 0$ for every $e \in E(H)$.

Since H is a minimal piece, the observations above imply the existence of a 3-free edge a_1b_1 in $H - z$. Then $t_H(a_1b_1) = 1$ (a_1b_1z is the only triangle of H containing e).

If $H = H_3$, the lemma follows. Suppose then that $H \neq H_3$. In this case, we must have $t_H(a_1z)$, $t_H(b_1z) > 1$, otherwise either $H = H_3$ or at least one of a_1, b_1 could be removed to obtain a smaller piece. Let a_1a_2z, b_1b_2z be new triangles containing a_1z, b_1z, respectively. Clearly, $a_2 \neq b_2$, otherwise $t_H(a_1b_1) > 1$. By the same reason, a_1b_2, $a_2b_1 \notin E(H)$.

Now consider vertex a_1. We cannot have $t_H(a_1a_2) = 1$, otherwise $H - a_1$ or $H - a_2$ would be a piece, contradicting minimality. Let therefore a_3 be a new vertex such that $a_1a_2a_3$ is a triangle containing a_1a_2. Observe that a_3 satisfies a_3z, $a_3b_1 \notin E(H)$.

The same arguments above lead to the existence of a new vertex b_3 adjacent to b_1, b_2 and not adjacent to a_1, z.

Let us show now that $a_2b_2 \notin E(H)$. Suppose to the contrary. Denote by $\delta_H(v)$ the degree of vertex v in graph H. Then $\delta_H(a_2) = \delta_H(b_2) = 4$ and, consequently, a_2b_3, $a_3b_2 \notin E(H)$. If $a_3b_3 \in E(H)$, there would exist a new vertex c adjacent to both a_3 and b_3 (since a_3b_3 cannot be a 3-free edge); but then both a_3 and b_3 would be bonds, a contradiction. Therefore $a_3b_3 \notin E(H)$ and $a_1, a_2, a_3, b_1, b_2, b_3, z$ induce G_7. However, if $H = G_7$ then H is not minimal; and if H properly contains G_7 then a_3 or b_3 would be a bond. In either case, a contradiction arises. We conclude that $a_2b_2 \notin E(H)$.

To complete the proof of the claim, we analyze two cases:

Case 1: $a_2b_3 \in E(H)$. In this case, $\delta_H(a_2) = 4$, and since a_2b_3 cannot be a 3-free edge, we must have $a_3b_3 \in E(H)$. Observe now that $a_3b_2 \in E(H)$, otherwise b_3 would be a bond. Therefore $a_1, a_2, a_3, b_1, b_2, b_3, z$ induce H'_7.

Case 2: $a_2b_3 \notin E(H)$. In this case, we must have $a_3b_2 \notin E(H)$, otherwise there should exist a new vertex c adjacent to both a_3 and b_2, implying $\delta_H(b_2) > 4$, a contradiction. We consider two subcases:

Case 2.1 $a_3b_3 \in E(H)$. Since a_3b_3 cannot be a 3-free edge, let c be a new vertex adjacent to both a_3 and b_3. Then c must also be adjacent to both b_2 and a_2 (otherwise a_3, b_3 would be bonds). Thus $a_1, a_2, a_3, b_1, b_2, b_3, z, c$ induce H''_8.

Case 2.2 $a_3b_3 \notin E(H)$. Since $t_H(a_1a_3) = t_H(b_1b_3) = 1$, we have $t_H(a_2a_3), t_H(b_2b_3) > 1$. This leads to the same situation as in the beginning of the proof of the Lemma, replacing a_1, z, b_1 by a_3, a_2, a_1 or b_3, b_2, b_1.

Since H is finite, at some point we will have two sequences of vertices $a_1, a_2, a_3, \ldots, a_j$ and $b_1, b_2, b_3, \ldots, b_k$ such that either

- Case 1 applies: $a_{j-1}b_k, a_jb_k, a_jb_{k-1} \in E(H)$, i.e., vertices $a_1, \ldots, a_j, b_1, \ldots, b_k, z$ induce H'_{j+k+1} or
- Case 2.1. applies: $a_{j-1}b_k, a_jb_{k-1} \notin E(H), a_jb_k \in E(H)$ and there exists c adjacent to vertices $a_{j-1}, a_j, b_{k-1}, b_k$, i.e., vertices $a_1, \ldots, a_j, b_1, \ldots, b_k, z, c$ induce H''_{j+k+2}.

$\square$

Lemma 16 *If H is a non-minimal piece then H is one of the graphs H_n $(n \geq 4)$, G_4, G_{5i} $(1 \leq i \leq 5)$, G_{6j} $(1 \leq j \leq 5)$, G_7.* (See Figure 12).

Proof: If H is a non-minimal piece with n vertices then there exists $z \in V(H)$ such that $H' = H - z$ is a piece with $n-1$ vertices. Since H_3 is the only piece with three vertices, H_4 and G_4 are the only possible pieces with four vertices. In general, by taking a piece H' with $n-1$ vertices, one can try to obtain pieces with n vertices by adding one vertex to H' in all possible ways. For instance, from H_4 we obtain H_5, G_{51}, G_{55}; and from G_4 we obtain G_{52}, G_{53}, G_{54}. Along this process, some maximal pieces (i.e., pieces not properly contained in pieces with one more vertex) are generated, such as G_{53}, G_{54}, G_{55}. To conclude the proof, we observe that, for $n \geq 6$, the only piece that can be generated from H_n by adding one vertex to it is H_{n+1}. $\square$

A direct consequence of Theorem 14 is:

Corollary 17 *Let G be a graph of maximum degree at most four containing no bonds. Then a minimum triangle-transversal of G can be obtained in polynomial time.*

Proof: After removing the 3-free edges of G, each of its connected components is a piece, for which a minimum triangle-transversal is easily obtained. $\square$

We analyze now graphs of maximum degree at most four that may contain bonds. We can restrict our analysis to connected graphs without 3-free edges. The following definition is useful.

Definition 18 *Let H be a piece in Figure 12, and let $v \in V(H)$. If $\delta_H(v) = 2$ then v is a* connector*, otherwise v is an* inner vertex.

Next, we describe a decomposition for graphs of maximum degree at most four containing no 3-free edges:

Definition 19 *Let G be a connected graph of maximum degree at most four without 3-free edges. The* piece decomposition *of G is the collection of pieces obtained by splitting each bond of G into two vertices, each having two adjacent neighbors, as shown in Figure 13. Each piece of the collection is also said to be a* **piece of *G***.

A piece decomposition of G can be obtained in polynomial time by locating its bonds. Observe that every bond is shared by two pieces of G. If v is a bond of G shared by two pieces H_1 and H_2 of G then v is a connector both of H_1 and H_2, since $\delta_{H_1}(v) = \delta_{H_2}(v) = 2$.

Pieces have an important property in relation to triangle-transversals:

Property 20 *Let G be a graph of maximum degree at most four containing no 3-free edges. Then, for every minimum triangle-transversal T of G and for every piece H of G, the number of inner vertices of H belonging to T is a function only of the number of connectors of H belonging to T.*

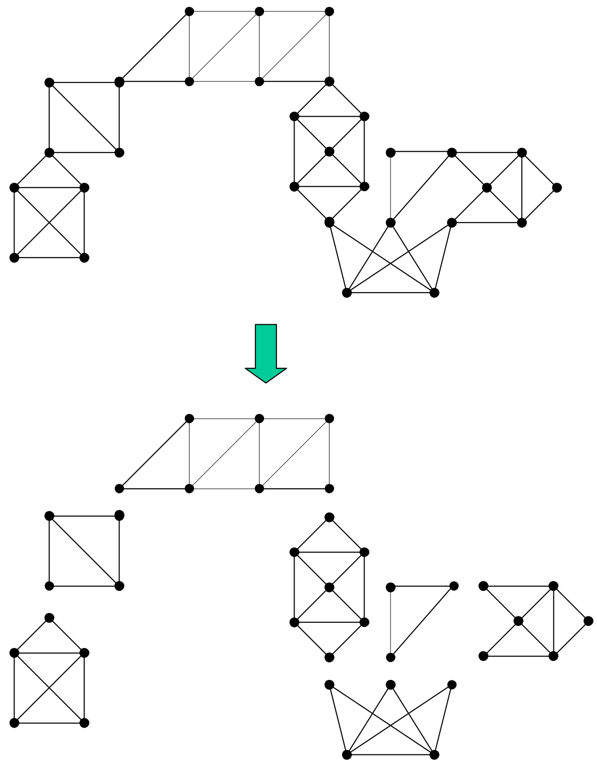

Fig. 13: Piece decomposition.

The above property is derived directly from the symmetry of connectors in every piece. Consider, for example, that G contains a piece H isomorphic to G_{61} (see Figure 12). If T contains exactly two connectors of H, no matter which of them, then T contains exactly one inner vertex of H. This leads us to the following definition.

Definition 21 *Let H be a piece. The* template *of H is a sequence $(t_0, \dots, t_k)$ such that:*

(1) k is the number of connectors of H;

(2) if H is a piece of a graph G of maximum degree at most four containing no 3-free edges, T is a minimum triangle-transversal of G, and ℓ is the number of connectors of H belonging to T, then t_ℓ is the number of inner vertices of H belonging to T.

For instance, the templates of G_{55} and G_{61} are, respectively, $(1, 1, 1, 0)$ and $(2, 1, 1, 1)$. The template of H_n, for $n \geq 4$, depends on the value of n: if $n = 3j$ then it is $(\frac{n}{3}, \frac{n-3}{3}, \frac{n-3}{3})$; if $n = 3j + 1$ then it is $(\frac{n-1}{3}, \frac{n-1}{3}, \frac{n-4}{3})$; and if $n = 3j + 2$ then it is $(\frac{n-2}{3}, \frac{n-2}{3}, \frac{n-2}{3})$.

The piece H_3 is special, since all of its vertices are connectors and the case $\ell = 0$ cannot occur for it. To be consistent with Definition 21, the template of H_3 is $(1, 0, 0, 0)$.

We remark that a piece with template of the form $(\alpha, \alpha, \ldots, \alpha)$ or $(\alpha, \alpha - 1, \ldots, \alpha - 1)$, for some $\alpha > 0$, will always have α of its vertices belonging to a minimum triangle-transversal T, one of which will be a connector.

Templates will be helpful to describe reduction rules that eliminate almost all types of pieces of an input graph G of maximum degree at most four.

Reduction rules. Let G be a graph of maximum degree at most four. Let T be a minimum triangle-transversal of G to be computed, initially empty. The following rules are used to transform G into a unique new *reduced graph* G^r. The rules are applied just once, in the given order.

1. Remove 3-free edges of G.

2. If there is a connected component C of G containing only one bond and only one piece then C is a *circular 3-ladder*, a graph obtained from H_n (for $n \geq 8$) by collapsing its degree-two vertices. In this case, it is easy to obtain a triangle-transversal T_C of C. Include in T the vertices of T_C and remove C from G.

3. If there is a connected component C of G containing as an induced subgraph a piece $H \in \{G_4, G_{51}, G_{53}, G_{54}, G_{63}, G_{64}, H'_n, H''_n\}$, then $C = H$, since H has no connectors. As in the previous rule, it is easy to obtain a triangle-transversal T_C of C in this case. Include in T the vertices of T_C and remove C from G.

4. For each piece H of G isomorphic to G_{52}, choose two inner vertices $v, w \in V(H)$ such that at least one of them has degree four. (Recall that the template of G_{52} is $(2, 2)$.) Include v, w in T, and remove from G all the inner vertices of H. An analogous procedure can be applied to any piece isomorphic to G_{65}, provided that v, w are not adjacent to the same connector of H.

5. For each piece H of G isomorphic to G_{62}, let v be the connector of H and w the inner vertex of H whose neighbors induce C_4. Include v, w in T, and remove all the vertices of H from G (including v).

6. The templates of G_7 and H_7 are identical. Thus, transform every piece H of G isomorphic to G_7 into another piece isomorphic to H_7, as follows: if v and w are the connectors of H, and xy is an edge of H such that x is adjacent to v and y is adjacent to w, then remove xy from G.

7. The template of G_{61} is $(2, 1, 1, 1)$. Note that it can be obtained by adding one to each t_i in the template of H_3. Thus, if H is a piece of G isomorphic to G_{61} where a, b, c are its connectors and v, w, x its inner vertices, remove v, w, x from G and add to G the edges ab, ac, bc. This corresponds to "replacing" H by a copy of H_3. It is easy to see that there exists a triangle-transversal of the new graph with size q if and only if there exists a triangle-transversal of the previous graph with size $q + 1$.

8. For $n = 3j + 2$ ($j \geq 1$), the template of H_n says that the number of inner vertices to be included in T is always the same. Thus, for each piece H isomorphic to H_{3j+2} for some $j \geq 1$, include in T a suitable subset of j inner vertices of H, and remove from G all the inner vertices of H. (In the case of H_5, for instance, the degree-four inner vertex must be included in T.)

9. For $n = 3j + 1$ ($j \geq 2$) the template of H_n can be obtained by adding $j - 1$ to each t_ℓ in the template of H_4. Thus, if H is a piece of G isomorphic to H_{3j+1} for some $j \geq 2$, remove from G all the

inner vertices of H except the two neighbors v, w of some connector of H; next, link v, w to the other connector of H. This corresponds to replacing H by a copy of H_4. Again, it is easy to see that there exists a triangle-transversal of the new graph with size q if and only if there exists a triangle-transversal of the previous graph with size $q + j - 1$.

10. For $n = 3j$ ($j \geq 2$) the template of H_n can be obtained by adding $j-1$ to each t_ℓ in the template of H_3 (for $\ell \leq 2$). Thus, if H is a piece of G isomorphic to H_{3j} for some $j \geq 2$, remove all the inner vertices of H and create a triangle using the connectors of H together with a new vertex x. This corresponds to replacing H by a copy of H_3. Since x is a degree-two vertex, we can assume that $x \notin T$. Hence, there exists a triangle-transversal of the new graph with size q if and only if there exists a triangle-transversal of the previous graph with size $q + j - 1$.

11. The only possible pieces of G are now H_3, H_4 and G_{55}. The pieces H_4 and G_{55} are called *crowns* (respectively, *2-crown* and *3-crown*). Then, for every 2-crown or 3-crown containing a vertex with degree two in G, add one of its inner vertices to T and remove all of its vertices from G, except the ones playing the role of bonds in G. The application of the rules is completed. □

We denote by G^r the graph obtained from G by the application of rules $1 - 11$. Graph G^r is called the *reduced graph of* G, which is uniquely defined from G. It will be useful to define an intersection graph $\mathcal{P}(G^r)$ as follows: the pieces of G^r (triangles or crowns) are the vertices of $\mathcal{P}(G^r)$, and two vertices of $\mathcal{P}(G^r)$ are adjacent if and only if they share a bond of G^r. The vertices representing triangles are called *t-vertices*, and the vertices representing crowns are called *c-vertices*. In addition, c-vertices representing 2-crowns (respectively, 3-crowns) are called *c^2-vertices* (respectively, *c^3-vertices*). Clearly, $\mathcal{P}(G^r)$ is a graph of maximum degree at most three.

We remark that the graph of maximum degree at most four constructed in the reduction of Theorem 11 contains only triangles and crowns as pieces. Hence, TRIANGLE-TRANSVERSAL remains NP-complete for graphs of maximum degree at most four containing only such pieces. By excluding the crowns, we have the result below.

Theorem 22 TRIANGLE-TRANSVERSAL *is polynomial time solvable for graphs G of maximum degree at most four for which G^r contains no piece isomorphic to a crown.*

Proof: If G^r contains no piece isomorphic to a crown then $\mathcal{P}(G^r)$ contains only t-vertices. Take a maximum matching M of $\mathcal{P}(G^r)$. Let S be the subset of M-unsaturated vertices of $\mathcal{P}(G^r)$. An optimal triangle-transversal T of G^r is formed as follows: for each edge $e \in M$, include in T the corresponding bond of G^r, and for each vertex of S include in T any vertex of the corresponding piece of G^r. □

We now analyze the performance of the following approximation algorithm $\mathscr{A}$: given a graph G of maximum degree at most four, compute G^r, and for each crown C of G^r whose inner vertices are u_C and v_C, include u_C in T and remove u_C, v_C from G^r; then apply to the resulting graph the method described in Theorem 22. Let $\mathscr{A}(G)$ be the size of the triangle-transversal obtained by the application of algorithm $\mathscr{A}$ to G, and denote by $OPT(G)$ the size of an optimal solution for G.

Theorem 23 *Let G be a graph of maximum degree at most four.*
(a) If G^r contains no 2-crowns then $\mathscr{A}(G) \leq \frac{4}{3}OPT(G)$.
(b) If G^r contains no 3-crowns then $\mathscr{A}(G) \leq \frac{3}{2}OPT(G)$.

Proof: We present only the proof of (a); the proof of (b) is similar.

We first observe that there exists an integer $c_G \geq 0$ such that $OPT(G) = OPT(G^r) + c_G$ and $\mathscr{A}(G) = \mathscr{A}(G^r) + c_G$. Thus, it suffices to show that $\mathscr{A}(G^r) \leq \frac{4}{3}OPT(G^r)$.

Let T^* be an optimal triangle-transversal for G^r, and let T be a triangle-transversal obtained by applying algorithm $\mathscr{A}$ to G^r. Say that a crown C of G^r is *unmatched* if $T^* \cap \{u_C, v_C\} = \emptyset$, where u_C, v_C are the inner vertices of C.

Define subsets $A, B \subseteq T^*$ as follows:

$$\begin{aligned} A &= \{v \in T^* \mid v \text{ is a bond of } G^r \text{ and } v \text{ belongs to two unmatched crowns of } G^r\}, \\ B &= \{v \in T^* \mid v \text{ is a bond of } G^r \text{ and } v \text{ belongs to exactly one unmatched crown of } G^r\}. \end{aligned}$$

Write $|A| = a$ and $|B| = b$, and let k be the number of unmatched crowns. Since G^r contains no 2-crowns, $2a + b = 3k$ (I). Thus, $k - a = a + b - 2k$. Moreover, from (I), $a + b \leq 3k$. By manipulating this inequality, $a + b - 2k \leq \frac{a+b}{3}$. We conclude that $k - a \leq \frac{a+b}{3}$ (II).

For each unmatched crown C, algorithm $\mathscr{A}$ selects u_C or v_C to include in T. Also, $T \cap A = \emptyset$. Thus $|T| \leq |T^*| + k - a$. By (II), $|T| \leq |T^*| + \frac{a+b}{3}$. Since $a + b \leq |T^*|$, we have $|T| \leq |T^*| + \frac{|T^*|}{3} = \frac{4}{3}|T^*|$. □

Let G_1 (respectively, G_2) be a graph formed by k disjoint copies of the graph in Figure 14(a) (respectively, Figure 14(b)). Then G_1 and G_2 are reduced graphs and satisfy $\mathscr{A}(G_1) = 4k, OPT(G_1) = 3k, \mathscr{A}(G_2) = 3k$ and $OPT(G_2) = 2k$. Thus, the bounds in Theorem 23 are tight.

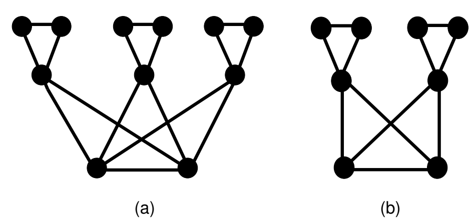

Fig. 14: Analyzing the approximation ratio.

We remark that running algorithm $\mathscr{A}$ on G^r and converting the triangle-transversal of G^r into a triangle-transversal of G yields an $\alpha(G)$-approximation algorithm with $\alpha \in [\frac{4}{3}, \frac{3}{2}]$. The value of $\alpha(G)$ depends on the ratio between the number of 3-crowns and the number of 2-crowns in G^r. In general, we have:

Theorem 24 *Let G be a graph of maximum degree at most four. Then there exists a $\frac{3}{2}$-approximation algorithm for computing a triangle-transversal of G that runs in polynomial time.* □

3.3 Extending the reduction rules

We can extend the reduction rules presented in the previous subsection in order to obtain some additional polynomial cases.

Extended reduction rules. Let G be a graph of maximum degree at most four. Set $\mathcal{G} = G^r$. It is clear that two adjacent t-vertices in $\mathcal{P}(\mathcal{G})$ represent two triangle-pieces sharing exactly one bond in $\mathcal{G}$. In addition, if a t-vertex x is adjacent to a c-vertex y in $\mathcal{P}(\mathcal{G})$ then the corresponding pieces in $\mathcal{G}$ also share exactly one bond (otherwise we would have the configuration of G_{65}).

12. If two adjacent c-vertices y_1, y_2 in $\mathcal{P}(\mathcal{G})$ correspond to two crowns C_1, C_2 sharing more than one bond in $\mathcal{G}$ then include in T two inner vertices, one for each crown. (Recall that T denotes a minimum triangle-transversal being computed.) Remove from $\mathcal{G}$ all the inner vertices of C_1, C_2. At this point, there is a one-to-one correspondence between edges of $\mathcal{P}(\mathcal{G})$ and bonds of $\mathcal{G}$. From now on, we do not distinguish a t-vertex x in $\mathcal{P}(\mathcal{G})$ and the corresponding triangle-piece in $\mathcal{G}$. The same applies to c-vertices in $\mathcal{P}(\mathcal{G})$ and corresponding crowns in $\mathcal{G}$. For two adjacent vertices in $\mathcal{P}(\mathcal{G})$, we also refer to the bond shared by them in $\mathcal{G}$.
13. If a t-vertex x is a pendant vertex of $\mathcal{P}(\mathcal{G})$ then we can include in T the bond v of $\mathcal{G}$ corresponding to the edge incident to x in $\mathcal{P}(\mathcal{G})$. Let X be the subset of vertices of $\mathcal{G}$ belonging to the t-vertex x. Remove X from $\mathcal{G}$.
14. A c^3-vertex y with degree strictly less than three in $\mathcal{P}(\mathcal{G})$ can be disregarded by including in T one of its inner vertices and removing from $\mathcal{G}$ all of its inner vertices. The same procedure can be applied to a c^2-vertex with degree one in $\mathcal{P}(\mathcal{G})$.
15. Suppose that $\mathcal{P}(\mathcal{G})$ contains a chordless cycle $\mathcal{C}$ of t-vertices isomorphic to C_k, such that exactly one of the t-vertices of $\mathcal{C}$, say x, has degree three in $\mathcal{P}(\mathcal{G})$. Let v be the bond of x corresponding to the edge incident to x in $\mathcal{P}(\mathcal{G})$ which does not belong to $E(\mathcal{C})$, and let Z be the subset of vertices of $\mathcal{G}$ belonging to the t-vertices forming C.
    (a) if k is even, include in T k suitable vertices of $\mathcal{G}$ belonging to alternating t-vertices of $\mathcal{C}$, and remove $Z \backslash \{v\}$ from $\mathcal{G}$.
    (b) if k is odd, include in T the vertex v plus k suitable vertices of $\mathcal{G}$ belonging to alternating t-vertices of $\mathcal{C}$, and remove Z from $\mathcal{G}$.

Rule 15 can be generalized in the following way:

16. Let $\mathcal{H}$ be an induced subgraph of $\mathcal{P}(\mathcal{G})$ such that exactly one of the vertices of $\mathcal{H}$, say x, has a neighbor outside $\mathcal{H}$. Let Z be the subset of vertices of G belonging to t-vertices or c-vertices of $\mathcal{H}$, and let $\mathcal{G}' = \mathcal{G}[Z]$. Finally, let v be the bond of $\mathcal{G}$ corresponding to the edge incident to x in $\mathcal{P}(\mathcal{G})$ which does not belong to $E(\mathcal{H})$. Then:
    (a) If $OPT(\mathcal{G}') = k$ and $OPT(\mathcal{G}' - v) = k$ then $OPT(G) = OPT(G - \ (Z \backslash \{v\})\ ) + k$;
    (b) if $OPT(\mathcal{G}') = k$ and $OPT(\mathcal{G}' - v) = k - 1$ then $OPT(\mathcal{G}) = OPT(\mathcal{G} - Z) + k$.

To conclude this section, we present another case that can be solved in polynomial time. A graph H of maximum degree at most three is a *pseudo-tree* if every induced subgraph H' of H satisfies at least one of the following properties: H' contains a pendant vertex; H' is a disjoint union of chordless cycles; H' contains a chordless cycle where exactly one vertex of the cycle has degree three. Using iteratively the rules above, we have the following result, stated without proof.

Theorem 25 TRIANGLE-TRANSVERSAL *is polynomial time solvable for graphs G of maximum degree at most four for which $\mathcal{P}(G^r)$ is a pseudo-tree.* □

4 Conclusions

In this paper we have studied the problem C_k-TRANSVERSAL for graphs of maximum degree at most Δ, for all values k and Δ. We have shown that the problem is polynomial-time solvable for $k \leq 4$ and $\Delta = 3$, and NP-complete otherwise; in particular, we have characterized graphs of maximum degree at most three containing no 4-free edges in terms of circular/Möbius ladders. A polynomial-time approximation algorithm for finding C_k-transversals, for $k = 3$, in graphs of maximum degree at most four was presented, based on a new decomposition theorem and reduction rules for such graphs. An interesting question is to devise approximation algorithms for other values of k.

Acknowledgments

The authors are very grateful to the two anonymous referees for providing us with constructive comments and suggestions, which greatly improved the presentation of this work.

References

[1] R. L. Brooks. On colouring the nodes of a network. *Proc. Cambridge Phil. Soc.* 37 (1941) 194–197.

[2] M. R. Cerioli, L. Faria, T. O. Ferreira, C. A. J. Martinhon, F. Protti and B. Reed. Partition into cliques for cubic graphs: planar case, complexity and an approximation algorithm. *Discrete Applied Mathematics* 156 (2008) 2270–2278.

[3] D. Cornaz and A. R. Mahjoub. The maximum induced bipartite subgraph problem with edge weights. Submitted manuscript.

[4] M. Dantas da Silva, F. Protti and J. L. Szwarcfiter. Applying modular decomposition to parameterized cluster editing problems. *Theory of Computing Systems* 44 (2009) 91–104.

[5] D. P. Dailey. Uniqueness of colorability and colorability of planar 4-regular graphs are NP-complete. *Discrete Mathematics* 30 (1980) 289–293.

[6] P. C. Fishburn. *Interval Orders and Interval Graphs*. Wiley, New York, 1985.

[7] A. Galluccio, P. Hell and J. Nešetřil. The complexity of H-colouring of bounded degree graphs. *Discrete Mathematics* 222 (2000) 101–109.

[8] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman, New York, 1979.

[9] J. Gramm, J. Guo, F. Hüffner, and R. Niedermeier. Graph-modeled data clustering: Fixed-parameter algorithms for clique generation. *Theory of Computing Systems* 38, 4 (2005) 373–392.

[10] D. Lichtenstein. Planar formulae and their uses. *SIAM Journal on Computing* 43 (1982) 329–393.

[11] B. Reed, K. Smith and A. Vetta. Finding odd cycle transversals. *Operations Research Letters* 32 (2004) 299–301.

[12] G. Manic and Y. Wakabayashi. Packing triangles in low-degree graphs and indifference graphs. *Proc. European Conference on Combinatorics, Graph Theory and Applications (EuroComb'05)*, Berlin, Germany, 2005. *Discrete Mathematics and Theoretical Computer Science (DMTCS)*, Vol. AE 2005, pp. 251–256.

[13] M. Yannakakis. Node- and edge-deletion NP-complete problems. *Proceedings of the Tenth Annual ACM Symposium on Theory of Computing – STOC'78*, pp. 253–264, 1978, ACM Press.