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Vertex-colouring edge-weightings with two edge weights

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An edge-weighting vertex colouring of a graph is an edge-weight assignment such that the accumulated weights at the vertices yields a proper vertex colouring. If such an assignment from a set S exists, we say the graph is S-weight colourable. It is conjectured that every graph with no isolated edge is $\{1, 2, 3\}$-weight colourable.

We explore the problem of classifying those graphs which are $\{1, 2\}$-weight colourable. We establish that a number of classes of graphs are S-weight colourable for much more general sets S of size 2. In particular, we show that any graph having only cycles of length $0 \bmod 4$ is S-weight colourable for most sets S of size 2. As a consequence, we classify the minimal graphs which are not $\{1, 2\}$-weight colourable with respect to subgraph containment. We also demonstrate techniques for constructing graphs which are not $\{1, 2\}$-weight colourable.

Keywords: edge weighting, graph colouring

1 Introduction

Let G be a simple graph and S be a set of real numbers. An S-*edge-weighting* of G is an assignment $w : E(G) \to S$. Given an S-edge-weighting, the *weighted degree* of a vertex v, denoted $w(v)$, is the sum of weights of the edges incident with v. An S-edge-weighting gives a *vertex colouring* if the weighted degrees of adjacent vertices are different. If an S-edge-weighting vertex colouring w exists, we also call w an S-*weight colouring* and we say G is S-*weight colourable*. For a positive integer k, we say G has a k-*weight colouring* or G is k-*weight colourable* if it is S-weight colourable for every set S of size k. The most commonly studied sets S are those of the form $\{1, \ldots, k\}$.

Problem 1 *Given a graph G with no isolated edges, find the minimum k such that G is $\{1, \ldots, k\}$-weight colourable.*

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It is not hard to verify that K_4 with a single leaf attached is $\{1,2\}$-weight colourable but is not $\{0,1\}$-weight colourable. It follows that the S-weight colourability of a graph is not only dependent on the size of S but also on the particular elements of S. However, if a graph G is S-weight colourable then there exists an $i_0 = i_0(G,S)$ such that for all $i > i_0$ the graph is also $\{s+i : s \in S\}$-weight colourable. One such value for i_0, though not necessarily the smallest, is $i_0 = |S| \cdot \Delta(G) \cdot \max\{|s| : s \in S\}$, where $\Delta(G)$ is the maximum degree of G.

Let us start by considering the 2-weight colourability of a simple class of graphs – paths. If a and b are non-zero real numbers, then every path of length at least 2 has an $\{a,b\}$-weight colouring. Assigning the edge weights $a,a,b,b,a,a,b,b,\ldots$, beginning with one leaf of the path, gives such a colouring. However, a path has a $\{0,a\}$-weight colouring if and only if it is not of length $1 \bmod 4$. The reader can easily check that paths of length 2, 3 and 4 have a $\{0,a\}$-weight colouring. However, if we let $P = e_1, e_2, e_3, e_4, e_5$ be a path of length 5 (we omit vertex labels) then if $w(e_2) = 0$ (or $w(e_4) = 0$) then the ends of e_1 (e_5) will have equal weight. Thus the only way to achieve a $\{0,a\}$-weight colouring of P is if $w(e_2) = w(e_4) = a$. However, this implies that the ends of e_3 will have the same weight, and hence a $\{0,a\}$-weight colouring cannot exist. These examples easily extend to longer paths; the details are left to the reader.

In general, it is unknown how difficult it is to decide if a given graph admits a $\{1,2\}$-weight colouring, or more generally an $\{a,b\}$-weight colouring. As such, we present the following question:

Problem 2 *Is it NP-complete to decide whether a given graph is 2-weight colourable?*

Returning to Problem 1, we state the following conjecture, due to Karoński, Łuczak, and Thomason [KŁT04], which motivates most of the known results on the $\{1,\ldots,k\}$-weight colourability of graphs.

Conjecture 1.1 *Every graph with no isolated edge is $\{1,2,3\}$-weight colourable.*

Karoński et al. [KŁT04] showed that the Conjecture 1.1 is true for 3-colourable graphs. They also proved that if S is any set of at least 183 real numbers which are linearly independent over the rational numbers then every graph with no isolated edge is S-weight colourable. Recently, Kalkowski et al. [KKP09] showed that every graph with no isolated edge is $\{1,\ldots,k\}$-weight colourable for $k = 5$. This result is an improvement on the previous bounds on k established by Addario-Berry et al. [ABDM$^+$07], Addario-Berry et al. [ABDR08], and Wang et al. [WY08], who obtained the bounds $k = 30$, $k = 16$, and $k = 13$, respectively.

Our work in this paper is similarly motivated by Conjecture 1.1. However, where most others have attempted to lower the best known value of k as described above, our focus is on establishing which graphs are $\{1,2\}$-weight colourable. Addario-Berry, Dalal and Reed [ABDR08] showed that asymptotically almost every graph is $\{1,2\}$-weight colourable, however it is not known which ones are not. Chang et al and Lu et al ([CLWY10], [LYZ10]) have made some progress in determining which classes of graphs are $\{1,2\}$-weight colourable, notably having shown that 3-connected bipartite graphs are one such class. A complete classification of such graphs would determine those graphs for which $k = 3$ is the smallest possible solution in Problem 1, and would reduce Conjecture 1.1 to just those graphs.

The results that follow are, for the most part, concerned with a more general problem than that of finding $\{1,2\}$-weight colourings, namely that of finding $\{a,b\}$-weight colourings for more general values of a and b. In such cases, the existence of a $\{1,2\}$-weight colouring follows as an unstated corollary. In Section 2, we establish a wide range of basic graphs which admit $\{a,b\}$-weight colourings. We also establish classes of graphs which do not admit $\{a,b\}$-weight colourings, but which do admit an $\{a,b\}$-edge weighting which is almost a proper colouring. These results provide building blocks for our results on the weight colourability of bipartite graphs in Section 3 and of other general classes of graphs, particularly direct products of graphs, in Section 4. Of note, we show in Section 3 that if every cycle of G is of length $0 \bmod 4$, then G is $\{1, 2\}$-weight colourable.

2 Building blocks: Weight colourings of basic graphs

We will use standard graph theory terminology; the reader may refer to [BM08] for clarification of any terms which are not specifically defined here.

The *length* of a path (walk) is defined to be the number of edges of the path (walk). A *thread* in a graph G is a walk connecting two vertices x and y, not necessarily distinct, such that the internal vertices are distinct from all others on the walk, all internal vertices have degree 2 in G, and $\deg(x), \deg(y) \geq 3$. If x and y are distinct, then the walk is in fact a path and in this case we may refer to the thread as an *ear*. If the condition that $\deg(x), \deg(y) \geq 3$ is changed to $\deg(x), \deg(y) \geq 2$ in either case, we have a *subthread* or *subear* respectively.

A *cut vertex* of a graph is one whose removal disconnects the graph. A graph is 2-connected if it has no cut vertex. A graph (not necessarily simple) is called *separable* if it can be decomposed into two nonempty subgraphs with exactly one vertex in common. A simple graph is separable if and only if it is not 2-connected. A maximal nonseparable subgraph of G is a *block* of G. Note that a block is isomorphic either to K_2 or to a 2-connected graph. An *end block* of G is a block which contains at most one cut vertex of G.

A graph is c-*colourable* if the vertices can be coloured with c colours so that adjacent vertices get different colours.

K_n and C_n, respectively, denote the complete graph and the cycle on n vertices. The *Cartesian product* of two graphs G and H, denoted by $G \,\square\, H$, is defined as the graph having vertex set $V(G) \times V(H)$ where two vertices (u, u') and (v, v') are adjacent if and only if either $u = v$ and u' is adjacent to v' in H or $u' = v'$ and u is adjacent to v in G.

We present a few simple observations.

Proposition 2.1 *Let a, b, t be nonzero real numbers and G a graph. Then*

(i) G is $\{a, b\}$-weight colourable if and only if G is $\{at, bt\}$-weight colourable, and

(ii) if G is $\{a, b\}$-weight colourable then G is $\{p, q\}$-weight colourable for any nonzero $p, q \in \mathbb{R}$ which are linearly independent over $\mathbb{Q}$.

Proof: (i) This follows from the fact that $w(u) \neq w(v)$ if and only if $t \cdot w(u) \neq t \cdot w(v)$. (ii) Note that if two adjacent vertices receive distinct linear combinations of a and b as weights, then the coefficients of these linear combinations will suffice for any two linearly independent nonzero reals. □

From Proposition 2.1 we deduce the following, adopting the convention that 0 and 1 are relatively prime integers:

Corollary 2.2 *A graph G is 2-weight colourable if and only if G is $\{a, b\}$-weight colourable for every pair of relatively prime integers a and b.*

Proposition 2.1 allows us to reduce our proofs of positive results on the existence of $\{a, b\}$-weight colourings of a graph to relatively prime integers. Results in which we show that G does not admit an $\{a, b\}$-weight colouring will not rely on such assumptions – we will prove them for all real a, b.

Proposition 2.3 *If G is d-regular and $\{a, b\}$-weight colourable for a fixed choice of a and b then (i) it is d-colourable, and (ii) it is 2-weight colourable.*

Proof: (i) The weighted degree of each vertex must be a number of the form $ta + (d - t)b$ for some $0 \leq t \leq d$, and a vertex of weighted degree da cannot be adjacent to a vertex of weighted degree db. Thus putting the vertices of weighted degree da or db in the same colour class gives a d-colouring.

(ii) In an $\{a, b\}$-edge weighting of a d-regular graph, the accumulated weight at any vertex is in a one-to-one correspondence with the number of incident edges of weight a. Thus if one choice of a and b gives a vertex colouring, then any other choice of a and b will as well. □

Corollary 2.4 *If $\chi(G) = \Delta(G) + 1$ or, equivalently (by Brooks theorem), if G is an odd cycle or a complete graph then G is not S-weight colourable for any set S of size 2.*

Even though the complete graph is not S-weight colourable for any set of size 2, it has an S-edge-weighting that is very close to being an S-weight colouring. This specific weighting will be useful in constructing families of 2-weight colourable graphs and non-2-weight colourable graphs in Section 4.

Lemma 2.5 *Given $n \geq 2$ and $a \neq b \in \mathbb{R}$, there is an $\{a, b\}$-edge-weighting of K_n such that the weighted degrees of all the vertices are distinct except for 2 of them. Furthermore, in any such $\{a, b\}$-edge-weighting, the degree sequence of the subgraph induced by the edges of weight a (as well as the subgraph induced by the edges of weight b) is either*

$$(1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1, \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor + 1, \ldots, n - 2, n - 1),$$

or

$$(0, 1, \ldots, \left\lceil \frac{n}{2} \right\rceil - 2, \left\lceil \frac{n}{2} \right\rceil - 1, \left\lceil \frac{n}{2} \right\rceil - 1, \left\lceil \frac{n}{2} \right\rceil, \ldots, n - 3, n - 2).$$

Proof: We prove the first part with an explicit construction. Choose any two vertices and assign weight a to the edge joining them. Choose a new vertex and assign weight b to all the edges joining this vertex to the previous two vertices. Choose another vertex and assign weight a to all the edges joining this vertex to the previous three vertices. By repeating this process until all vertices are exhausted, we achieve the desired edge-weighting since the two vertices chosen first will have the same weight while the remainder of the graph is properly coloured. Note that we achieve the same result by swapping a and b in this argument.

We prove the second part of the lemma by induction on n. Suppose w is such an edge-weighting of K_n and let $w(u) = w(v)$. It is easy to verify the claim for $n = 2$ and $n = 3$. If $w(x) \notin \{(n-1)a, (n-1)b\}$ for every vertex x then $w(x)$ can only take $n - 2$ values, a contradiction to the choice of w. If $w(u) = w(v) \in \{(n-1)a, (n-1)b\}$ then by removing u and v, w induces an $\{a, b\}$-weight colouring of K_{n-2}, a contradiction to Corollary 2.4. Thus there exists a vertex $x \neq u, v$ such that $w(x) \in \{(n-1)a, (n-1)b\}$. The claim follows by induction on $K_n - x$. □

The following technical lemmata will be useful for the rest of the paper, since they establish useful tools for finding edge-weighting vertex colourings of graphs with specific structural properties.

Lemma 2.6 *Suppose G has a vertex v with a set of leaf neighbours L where $|L| \geq \lceil \deg(v)/2 \rceil$. Let $a \neq b$ be real numbers with $ab > 0$. If $G \setminus L$ is $\{a, b\}$-weight colourable, then so is G.*

Proof: As mentioned, Proposition 2.1 allows us to only consider $a, b \in \mathbb{Z}^+$. Suppose w is an $\{a, b\}$-weight colouring of $G \setminus L$. The possible extensions of w to G give exactly $|L| + 1$ possible weights for v. Since v has at most $|L|$ neighbours in $G \setminus L$, in at least one of the extensions, the weighted degree of v is different from the weighted degrees of the neighbours of v in $G \setminus L$. The weighted degree of v is also different from the weighted degrees of the neighbours of v in L, since $ab > 0$. □

Corollary 2.7 *Every tree with at least* 3 *vertices is* $\{a, b\}$*-weight colourable, where* $a \neq b$ *are real numbers with* $ab > 0$*.*

Proof: The statement holds for any star, $K_{1,n-1}$, since the assignment of a to all edges achieves the desired result. As such the result holds for $n = 3$ since the unique tree on 3 vertices is a star. Let T be a tree on n vertices which is not a star and assume the result holds for any tree with fewer than n vertices. Every tree has a vertex v that has at least $\lceil \deg(v)/2 \rceil$ leaf neighbours. Since T is not a star, removing the leaf neighbours of v gives a subtree T' on at least 3 vertices. By the induction hypothesis T' has an $\{a, b\}$-weight colouring. By Lemma 2.6, T does as well. □

The following lemma establishes that we may contract long threads in a way that maintains weight colourability.

Lemma 2.8 *Let* G *be a graph,* $P = v_0, e_1, v_1, e_2, v_2, e_3, v_3, e_4, v_4, e_5, v_5$ *be a subthread of* G*, and* $a \neq b$ *be any two real numbers. Let* $G' = G/\{e_1, e_2, e_3, e_4\}$ *Then,*

(i) If w *is an* $\{a, b\}$*-weight colouring of* G*, then* $w(e_1) = w(e_5) \neq w(e_3)$*.*

(ii) If G' *is* $\{a, b\}$*-weight colourable, then so is* G*.*

(iii) If $\deg(v_0) = 2$ *or* $\deg(v_5) = 2$*, then* G *is* $\{a, b\}$*-weight colourable if and only if* G' *is* $\{a, b\}$*-weight colourable.*

Proof: (i) If $w(e_1) \neq w(e_5)$ then either one of the two choices for $w(e_3)$ results in an improper colouring at e_2 or e_4. Hence $w(e_1) = w(e_5)$ and $w(e_3)$ must be distinct.

(ii) For convenience, we still denote the vertex obtained from the contraction by v_0. Suppose w' is an $\{a, b\}$-weight colouring of G'. Then $w'(v_0) \neq w'(v_5)$. Without loss of generality assume $w'(v_0v_5) = a$. Let $w(e) = w'(e)$ for each $e \notin \{e_1, e_2, e_3, e_4, e_5\}$, $w(e_1) = w(e_5) = a$ and $w(e_3) = b$. There are two possibilities for the weights of e_2 and e_4. Assigning $w(e_2) = a$ and $w(e_4) = b$ does not yield a proper vertex colouring of G if and only if either $w(v_0) = 2a$ or $w(v_5) = a + b$. Similarly, defining $w(e_2) = b$ and $w(e_4) = a$ does not yield a proper vertex colouring of G if and only if either $w(v_0) = a + b$ or $w(v_5) = 2a$. Suppose that neither weighting works. If the first possibility gives $w(v_0) = 2a$, then the second must give $w(v_5) = 2a$. If the first possibility gives $w(v_5) = a + b$, then the second gives $w(v_0) = a + b$. In either case $w(v_0) = w(v_5)$, a contradiction.

(iii) Assume $\deg(v_0) = 2$ and let e_0 be the other edge incident with v_0. Suppose w is an $\{a, b\}$-weight colouring of G. By (i) we have $w(e_0) = w(e_4)$ and $w(e_1) = w(e_5)$. Hence $w(v_0) = w(v_4) \neq w(v_5)$. Thus, by assigning the common weight of e_1 and e_5 to the edge v_0v_5, we get an $\{a, b\}$-weight colouring of G'. □

The degree condition on the ends of P in Lemma 2.8 (iii) cannot be dropped. For example, by taking G to be the path of length 5, $a = 1$, and $b = 2$, Lemma 2.8 (iii) fails.

From this lemma we may deduce necessary and sufficient conditions for the existence of $\{a,b\}$-weight colourings of cycles.

Proposition 2.9 *Let a and b be any distinct real numbers. Then C_n is $\{a,b\}$-weight colourable if and only if $n \equiv 0 \pmod 4$.*

In lieu of a proof, we simply note that, by Lemma 2.8 (iii), the proof of this proposition may be reduced to the cases C_3, C_4, C_5 and C_6. The details are left to the reader. There are $\{a,b\}$-edge weightings of other cycles of length $4k+1, 4k+2$ and $4k+3$ which give vertex colourings with as few conflicts as possible. These results are largely technical, though not difficult to prove.

Proposition 2.10 *Let a and b be any distinct real numbers. Then C_{2k+1} has an $\{a,b\}$-edge weighting w such that only one edge $e = uv$ has the property that $w(u) = w(v)$.*

Proposition 2.11 *Let a and b be any distinct real numbers. Then C_{4k+2} has an $\{a,b\}$-edge weighting w such that precisely two edges $e = uv$ and $e' = u'v'$ have the property that $w(u) = w(v)$ and $w(u') = w(v')$. Furthermore,*

- *the distance between e and e' is even,*
- *e and e' may be chosen to be any two edges at an even distance, and*
- *if f_1 and f_2 are the edges incident to e, then their weights are equal and can be chosen to be either a or b (similar for e').*

We present a specific consequence of Proposition 2.11 which we will find useful.

Proposition 2.12 *Let k be an integer, $k \geq 1$. Then C_{4k+2} has an $\{a,b\}$-edge-weighting such that three consecutive vertices have equal weight and the rest of the cycle is properly coloured. Furthermore, the edge-weighting can be chosen so that the weights of the four edges which contribute to the weights of those three vertices will all be a, all b, or alternate between a and b.*

Let $\Theta_{(m_1,\ldots m_d)}$, $d \geq 3$, be the graph constructed from d internally disjoint paths between distinct vertices x and y, where the i-th path has of length m_i. For simplicity, we assume $m_1 \leq m_2 \leq \cdots \leq m_d$. Such graphs will be referred to as *theta graphs*. We present necessary and sufficient conditions for theta graphs to be 2-weight colourable.

Theorem 2.13 *Let $d \geq 3$ and let a, b be real numbers. The graph $\Theta_{(m_1,m_2,\ldots,m_d)}$ is 2-weight colourable if and only if it is not of the form $\Theta_{(1,4k_2+1,\ldots,4k_d+1)}$.*

Proof: Let x and y be the two vertices of degree greater than two, and let $\{P_i | 1 \leq i \leq d\}$ be the d internally disjoint paths between x and y.

Suppose w is an $\{a,b\}$-weight colouring of $\Theta_{(1,4k_2+1,\ldots,4k_d+1)}$. By applying Lemma 2.8 (i) to each P_i, we observe that on any of the d disjoint paths between x and y the first and last edges must receive same weight. Thus $w(x) = w(y)$, a contradiction since x and y are adjacent. Hence $\Theta_{(1,4k_2+1,\ldots,4k_d+1)}$ is not $\{a,b\}$-weight colourable for any a, b.

Consider $\Theta_{(m_1,m_2,\ldots,m_d)} \ncong \Theta_{(1,4k_2+1,\ldots,4k_d+1)}$. We can assume that $|a| \geq |b|$. Let n_j be the number of paths that have length equivalent to $j \bmod 4$. Note that $n_0 + n_1 + n_2 + n_3 = d = \deg(x) = \deg(y)$. For each path P_i, weight each edge according to Lemma 2.8 so that the edges incident with x are weighted

a. Then $w(x) = da$ so it has no conflicts with its neighbours since $d \geq 3$ and the condition on the magnitudes of a and b gives $da \notin \{2a, a+b\}$. Note that, if $|P_i| \geq 2$, there are two choices for the next edge's weight on P_i which determines the rest of the weights. Given one such weighting of a path P_i, the effects of switching to the alternate weighting where the edge incident to x receives weight a depend on the parity of the length of the path. If $|P_i|$ is even, the weight of the edge incident to y and the vertex weights of the neighbours of x and y on P_i all change. If $|P_i|$ odd, the weight of the edge incident to y remains unchanged, but the vertex weights of the neighbours of x and y on P_i do change. In all cases the only possible weights on path-neighbours of x or y are $2b$, $a+b$ and $2b$. We prove, by cases, that there is an appropriate set of choices which make $w(y)$ distinct from its neighbours.

$\mathbf{n_0 + n_2 \geq 4}$: Our choice of weightings for even P_i's give at least 5 possible values for $w(y)$, so there is a choice such that $w(y) \notin \{2b, a+b, 2a, da\}$.

$\mathbf{n_0 + n_2 = 3}$: If no P_i has length 1, da is not a forbidden weight for y. Also, if $n_3 \geq 1$ then there is an edge incident to y with weight b, and $w(y) \neq da$. In either case there is a choice of weightings so that $w(y) \notin \{2b, a+b, 2a\}$.

So, assume that $m_1 = 1$ and $n_3 = 0$. If the initial weighting fails then we must have

$$\{2b, a+b, 2a, da\} = \{(d-3)a + 3b, (d-2)a + 2b, (d-1)a + b, da\}$$

which implies that $b = -(d-3)a$ and $d \geq 4$. The fact that $|a| \geq |b|$ gives that $d = 4$, implying $n_1 = 1$ and $b = -a$. We weight all edges explicitly. The single edge on the path of length 1 receives weight a. If $n_0 = 3$ then weight the edges of one even path $a, a, \ldots, -a, -a$ and the other two $a, -a, \ldots, -a, a$. If $n_0 = 2$ and $n_2 = 1$ weight the edges of the paths of length $0 \bmod 4$ with $a, a, \ldots, -a, -a$ and the other even path with $a, a, \ldots, a, a$. If $n_0 = 1$ and $n_2 = 2$ weight the edges of the path of length $0 \bmod 4$ with $a, -a, \ldots, -a, a$ and the two other even paths with $a, -a, \ldots, a, -a$. Finally if $n_0 = 0$ and $n_2 = 3$ weight the edges of all even paths with $a, -a, \ldots, a, -a$. Each weighting gives a vertex-colouring for its respective case.

$\mathbf{n_0 + n_2 = 2}$: If $n_3 = 0$ and $n_0 > 0$ then assign weights to the edges of one path which is length $0 \bmod 4$ so that the weights of the first and last edges are both a. Weight the edges of the other even path so that the edge incident to x is weighted a and the edge incident to y is weighted b. If $n_3 = n_0 = 0$ but either $d > 3$ or $b \neq 0$ then assign weights to the edges of both even paths so that their edges incident with x are weighted a, one of the edges incident with y is weighted a and the other is weighted b. In both cases weight the edges of the paths of length $1 \bmod 4$ so the weights are, in order beginning with the edge incident with x, $a, a \ldots b, a$ (if the path is a single edge, give it weight a). In the case when $n_3 = n_0 = 0$, $d = 3$ and $b = 0$ weight the edges of the two even paths $a, 0, 0, \cdots a, a, 0$ and the single odd path with $0, 0, a, a, \cdots a, 0$ (beginning with the edge incident with x in each case). The weighting given in each case gives a proper vertex colouring.

Assume $n_3 \geq 1$. If $n_0 \neq n_2$ then choose weightings for each P_i so that $w(x) = da$ and each remaining neighbour of y has accumulated weight $a+b$. Then $w(y) = an_0 + an_1 + bn_2 + bn_3$. Since $n_3 \geq 1$ we have $w(x) \neq w(y)$, so the only possible conflict is if $w(y) = a+b$. In this case change both even P_i's to their alternate weighting, maintaining $w(x) = da$ and producing a new weight at y:

$$\begin{aligned} w'(y) &= bn_0 + an_1 + an_2 + bn_3 \\ &= w(y) + (a-b)(n_2 - n_0) \\ &= a + b + (a-b)(n_2 - n_0) \\ &= \begin{cases} 3a - b & \text{if } n_0 = 0, n_2 = 2 \\ 3b - a & \text{if } n_0 = 2, n_2 = 0 \end{cases} \end{aligned}$$

In either case $w'(y) \neq a+b$. If $n_0 = 2$ then y has neighbours with weights $2b$, and $3b-a \neq 2b$. Similarly if $n_2 = 2$ then the weight at y avoids conflict with its neighbours with weight $2a$.

If $n_0 = n_2 = 1$ we start again with choices from the basic strategy that leave all path-neighbours with weight $a+b$. We have $w(x) = da \neq (n_1+1)a + (n_3+1)b = w(y)$. Thus the only conflict can again be if $w(y) = a+b$ or equivalently, $an_1 + bn_3 = 0$. In this case we weight the edges of P_i's of lengths equivalent to 0, 1, 2, and 3 $\bmod 4$ with $\{a, a, \ldots, b, b\}$, $\{a, b, \ldots, a, a\}$, $\{a, a, \ldots, a, a\}$ and $\{a, b, \ldots, b, b\}$ respectively. We still have that $w(y) = a + b \neq da = w(x)$ and no neighbour of y has weight $a+b$.

$\mathbf{n_0 + n_2 = 1}$: If $n_3 = 0$ then weight the edges of the even path so that the edge incident with x receives weight a and the edge incident with y receives weight b. Weight the edges of the paths of length $1 \bmod 4$ so the weights are, in order beginning with the edge incident with x, $a, a \ldots b, a$ (if the path is a single edge, give it weight a). This weighting gives a proper vertex colouring. Assume $n_3 \geq 1$. Again, weight the edges of each P_i so that $w(x) = da$ and each neighbor of y (distinct from x) has accumulated weight $a+b$. Since $n_3 \geq 1$ we have that $w(x) = da \neq w(y)$. If $w(y) \neq a+b$, then w is an $\{a,b\}$-weight colouring. Suppose $w(y) = a+b$. Equivalently

$$(n_0 + n_1 - 1)a + (n_2 + n_3 - 1)b = 0. \tag{1}$$

Change the edge weights of the even length path to begin with b, a. Call this weighting w'. We now have $w'(x) = (d-1)a + b$ and $w'(y) \neq a+b$. All neighbours of y still have weight $a+b$, so the only possible conflicts are between x and its neighbours. We reduce all potential conflicts to one of four cases, which are solved explicitly.

If $w'(x) = w'(y)$ then since $w'(x) = (d-1)a + b$, y is incident with precisely one edge with weight b. Since $n_3 \geq 1$, the edge with weight b comes from a path of length $3 \bmod 4$. This gives $n_0 = 0$, $n_2 = 1$ and $n_3 = 1$ and then Equation 1 and $|a| \geq |b|$ gives either

- $n_0 = 0, n_1 = 1, n_2 = 1, n_3 = 1$ and $b = 0$ (**case iii.** below).
- $n_0 = 0, n_1 = 2, n_2 = 1, n_3 = 1$ and $b = -a$ (**case iv.** below).

The neighbours of x have accumulated weights either $a+b$ or $2a$. If $w(x) = (d-1)a + b = a + b$ then this implies that $d = 2$ but the hypotheses of the theorem include $d \geq 3$. If $w(x) = (d-1)a + b = 2a$ then $b = -(d-3)a$. The fact that $d \geq 3$ and $|a| \geq |b|$ now give either

- $n_0 = 1, n_1 = 0, n_2 = 0, n_3 = 2$ and $b = 0$ which is dealt with in **case i.** below.
- $n_0 = 1, n_1 = 1, n_2 = 0, n_3 = 2$ and $b = -a$ which is dealt with in **case ii.** below.
- $n_0 = 0, n_1 = 1, n_2 = 1, n_3 = 1$ and $b = 0$ which is dealt with in **case iii.** below.

- $n_0 = 0, n_1 = 2, n_2 = 1, n_3 = 1$ and $b = -a$ which is dealt with in **case iv.** below.

case i. In this case x and y are not adjacent. Weight the edges of the path of length equivalent to $0 \bmod 4$ with $0, 0, \ldots, a, a$ and the two odd paths with $a, 0, \ldots, 0, 0$.
case ii. In this case x and y are not adjacent. Weight the edges of the paths of lengths equivalent to $0 \bmod 4$, $1 \bmod 4$ and $3 \bmod 4$ with $-a, -a, \ldots, a, a$, $a, -a, \ldots, a, a$ and $a, -a, \ldots, -a, -a$ respectively.
case iii. In this case x and y may be adjacent. Weight the edges of the paths of lengths equivalent to $1 \bmod 4$, $2 \bmod 4$ and $3 \bmod 4$ with $a, a, \ldots, 0, a$, $0, 0, \ldots, 0, 0$ and $0, 0, \ldots, 0, a$ respectively.
case iv. In this case x and y may be adjacent. Weight the edges of the paths of lengths equivalent to $1 \bmod 4$, $2 \bmod 4$ and $3 \bmod 4$ with $a, -a, \ldots, a, a$, $-a, -a, \ldots, -a, -a$ and $a, -a, \ldots, -a, -a$ respectively.

Each of these edge-weightings gives a proper vertex colouring.

$\mathbf{n_0 + n_2 = 0}$: Every weighting of the paths P_i which gives $w(x) = da$ must give $w(y) = an_1 + bn_3$. If $m_1 = 1$ then, since our graph is not $\Theta_{(1,4k_2+1,\ldots,4k_d+1)}$, we have $n_3 \geq 1$ and thus $w(x) \neq w(y)$. Suppose $m \neq 1$. For each P_i we have two choices for y's neighbour. Each choice leaves $w(y)$ constant. Thus there is a choice for each path which gives an edge-weighting vertex-colouring. □

3 Bipartite graphs

We begin the section by noting that the property of being $\{a, b\}$-weight colourable is not one that is inherited by subgraphs, nor is the property of being non-$\{a, b\}$-weight colourable. For example, the graph consisting of K_4 with a leaf attached is $\{1, 2\}$-weight colourable, however K_4 is not $\{a, b\}$-colourable for any choice of a and b. Similarly K_4 contains the subgraph C_4 which is 2-weight colourable.

We can, however, characterize the minimal graphs with respect to subgraph containment in the class of graph which are not $\{a, b\}$-weight colourable for many pairs $\{a, b\}$ (in particular, $\{1, 2\}$). In Theorem 3.9 we establish that any graph which is not $\{a, b\}$-weight colourable must contain C_{2k+1} or C_{4k+2} as a subgraph for some positive integer k.

Definition 3.1 *A graph G is* round *if every cycle of G has length* $0 \bmod 4$.

The class of round graphs is much richer than merely those obtained by taking a graph and subdividing each edge into a path of length 4. For example, $\Theta_{(2,2,2)} \cong K_{2,3}$ is a round graph which is not obtained in this way.

The following lemma establishes a useful subgraph condition of round graphs which we will use in our study of the $\{a, b\}$-weight colourability of round graphs.

Proposition 3.2 *If G is a round graph and $\Theta_{(i,j,k)}$ is a subgraph of G, then i, j and k are even and $i \equiv j \equiv k \pmod 4$.*

Proof: Let $\Theta_{(i,j,k)}$ be a subgraph of G and let P_i, P_j and P_k be the corresponding paths of length i, j and k respectively. Since G is round, $i + j \equiv i + k \equiv j + k \equiv 0 \pmod 4$. The result follows. □

Before proceeding with our results on bipartite graphs we present the following definition which we adopt throughout this section (and this section only). In any $\{a, b\}$-edge-weighting of a graph, the weighted degree of every vertex is of the form $ra + sb$ for some nonnegative integers r, s. We will call a weighted vertex *even (odd)* if its weighted degree is $ra + sb$ with r even (odd). Note that the parity of a weighted vertex does not necessarily refer to the parity of its weight. However, by Proposition 2.1, if a and b are not independent over $\mathbb{Q}$ then we will assume that they are relatively prime integers, and so we will assume that a is an odd integer in this case. If b is even, which will be the case in a number of the following results, then the parity of a weighted vertex does coincide with the parity of its weight.

Since a number of our arguments rely on this notion of parity, we often exclude those pairs of numbers whose ratio may be reduced to a ratio of odd integers. We define the sets:

$$\mathcal{E} = \left\{ \{a,b\} \;\middle|\; \frac{a}{b} = \frac{p}{q},\ p,q \text{ odd integers} \right\}$$
$$\mathcal{N} = \left\{ \{a,b\} \;\middle|\; \frac{a}{b} = \frac{p}{q},\ p,q \in \mathbb{Z},\ pq \leq 0 \right\}$$

We have already seen examples of bipartite graphs which are 2-weight colourable (C_{4k} for any $k \geq 1$, bipartite theta graphs except $\Theta_{(1,4k_2+1,\ldots,4k_d+1)}$) and some which are not (C_{4k+2} for any $k \geq 1$). From these examples, we note that a bipartite graph G with both parts of odd size is not necessarily $\{a,b\}$-weight colourable. However, if G has one part of even size, we are able to prove G is $\{a,b\}$-weight colourable for particular values of a and b.

Theorem 3.3 *Let $a, b \in \mathbb{R}$ be such that $\{a,b\} \notin \mathcal{E}$. If G is a connected bipartite graph with at least one part being of even size, then G is $\{a,b\}$-weight colourable.*

Proof: Let $V(G) = X \cup Y$ be a bipartition of the vertices of G with $|X|$ even. By Corollary 2.2 and since $\{a,b\} \notin \mathcal{E}$, we may assume that a is an odd integer and b is an even integer. We assign the weight b to each edge of G. Clearly v is even for each $v \in V(G)$. Let $V(X) = \{x_1, x_2 \ldots x_{2k}\}$ and let P_i be an $x_{2i-1}x_{2i}$-path in G. By changing every edge weight along P_1 we only change the parity of x_1 and x_2. By repeating this process for each P_i we have that every vertex of X has odd parity and every vertex of Y has even parity.

Call the resulting edge-weighting w. Suppose that w is not an $\{a,b\}$-weight colouring. Then there are adjacent vertices x and y such that $w(x) = w(y)$. Thus there exist integers r, r', s, s' such $w(x) = ra + sb$ where r is odd, $w(y) = r'a + s'b$ where r' is even, and $ra + sb = r'a + s'b$. If a and b are linearly independent over $\mathbb{Q}$, we must have $r = r'$, a contradiction. Hence $b = (p/q)a$ for some $p, q \in \mathbb{Z}$ with $\gcd(p,q) = 1$. Thus $rq + sp = r'q + s'p$. Since r is odd and r' is even, p even implies q must be even, a contradiction. Hence p is odd. Similarly, q is odd. Therefore, $b/a = p/q$ with p, q odd, contradicting our choice of a and b. Thus w is an $\{a,b\}$-weight colouring of G. $\square$

Corollary 3.4 *Let $a, b \in \mathbb{R}$ be such that $\{a,b\} \notin \mathcal{E} \cup \mathcal{N}$. Let $G \neq K_2$ be a connected bipartite graph with a vertex of degree 1. Then G is $\{a,b\}$-weight colourable. In particular, trees are $\{a,b\}$-weight colourable.*

Proof: Let $V(G) = X \cup Y$ be a bipartition of the vertices of G. Let $x \in X$ be a vertex of degree 1 and let $y \in Y$ be its neighbour. If $|X|$ or $|Y|$ is even, then G is $\{a,b\}$-weight colourable by Theorem 3.3. If $|X|$ is odd, then $G - x$ has an $\{a,b\}$-weight colouring by Theorem 3.3, say w', such that vertices in $X \setminus \{x\}$ are odd and vertices in Y are even. By assigning b to the edge xy we maintain the parity of all the vertices. Also, since $\{a,b\} \notin \mathcal{N}$ we have $w'(y) \neq 0$ and so x and y will receive different weights, thus giving an $\{a,b\}$-weight colouring of G. $\square$

Theorem 3.5 *Let $a, b \in \mathbb{R}$ be such that $\{a, b\} \notin \mathcal{E} \cup \mathcal{N}$. Let G be a connected bipartite graph with a thread of even length P and let U be the internal vertices of P. If $G - U$ is connected then G is $\{a, b\}$-weight colourable.*

Proof: We may assume that a is a positive odd integer and b is a positive even integer. If $X \cup Y$ is the bipartition of $V(G)$ and either $|X|$ or $|Y|$ is even, then G is $\{a, b\}$-weight colourable by Theorem 3.3. Assume both parts of G are of odd size. Let x and y be the ends of P. We first assume that x and y are distinct. By Lemma 2.8, we may assume that P is a path of length either 2 or 4.

Consider the case that P is of length 2, say $P = xvy$. Let G' be the bipartite graph obtained from G by deleting v and adding two leaves, v_1 adjacent to x and v_2 adjacent to y. Now G' is connected and bipartite with an even side, where v_1 and v_2 both belong to the even side. Theorem 3.3 gives an $\{a, b\}$-weight colouring of G', say w', so that v_1 and v_2 are both odd vertices. Hence xv_1 and yv_2 must both receive a as their weight. Let w be an $\{a, b\}$-edge-weighting of G, where $w(xv) = w'(xv_1) = a$, $w(yv) = w'(yv_2) = a$ and $w(e) = w'(e)$ for all other edges $e \in E(G)$. If w is not an $\{a, b\}$-weight colouring of G, then either $w(x) = 2a$ or $w(y) = 2a$. Without loss of generality, suppose $w(x) = ra + sb = 2a$ (a similar argument will hold for y). Since $w(xv) = a$ and r even, we have $r \geq 2$. If $r = 2$, then $sb = 0$ which implies $s = 0$ or equivalently $\deg_G(x) = 2$, a contradiction. If $r \geq 3$, then $sb < 0$ which gives $b < 0$, a contradiction. Thus w is an $\{a, b\}$-weight colouring of G.

Suppose $|P| = 4$. Let $P = xv_1v_2v_3y$ and let $G' = G - v_2$. Now G' is bipartite with an even side X', and $x, y \in X'$. Theorem 3.3 gives an $\{a, b\}$-weight colouring of G', say w', so that v_1 and v_3 are both even vertices. Hence xv_1 and yv_3 must both receive b as their weight. Let w be an $\{a, b\}$-edge-weighting of G, where $w(v_1v_2) = w(v_2v_3) = a$ and $w(e) = w'(e)$ for all other edges $e \in E(G)$. If w is not an $\{a, b\}$-weight colouring of G, then either $w(x) = a+b$ or $w(y) = a+b$. Suppose $w(x) = ra+sb = a+b$. Then $(r-1)a = -(s-1)b$, and thus r is odd. Again, we have that a and b are positive integers. Thus either $r - 1 < 0$ or $s - 1 < 0$. However, since $w(xv_1) = b$, we have $s \neq 0$, and since r is odd, $r \neq 0$. Thus w is an $\{a, b\}$-weight colouring of G.

Now, suppose x and y are not distinct (call this vertex x). Then P is a cycle which is an end block of G and x is a cut vertex of G. Let z_1 and z_2 be the neighbours of x in P. Since $G' = G - U$ is a connected bipartite graph with one part having even size, then by Theorem 3.3 there is an $\{a, b\}$-weight colouring of G', say w'. We give an edge weighting w'' of P as follows:

- if P has length $2 \pmod 4$, then by Proposition 2.12 we may define an $\{a, b\}$-weight colouring of P, w'', so that $w''(z_1) = w''(x) = w''(z_2) = 2a$ and P is properly coloured elsewhere;
- if P has length $0 \pmod 4$, then by Proposition 2.9 we may define an $\{a, b\}$-weight colouring of P, w'', so that $w''(x)$ is the larger of $2a$ and $2b$ and P is properly coloured.

Let w be the weighting obtained by combining w' and w''. Then $w(x) > w(z_1), w(z_2)$ and x has the same parity under w as under w'. Hence the weight of x is distinct from its neighbours in G. Since all other vertices are properly coloured by w' or w'', w gives an $\{a, b\}$-weight colouring of G. □

Theorem 3.6 *If G is a 2-connected round graph which is not a cycle then G contains at least 2 even ears.*

Proof:

We first claim that G contains no proper 2-connected subgraph which contains all even ears of G. Toward a contradiction suppose H is a 2-connected maximal proper subgraph of G that contains all even ears of G. There exist two vertices of H, say x and y, which are connected by a path P such that $H \cap P = \{x, y\}$. Since H is 2-connected, there are also 2 edge disjoint paths P' and P'' in H between x and y. Thus $P \cup P' \cup P''$ is a theta graph, and by Lemma 3.2 P must be of even length. Since H already contains all even ears of G, $H' = H \cup P$ must be a proper subgraph of G but H' is also 2-connected which contradicts the maximality of H.

Now, if G has no even ear, then any cycle of G is a 2-connected subgraph containing all the even ears and this is a contradiction as G is not a cycle. If G has only one ear, let T be the ear and let x and y be the two ends of T. Then there are 2 edge disjoint paths connecting x and y, one of which must be edge disjoint from T. This path together with T forms a cycle that contains all the even ears of G, a contradiction. □

Corollary 3.7 *If G is a round graph and all threads of G are odd, then G has at least two leaves.*

We are now able to prove that round graphs can be edge-weight vertex-coloured with most sets of size 2.

Theorem 3.8 *Every round graph is $\{a, b\}$-weight colourable for $\{a, b\} \notin \mathcal{E} \cup \mathcal{N}$.*

Proof: Let G be a round graph. Let B be an end block with vertex of attachment v. If B is isomorphic to K_2, then G is a bipartite graph with a leaf and thus is $\{a, b\}$-weight colourable by Corollary 3.4. If B is a cycle, then B is an even thread and G is $\{a, b\}$-weight colourable by Theorem 3.5. Otherwise, if B is a 2-connected graph which is not a cycle, then by Theorem 3.6, B has at least two even ears and thus B has at least one even ear, say P, which does not contain v as an internal vertex. Let U be the internal vertices of P. Since $G - U$ is connected, G is $\{a, b\}$-weight colourable by Theorem 3.5. □

Theorem 3.8, together with Proposition 2.9, gives a class of minimal subgraphs with respect to containment which cannot be $\{a, b\}$-weight coloured for the pairs $\{a, b\}$ on which we have focused.

Corollary 3.9 *Let a and b be real numbers such that $\{a, b\} \notin \mathcal{E} \cup \mathcal{N}$. Any graph which is not $\{a, b\}$-weight colourable must contain a cycle of length $1, 2$ or $3 \bmod 4$.*

We end this section with the following problem.

Problem 3 *Is it true that all bipartite graphs except C_{4k+2} and $\Theta_{(1,4k_1+1,4k_2+1,\ldots,4k_d+1)}$ are 2-weight colourable?*

4 More families of graphs with determined 2-weight colourability

We have given a number of examples of $\{a, b\}$-weight colourable graphs for values of a and b subject to particular restrictions. However we have seen few examples of graphs for which a and b can be any distinct real numbers. We note that the Petersen graph provides such an example of a 2-weight colourable graph. One such edge-weighting is given in Figure 1. By Proposition 2.3, note that any 2-weight colouring of the Petersen graph gives a 3-colouring of it, which is also an optimal proper vertex colouring.

In the rest of this section we describe more families of 2-weight colourable graphs as well as a class of nonbipartite graphs which are $\{a, b\}$-weight colourable when $ab > 0$. In particular we show that all

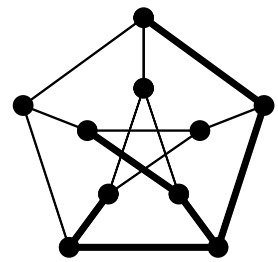

Fig. 1: An $\{a,b\}$-weight colouring of the Petersen graph. Bold edges are to receive weight b.

unicyclic graphs except cycles of length $1, 2, 3 \bmod 4$ are 2-weight colourable. We also provide a number of results on Cartesian products of graphs. Finally, we explore techniques for constructing graphs which do not admit $\{a,b\}$-weight colourings for any choice of a and b.

We begin with our result on unicyclic graphs.

Theorem 4.1 *Every connected unicyclic graph except C_{2k+1} and C_{4m+2} is $\{a,b\}$-weight colourable, where a and b are real numbers with $ab > 0$.*

Proof: We may assume that $0 < a < b$. By contradiction, let G be the smallest counterexample to our claim. Let C be the only cycle of G. We first note that by Lemma 2.6, we may assume that every vertex of G is either on C or is adjacent to a vertex of C. We may also assume that every vertex of G has degree at most 3. Next, we claim that there are at least two vertices of degree at least 3 on C. If not, let v be the only vertex of degree at least 3 on C. Let x and y be the neighbours of v on C. It is easy to find an edge-weighting w of C which yields a proper colouring on $C - v$ and $w(v) \geq w(x), w(y)$. By assigning b to the other edge incident with v, we get an $\{a,b\}$-weight colouring of G, a contradiction.

Next, we claim that G has at most one ear of length at least 2. If not, then we choose some maximal path of degree 3 vertices on C, $x_1, \ldots, x_k$, and remove all leaves of G adjacent to those vertices. Call this subgraph G'. By minimality of our choice of G, we can assign an $\{a,b\}$-weight colouring w' to G'. Let w be the weighting of $E(G)$ given by $w(e) = w'(e)$ if $e \in E(G')$ and $w(e) = b$ otherwise. The only possible conflicts are between x_1 and its neighbour on C which is not x_2, say y (or, similarly, between x_k and it's neighbour on C which is not x_{k-1}). However, since $w(x_1) \geq a + b + w(x_1y)$ and $w(y) \leq w(x_1y) + b$ (similar for x_k), w is an $\{a,b\}$-weight colouring of G which contradicts our choice of G.

If G has exactly one ear of length at least 2, let $e = rs$ and $e' = r's'$ be the two edges that have exactly one endpoint of degree 2. Specifically, let $\deg(r) = \deg(r') = 2$, $\deg(s) = \deg(s') = 3$. Note that r and r' need not be distinct, but, since there are at least 2 vertices on C of degree 3, s and s' are distinct. We construct an $\{a,b\}$-weight colouring of G based on the length of C mod 4.

- Suppose $|C|$ is odd. By Proposition 2.10, C has an $\{a,b\}$-edge weighting w' which gives a proper vertex colouring except across rs. Let $w(e) = w'(e)$ if $e \in E(C)$. If $w'(r') - w'(s') = a$, let $w(e) = b$ for all $e \in E(G) \setminus E(C)$. Otherwise, let $w(e) = a$ for all $e \in E(G) \setminus E(C)$. Clearly each leaf's neighbour has a weight strictly greater than its own. Since w' gives a proper colouring of C except for r and s, the only adjacent vertices of G which might not be properly coloured are r

and s or r' and s'. However, our choice of weights for the leaves of G guarantees that r, s, r', s' are properly coloured as well. Thus w is an $\{a, b\}$-weight colouring of G.

- Suppose $|C| \equiv 0 \pmod 4$. By Proposition 2.9, C has an $\{a, b\}$-weight colouring w' such that $w(r) = 2a$ and $w(s) = a + b$. Let $w(e) = w'(e)$ if $e \in E(C)$. If $w'(r') - w'(s') = a$, let $w(e) = b$ for all $e \in E(G) \setminus E(C)$. Otherwise, let $w(e) = a$ for all $e \in E(G) \setminus E(C)$. By the same argument as above, w is an $\{a, b\}$-weight colouring of G.

- Suppose $|C| \equiv 2 \pmod 4$. Let t be the other neighbour of s on C and let t' be the other neighbour of r' on C. By Proposition 2.11, there is an $\{a, b\}$-edge weighting such that all vertices are properly coloured except r, s and t, and such that $w(t'r') = w(r's') = a$. Let $w(e) = w'(e)$ for all $e \in E(C)$. Let f be the edge between s and its leaf, and let $w(f) = a$. For each $e \in E(G) \setminus E(C) \setminus \{f\}$, let $w(e) = b$. The only possible improperly coloured pairs of vertices are r and s, s and t or r' and s'. However,

$$\begin{aligned} w(r) &= w'(r) = w'(s) < w(s) \\ w(s) &= w'(s) + a = w'(t) + a < w'(t) + b = w(t) \\ w(r') &= 2a < a + 2b = w(s') \end{aligned}$$

  and so w is an $\{a, b\}$-weight colouring of G.

The only remaining case is that every vertex of C has degree 3. If $|C|$ is even, assign the same weight to all the edges on the cycle and alternating weights to the leaf edges. The reader can verify that a solution for the cases when $|C| = 3$ or $|C| = 5$ exists. Each of these cases can be extended to larger odd cycle by making the replacement indicated in Figure 2. Note that the variables $\bar{\ell}$ and $\bar{n}$ refer to the weights different from ℓ and n, respectively.

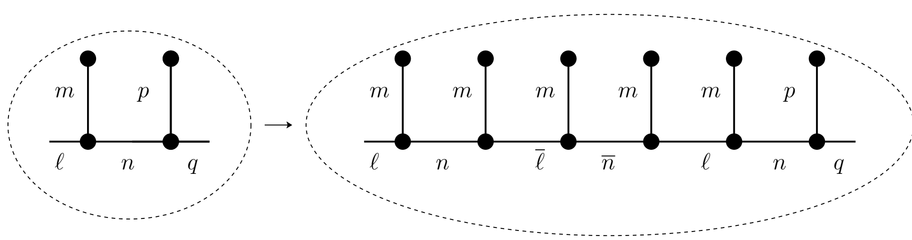

Fig. 2: Replacement operation to expand 2-weight colourings to larger cycles.

Thus, no minimal counterexample G exists. $\square$

Proposition 4.2 *For $n \geq 4$, the graph $K_2 \square K_n$ is 2-weight colourable.*

Proof: Let K_n and K'_n be the two copies of the complete graph. Denote the vertices of K_n and K'_n, respectively by

$$\{u_1, u_2, \ldots, u_{\lfloor n/2 \rfloor}, v_{\lfloor n/2 \rfloor}, v_{\lfloor n/2 \rfloor + 1}, \ldots, v_{n-2}, v_{n-1}\},$$
$$\{u'_1, u'_2, \ldots, u'_{\lfloor n/2 \rfloor}, v'_{\lfloor n/2 \rfloor}, v'_{\lfloor n/2 \rfloor + 1}, \ldots, v'_{n-2}, v'_{n-1}\}.$$

Let p be a derangement (permutation with no fixed points) of $\{1, \lfloor n/2 \rfloor\}$ and π be a derangement of $\{\lfloor n/2 \rfloor, n-1\}$. Let u_i be adjacent to $u'_{p(i)}$ for all $1 \le i \le \lfloor n/2 \rfloor$ and v_i be adjacent to $v'_{\pi(i)}$ for $\lfloor n/2 \rfloor \le i \le n-1$.

Since the graph is n-regular, if adjacent vertices have distinct weights then they have distinct numbers of incident edges having weight b. Using Lemma 2.5, we may weight the edges of K_n and K'_n so that the subscript of the vertex is precisely equal to the number of edges weighted b incident to that edge in K_n. Label $u_i u'_{p(i)}$ with a for all $1 \le i \le \lfloor n/2 \rfloor$ and weight $v_i v'_{\pi(i)}$ with b for $\lfloor n/2 \rfloor \le i \le n-1$. Then any two vertices that are adjacent have a distinct number of incident edges weighted b and thus $K_2 \,\square\, K_n$ is 2-weight colourable. $\square$

Figure 3 gives an illustration of this construction.

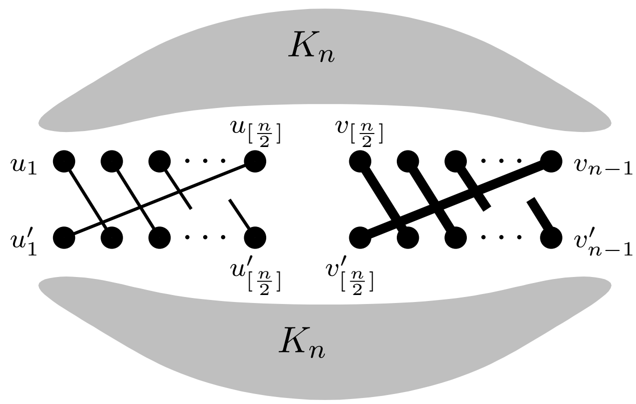

Fig. 3: An $\{a, b\}$-weight colouring of $K_2 \,\square\, K_n$. Bold edges are to receive weight b.

Proposition 4.3 *The graph $K_2 \,\square\, C_n$ is 2-weight colourable if and only if $n \ge 4$ and $n \ne 5$.*

Proof: If n is even, then give every edge of one copy of C_n weight a and every edge of the other copy weight b. By alternating the weights of the images of K_2 between a and b along the cycles, we have the desired $\{a, b\}$-weight colouring.

An example of an $\{a, b\}$-weight colouring of $K_2 \,\square\, C_7$ is given in Figure 4. It can be extended to an $\{a, b\}$-weight colouring of $K_2 \,\square\, C_9$ by replacing the left subgraph in Figure 5 with the right graph. Note that the right subgraph contains the left one, and thus this operation may be repeated as many times as needed to give an $\{a, b\}$-weight colouring for any $K_2 \,\square\, C_{2k+1} (k \ge 3)$ The reader may verify that no $\{a, b\}$-weight colouring of $K_2 \,\square\, C_3$ or $K_2 \,\square\, C_5$ exists. $\square$

Theorem 4.4 *Let G be a graph and H be a regular bipartite graph. If $G \,\square\, K_2$ is 2-weight colourable, then $G \,\square\, H$ is 2-weight colourable.*

Proof: Let w be an $\{a, b\}$-weight colouring of $G \,\square\, K_2$. Denote the two copies of G by G_1 and G_2 and denote the vertices of K_2 by t_1 and t_2. Since H is regular (say d-regular) and bipartite, Hall's Theorem guarantees a perfect matching M of H. Let X and Y be the parts of $V(H)$.

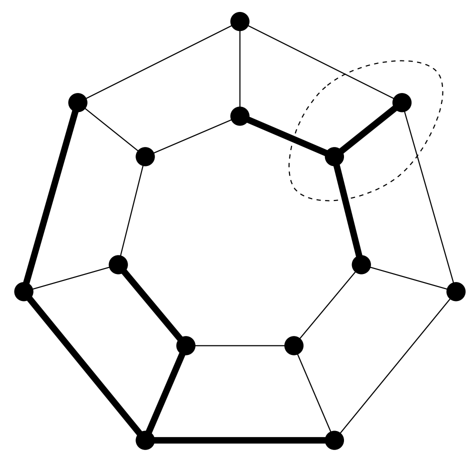

Fig. 4: An $\{a, b\}$-weight colouring of $K_2 \,\square\, C_7$. Bold edges are to receive weight b.

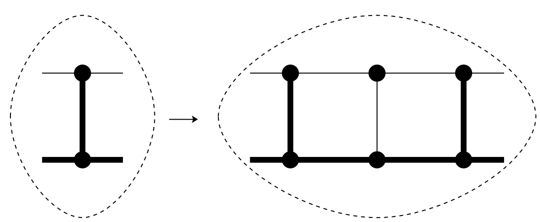

Fig. 5: Replacement operation for obtaining an $\{a, b\}$-weight colouring of $K_2 \,\square\, C_{2k+1}$ for $k \geq 4$.

Define an edge-weighting of $G \,\square\, H$ as follows. For each edge $e = xy \in M$ where $x \in X$ and $y \in Y$, weight the edges of the subgraph $G \,\square\, e$ by w so that each vertex $(u_G, x) \in V(G \,\square\, H)$ has weight $w(u_G, t_1)$ and $(u_G, y) \in V(G \,\square\, H)$ has weight $w(u_G, t_2)$. Assign every other edge of $G \,\square\, H$ weight a. Call this weighting ϕ.

We have that $\phi(u) = w(u_G, t_1) + (d-1)a$ if $u_H \in X$ and $\phi(u) = w(u_G, t_2) + (d-1)a$ if $u_H \in Y$. Two vertices are adjacent if either their H-coordinates agree and they are adjacent in a copy of G or if their G-coordinates agree and they are adjacent in a copy of H. In the former case, their weights are distinct under ϕ since they are distinct under w. In the latter, consider two adjacent vertices $u = (u_G, u_H)$ and $u' = (u_G, u'_H)$ where $u_H \in X, u'_H \in Y$. Then, $w(u_G, t_1) \neq w(u_G, t_2)$ by choice of w, which implies that $\phi(u) \neq \phi(u')$. Thus ϕ is an $\{a, b\}$-weight colouring of $G \,\square\, H$. $\square$

Corollary 4.5 *If G and H are regular bipartite graphs, then the following graphs are 2-weight colourable:*

(i) $K_n \,\square\, H$, *if* $n \geq 4$

(ii) $C_n \,\square\, H$ *if* $n \geq 4, n \neq 5$

(iii) $G \,\square\, H$

Proof: Applying Theorem 4.4 to Propositions 4.2 and 4.3 immediately gives results (i) and (ii) respectively. For (iii), since $K_2 \,\Box\, K_2 \cong C_4$, $K_2 \,\Box\, K_2$ is 2-weight colourable by Proposition 2.9. By Theorem 4.4, $K_2 \,\Box\, H$ is 2-weight colourable; applying Theorem 4.4 again gives us that $G \,\Box\, H$ is 2-weight colourable. □

In order to construct non-2-weight colourable graphs below, we make use of a class of "gadget" graphs. These gadgets are themselves 2-weight colourable, but they have the property that in any of their 2-weight colourings, certain edges receive a predetermined weight.

Define the graph $\widehat{K}_n$ to be the graph obtained from K_n by subdividing one edge exactly once.

Proposition 4.6 *For $n \geq 4$, the graph $\widehat{K}_n$ is 2-weight colourable. Moreover, in any 2-weight colouring of $\widehat{K}_n$, the edges incident to its degree 2 vertex must receive the same colour.*

Proof: Let x be the vertex of $\widehat{K}_n$ of degree 2 and let u, v be its neighbours. An $\{a, b\}$-weight colouring of $\widehat{K}_4$ is given in Figure 6. So assume $n \geq 5$. Let K_n be obtained by adding the edge uv to $\widehat{K}_n - x$. By Lemma 2.5, there exists an edge-weighting w of K_n such that all the vertices have distinct weighted degrees except for u and v. Moreover, $w(u) = w(v) = ra + (n-1-r)b$, where $r \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil - 1\}$. Assign the weight $w(uv)$ from K_n to the edges xu and xv in $\widehat{K}_n$. Note that w is an $\{a, b\}$-weight colouring as long as $w(u) = w(v) \neq w(x)$. We have $w(x) \in \{2a, 2b\}$. Since a and b may be swapped in Lemma 2.5, we assume that $w(x) = 2a$. If $w(u) \neq 2a$, we are done. Suppose $w(u) = 2a$. We consider two cases:

- If n is odd, then the edge weighting w' given by swapping every edge's weight gives $w'(u) = w(u) = 2a \neq 2b = w'(x)$.
- If n is even then, by the construction of the weighting in Lemma 2.5, $w(u) = \frac{n}{2}a + (\frac{n}{2} - 1)b$. So, $2a = \frac{n}{2}(a+b) - b$. If the edge weighting w' given by swapping every edge's weight gives a conflict between u and x, then $2b = \frac{n}{2}(a+b) - a$. Together, these imply that $a = b$, a contradiction.

Thus $\widehat{K}_n$ admits an $\{a, b\}$-weight colouring.

To prove the second part, toward a contradiction, suppose $\widehat{K}_n$ is the smallest counterexample for which there exists an $\{a, b\}$-weight colouring w such that $w(xu) \neq w(xv)$. By inspection, we may check that $\widehat{K}_4$ does not admit such edge-weighting. So assume $n \geq 5$. Note that there exists no vertex $y \neq u, v$ such that $w(y) \in \{(n-1)a, (n-1)b\}$, otherwise $\widehat{K}_n - y$ would be a smaller counterexample. Therefore, since w induces a vertex colouring and all the weighted degrees (except for x) are of the form $ra + (n-1-r)b$ for some $0 \leq r \leq n-1$, we must have $w(u), w(v) \in \{(n-1)a, (n-1)b\}$. But then by removing u, v, and x we get an $\{a, b\}$-weight colouring of K_{n-2}, a contradiction to Corollary 2.4. □

Corollary 4.7 *Given a graph G, let G' be obtained from identifying a vertex of G with the degree 2 vertex of $\widehat{K}_n$. Then in any 2-weight colouring of G', edges in $\hat{K}_n$ incident to its degree 2 vertex must receive same colour.*

Proof: Since the proof of Proposition 4.6 did not depend in any way on the accumulated weight at vertex x, then regardless of graph joined to $\hat{K}_n$ at x, the two edges incident with x in $\hat{K}_n$ must still receive the same weight. □

An example is given on the left of Figure 6. In the case $G = K_2$ and $n = 4$, the weight of the leaf's edge is forced to be equal to that of its incident edges; this is another useful gadget. It is shown on the right of Figure 6.

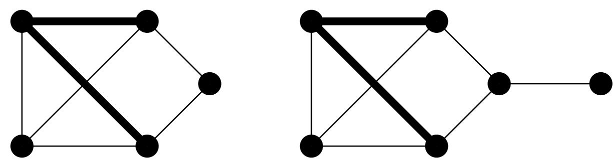

Fig. 6: The graphs $\hat{K}_4$ and $\hat{K}_4$ with a leaf are 2-weight colourable. Bold edges represent one weight-class.

We use Proposition 4.6, which established the weight colourability of $\hat{K}_n$, to construct the following examples of non 2-weight colourable graphs.

Example 4.8 *The following graphs cannot be 2-weight coloured:*

(i) Join two copies of $\hat{K}_4$ by an edge attached at their vertices of degree 2.

(ii) Join $2n+1$ copies of $\hat{K}_4$ to a C_{2n+1} by an edge attaching the degree 2 vertex in each copy of $\hat{K}_4$ to a distinct cycle vertex.

To see why the graph defined in (ii), which we denote H, cannot be S-weight coloured for any set S of size 2, consider the accumulated weight at one of the cycle vertices, say v. Since H is 3-regular graph, $w(v) \in \{3a, 2a+b, a+2b, 3b\}$. If $w(v) = 3a$, then the noncycle edge, e, incident with v must have weight a and, as shown in Figure 6, so must the two edges in the copy of $\hat{K}_4$ joined to v by an edge. Thus both endpoints of e would have weight $3a$. A similar argument shows that $w(v) \neq 3b$. Thus the only possible accumulated weights on cycle vertices are $2a + b$ and $2b + a$. Since an odd cycle cannot be properly 2-coloured, we see that H cannot be 2-weight coloured.

Our next family of gadget graphs are described below.

Proposition 4.9 *Let $0 \leq a \in \mathbb{Z}$ and $d \mid a$. Let H be a graph and G be a graph obtained from identifying a vertex u of H with a vertex of a K_n (all other vertices of H and K_n being disjoint). If*

$$\deg_H(u) < \left(\frac{d}{a+d}\right) \left\lfloor \frac{n-1}{2} \right\rfloor,$$

then G is not $\{a, a+d\}$-weight colourable. Furthermore, if

$$\deg_H(u) = \left(\frac{d}{a+d}\right) \left\lfloor \frac{n-1}{2} \right\rfloor,$$

then in any $\{a, a+d\}$-weight colouring of G, all edges in H incident to u must receive weight $a+d$.

Proof: We first prove the statement for $d = 1$. Toward a contradiction, suppose w is an $\{a, a+1\}$-weighting colouring of G. Every vertex of $K_n - u$ has weighted degree $ra + (n-1-r)(a+1) = (n-1)(a+1) - r$ for some $0 \leq r \leq n-1$ and both of the weights $(n-1)a$ and $(n-1)(a+1)$ cannot appear simultaneously on $K_n - u$. If $w(u) < (n-1)(a+1)$ then there are only $n-1$ colours available for the vertices of K_n, a contradiction. So $w(u) \geq (n-1)(a+1)$.

Let $w|_{K_n}$ be the edge-weighting of K_n induced by w. By Corollary 2.4, K_n is not $\{a, a+1\}$-edge-weight colourable. Thus, there must be exactly two vertices of K_n with the same weight given by $w|_{K_n}$ and u must be one such vertex. By Lemma 2.5, we get $w|_{K_n}(u) = ra + (n-1-r)(a+1) = (n-1)(a+1) - r$, where $r \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil - 1\}$. Note that $r \geq \lfloor (n-1)/2 \rfloor$. If u is incident with s edges of weight a in H, then we have

$$(n-1)(a+1) \leq w(u) = ra + (n-1-r)(a+1) + sa + (\deg_H(u) - s)(a+1),$$

which simplifies to $\deg_H(u) \geq (\frac{1}{a+1})(r+s)$. Hence

$$\deg_H(u) \geq \left(\frac{1}{a+1}\right)\left(\left\lfloor \frac{n-1}{2} \right\rfloor + s\right).$$

This is a contradiction since $\deg_H(u) < (\frac{1}{a+1}) \left\lfloor \frac{n-1}{2} \right\rfloor$. Also, if $\deg_H(u) = (\frac{1}{a+1}) \left\lfloor \frac{n-1}{2} \right\rfloor$ then we must have $s = 0$, proving the second claim.

Now, let d be any positive divisor of a. By Proposition 2.1, G has an $\{a, a+d\}$-weight colouring if and only if it has a $\left\{\frac{a}{d}, \frac{a}{d}+1\right\}$-weight colouring. If $\deg_H(u) < \left(\frac{1}{a/d+1}\right) \left\lfloor \frac{n-1}{2} \right\rfloor$ then G has no $\left\{\frac{a}{d}, \frac{a}{d}+1\right\}$-weight colouring by the above argument. Hence, if $\deg_H(u) < (\frac{d}{a+d}) \left\lfloor \frac{n-1}{2} \right\rfloor$ then G has no $\{a, a+d\}$-weight colouring. The second result follows similarly. □

We may use Proposition 4.9 to construct many graphs which are not $\{a, a+1\}$-weight colourable and so, in particular, are not $\{1, 2\}$-weight colourable. In fact, if H is any graph and u any vertex of H, then there is an n large enough so that attaching K_n to u (and only to u) gives a graph which is not $\{a, a+1\}$-weight colourable. We can also use the equality condition to construct graphs with no $\{a, a+1\}$-weight colouring (for a specific a). For example, the graph obtained by joining two copies of K_n, $n \geq 5$, with a path of length 3, say e_1, e_2, e_3, is not $\{a, a+1\}$-weight colourable for $a = \lfloor \frac{n-1}{2} \rfloor - 1$ since the weights of e_1 and e_3 are forced to be $a+1$, and thus the ends of e_2 will receive the same weight.

We finish this section with a problem whose solution would be useful in the study of $\{1, 2, 3\}$-weight colourable graphs.

Problem 4 *Does there exist a graph which is either*

- *uniquely* $\{1, 2, 3\}$*-weight colourable (up to isomorphism), or*
- $\{1, 2, 3\}$*-weight colourable where any such colouring forces certain edges of G to receive a particular weight?*

Moreover, does there exist a graph with either of these properties which maintains that property when it is attached in some way to another graph?

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