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A bound on the number of perfect matchings in Klee-graphs[†]

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The famous conjecture of Lovász and Plummer, very recently proven by Esperet et al. (2011), asserts that every cubic bridgeless graph has exponentially many perfect matchings. In this paper we improve the bound of Esperet et al. for a specific subclass of cubic bridgeless graphs called the *Klee-graphs*. We show that every Klee-graph with $n \geq 8$ vertices has at least $3 \cdot 2^{(n+12)/60}$ perfect matchings.

Keywords: Klee-graphs, perfect matchings, Lovász-Plummer conjecture

1 Introduction

In this paper, we focus on a specific class of planar cubic bridgeless graphs, namely the Klee-graphs, defined as follows. A graph G is a Klee-graph if $G = K_4$ or there exists a Klee-graph G' such that G can be obtained from G' by replacing a vertex by a triangle (see Figure 1).

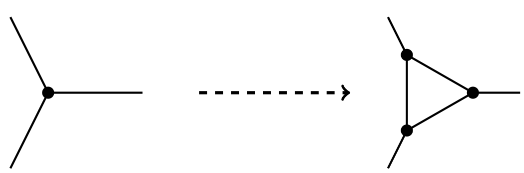

Fig. 1: Replacing a vertex by a triangle in a cubic graph.

For a given undirected graph G, let $V(G)$ be the set of its vertices and let $E(G)$ be the set of its edges. A *matching* in a graph G is a set $M \subseteq E(G)$ such that every vertex is an endvertex of at most one edge

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in M. A *perfect matching* is a matching where every vertex is an endvertex of exactly one edge in the matching.

Let us now mention a well-known conjecture of Lovász and Plummer [8, Conjecture 8.1.8]

Conjecture 1.1 (Lovász-Plummer) *Every bridgeless cubic graph G has exponentially many (as a function of $|V(G)|$) perfect matchings.*

In 1982, Edmonds et al. [2] proved that every bridgeless cubic graph on n vertices has at least $n/4+2$ perfect matchings. This linear lower bound was improved by Král' et al. [7] to $n/2$ and then by Esperet et al. [5] to $\frac{3}{4}n-10$. In 2009 Esperet et al. [4] came up with a deep and complex proof of a first known superlinear bound.

Meanwhile, the Lovász-Plummer conjecture has been proven for cubic bipartite graphs by Voorhoeve [10] and extended to all regular bipartite graphs by Shrijver [9]. Recently Kardoš et al. [6] proved that fullerene graphs, i.e., planar cubic 3-connected graphs with only pentagonal and hexagonal faces, have a very large number of perfect matchings — their lower bound is $2^{(n-380)/61}$. A result by Chudnovsky and Seymour [1] from 2008 shows that the conjecture is true for all planar cubic bridgeless graphs. However, their exponential bound is very small, namely, they assert that every n-vertex planar cubic graph G has at least $2^{n/655978752}$ perfect matchings.

The conjecture of Lovász and Plummer was finally proven in 2010 by Esperet et al. [3], who give a $2^{n/3656}$ lower bound for the number of perfect matchings in an arbitrary n-vertex cubic bridgeless graph.

On the other hand, Klee-graphs play important role in the matchings and cubic graphs theory and therefore obtaining a better bound for the number of perfect matchings in Klee-graphs seems interesting. Every 3-edge-connected cubic graph that is not a Klee-graph is double-covered (i.e., every edge belongs to at least two perfect matchings) [5]. Thus, one may expect that Klee-graphs are the 3-edge-connected cubic graphs with fewest perfect matchings.

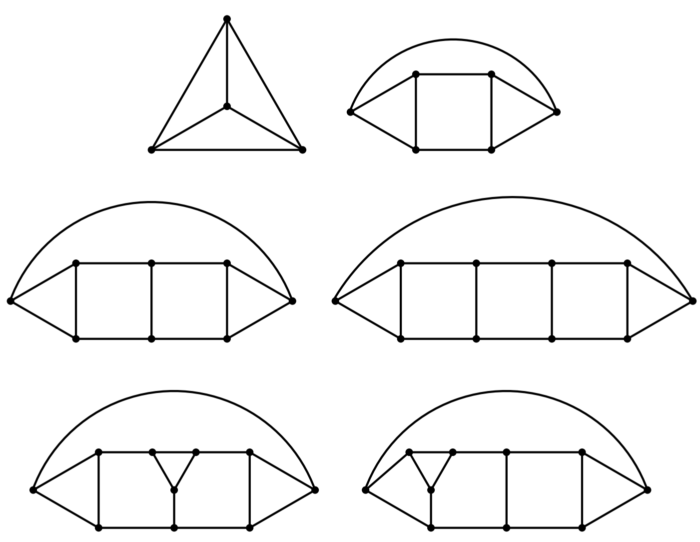

Fig. 2: All Klee-graphs with at most 10 vertices.

We start by introducing some notation in Section 2, then we prove a few basic lemmas in Section 3 and finally we do calculations and show that every Klee-graph with $n \geq 8$ vertices has at least $3 \cdot 2^{(n+12)/60}$ perfect matchings in Section 4. In Section 5 we provide an upper bound for the number of perfect matchings in Klee-graphs. More precisely, we show an infinite family of Klee-graphs with at most $c\,2^{n/17.285}$ perfect matchings, where c is some absolute constant.

2 Notation and definitions

Let us start with some notation. For a given set $D \subseteq V(G)$ let $\varepsilon_G(D) \subseteq E(G)$ be the set of edges connecting D with $V(G) \setminus D$, and let $N_G(D) \subseteq V(G) \setminus D$ be the set of neighbors of D. Denote by $G[D]$ the subgraph induced by D.

The following lemma helps us fix a naming convention on the edges of a Klee-graph.

Lemma 2.1 *For any Klee-graph G, the edge set $E(G)$ can be uniquely partitioned into three pairwise disjoint perfect matchings M^1, M^2 and M^3. In other words, any Klee-graph G is 3-edge-colorable, and the coloring is unique up to a permutation of the colors.*

Proof: We prove it by induction on the number of vertices of G. The claim is obvious in the unique 4-vertex Klee-graph, K_4.

Now we focus on the Klee-graph construction step, i.e., we replace a vertex x in a graph G' by a triangle $a_1a_2a_3$ obtaining a graph G. To prove the lemma, it is sufficient to show a bijection between partitions of $E(G')$ and $E(G)$ into three perfect matchings.

Assume $N_{G'}(x) = \{v_1, v_2, v_3\}$ and $a_iv_i \in E(G)$ for $i = 1, 2, 3$. For a partition of $E(G')$ into three perfect matchings $M^1_{G'}$, $M^2_{G'}$ and $M^3_{G'}$, note that the edges xv_1, xv_2 and xv_3 are incident to x and therefore they belong to different sets $M^i_{G'}$. Without loss of generality, assume that $xv_i \in M^i_{G'}$ for $i = 1, 2, 3$. Then $M^i_G = M^i_{G'} \setminus \{xv_i\} \cup \{a_iv_i, a_{i+1}a_{i+2}\}$ for $i = 1, 2, 3$ is a partition of $E(G)$ into three perfect matchings (we assume $a_4 = a_1$ and $a_5 = a_2$).

In what follows we prove that the 3-edge coloring is unique up to a permutation of colors. For a partition of $E(G)$ into three perfect matchings M^1_G, M^2_G, M^3_G, observe that the edges of the triangle $a_1a_2a_3$ belong to different sets M^i_G. Without loss of generality, assume that $a_{i+1}a_{i+2} \in M^i_G$ for $i = 1, 2, 3$ (again, $a_4 = a_1$ and $a_5 = a_2$). We infer that $a_iv_i \in M^i_G$ and $M^i_{G'} = M^i_G \setminus \{a_iv_i, a_{i+1}a_{i+2}\} \cup \{xv_i\}$ for $i = 1, 2, 3$ is a partition of $E(G')$ into three perfect matchings. □ From this point, if we consider a Klee-graph G, by M^1_G, M^2_G and M^3_G we denote the unique (up to a permutation) partition of $E(G)$ into three perfect matchings.

Let us now formalize the notation of the Klee-graph construction procedure, so we can later use arguments based on the below definition. We introduce a *construction tree* T of a Klee-graph G. It is an ordered rooted tree (i.e., for each node, its children are ordered) with subsets of $V(G)$ as labels of the nodes. The root of T has four children, whereas all other nodes are either leaves or have three children. In other words, T is a 4-regular tree rooted at a non-leaf node. The labels of the leaves are singletons, the label of a node is the union of the labels of its children and the label of the root is $V(G)$. We sometimes abuse the notation and identify the nodes with theirs labels.

Moreover, in the definition we maintain the following property: for every non-root node with label W in T the set $\varepsilon_G(W)$ has three elements, and they belong to different sets in the partition of $E(G)$ into three perfect matchings. We denote the edge of $\varepsilon_G(W)$ that belongs to M^i_G by e^i_W. We denote also $e^i_{\{x\}}$ by e^i_x for short. Inductively, we define the construction tree as follows:

1. A *construction tree* of K_4 with vertex set $V(K_4) = \{a, b, c, d\}$ is a tree with its root labeled $\{a, b, c, d\}$ and four leaves labeled $\{a\}$, $\{b\}$, $\{c\}$ and $\{d\}$. If the leaves are ordered in this way, we assume that the edges are labeled as follows: $e^1_a = e^1_b = ab$, $e^1_c = e^1_d = cd$, $e^2_a = e^2_c = ac$, $e^2_b = e^2_d = bd$, $e^3_a = e^3_d = ad$ and $e^3_b = e^3_c = bc$. That is, we partition the edge set $E(K_4)$ into perfect matchings $M^{K_4}_1 = \{ab, cd\}$, $M^{K_4}_2 = \{ac, bd\}$ and $M^{K_4}_3 = \{ad, bc\}$.

2. Assume that a Klee-graph G is created from a Klee-graph G' by replacing a vertex $x \in V(G')$ by a triangle $a_1a_2a_3$. Let T' be a construction tree of G' and let $e^i_x = xv_i$ for $i = 1, 2, 3$. Assume that the vertex v_i is connected to a_i in G. Then the tree T is obtained from T' as follows:

   - The leaf $\{x\}$ in T' is replaced by a node $\{a_1, a_2, a_3\}$ and with three new leaves $\{a_1\}$, $\{a_2\}$, $\{a_3\}$ connected in this order as its children.
   - Every other label $W_{T'}$ in T' is replaced by a label W_T defined as follows:

$$W_T = \begin{cases} W_{T'} \setminus \{x\} \cup \{a_1, a_2, a_3\} & \text{if } x \in W_{T'}, \\ W_{T'} & \text{otherwise.} \end{cases}$$

   We note that the edge labels behave as in the proof of Lemma 2.1. That is, for all $i = 1, 2, 3$ we have $e^i_{a_i} = e^i_{\{a_1a_2a_3\}} = a_iv_i$. Moreover $e^3_{a_1} = e^3_{a_2} = a_1a_2$, $e^2_{a_1} = e^2_{a_3} = a_1a_3$ and $e^1_{a_2} = e^1_{a_3} = a_2a_3$. The edge numbering is depicted in Figure 3. A number i near an edge $e = uv$ means that $e = e^i_v = e^i_u \in M^i_G$. We note also that for any non-root node $W_{T'}$ in T' we have the edges $e^i_{W_T}$ for $i = 1, 2, 3$ as follows:

$$e^i_{W_T} = \begin{cases} a_jv_j & \text{if } e^i_{W_{T'}} = e^j_x, \\ e^i_{W_{T'}} & \text{otherwise.} \end{cases}$$

Let us now check if the definition of T really maintains the aforementioned property that for every non-root node with label W_T in T, $\varepsilon_G(W_T)$ is a 3-edge-cut (i.e., a set of three edges, such that their removal disconnects the graph), and the edges belong to different perfect matchings in the partition M^1_G, M^2_G, M^3_G. Indeed, if:

- if $|W_T| = 1$ then the claim is obvious, since G is cubic,
- if $W_T = \{a_1, a_2, a_3\}$ then $a_1a_2a_3$ is a triangle, $\varepsilon_G(W_T) = \{a_1v_1, a_2v_2, a_3v_3\}$ and, since the edges of the triangle $a_1a_2a_3$ belong to different sets in the partition M^1_G, M^2_G, M^3_G, so do the edges of $\varepsilon_G(W_T)$,
- otherwise, if the label W_T was constructed from $W_{T'}$, then $\varepsilon_G(W_T)$ is constructed from $\varepsilon_{G'}(W_{T'})$ by replacing all edges of the form xv_i by a_iv_i and $xv_i \in M^{G'}_i$, $a_iv_i \in M^i_G$.

Note that, given a Klee-graph G, its construction tree is not defined uniquely. For instance the unique Klee-graph with 8 vertices (see Figure 2) has some freedom in choosing its construction tree: we can create this graph by replacing two vertices of K_4 with triangles — obtaining a construction tree of depth 2, or we can replace one vertex of K_4 with a triangle and then replace one of the new vertices with another triangle — obtaining a construction tree of depth 3.

Fix a Klee-graph G and any its construction tree T. Let W be one of the non-root nodes in T and let v_i be the endvertex of e^i_W that is not contained in W ($i = 1, 2, 3$). The subgraph $G[W]$ is called a *tripod*

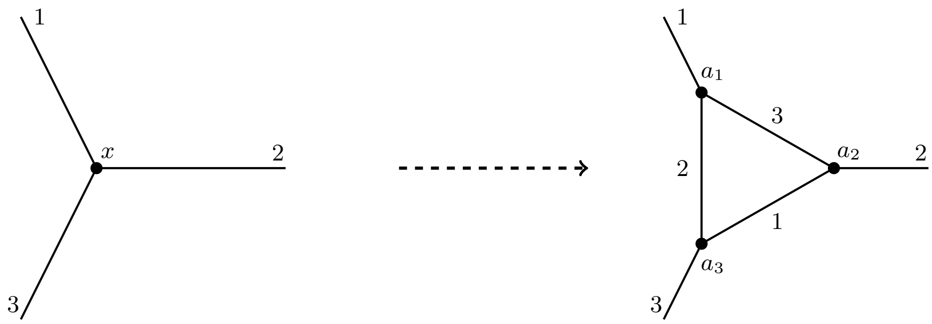

Fig. 3: Labeling edges after replacing a vertex by a triangle. A number i near an edge $e = uv$ means that $e = e_v^i = e_u^i \in M_G^i$.

and the edges $\varepsilon_G(W) = \{e_W^1, e_W^2, e_W^3\}$ are the *legs* of the tripod. A *tripod graph* G_W is the graph $G[W]$ extended by vertices $\{v_1, v_2, v_3\}$ and edges $\varepsilon_G(W) \cup \{v_1v_2, v_2v_3, v_3v_1\}$. The *size* of the tripod is $|W|$. We extend the definition of a construction tree to tripods: for a tripod W its construction tree, denoted as $T(W)$, is the subtree of T rooted at W. Note that if $T(W)$ has k non-leaf nodes, then it has $2k+1$ leaves and $|W| = 2k+1$, so the tripod size is always an odd integer.

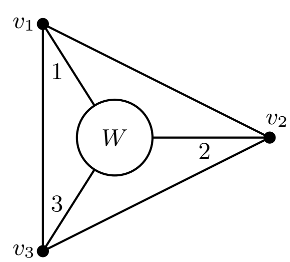

Fig. 4: Tripod graph G_W. A number i near an edge e means that $e = e_W^i \in M_G^i$.

Note that if a tripod W is not a single vertex, it consists of three smaller tripods, namely the children W_1, W_2, W_3 of W in the construction tree. It is straightforward from the definition of the construction tree that the legs of tripods are enumerated as in Figure 5. Later on, when we consider a tripod in a Klee-graph G, we implicitly assume that we are given a fixed construction tree for G.

3 Klee-graphs structure

In this section we gather a few structural results about Klee-graphs.

Lemma 3.1 *Let G be a Klee-graph with a construction tree T and let $W \subseteq V(G)$ be a tripod in G. Then the tripod graph G_W is a Klee-graph, too.*

Proof: It is sufficient to build a construction tree T' for the tripod graph G_W, which can be obtained by attaching to the root node the subtree $T(W)$ and three leaves with labels containing the vertices of the

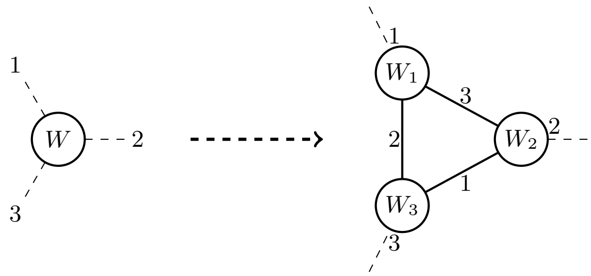

Fig. 5: Labeling of legs of a tripod and its children. A number i near an edge e that has an endpoint in a tripod $Y \in \{W, W_1, W_2, W_3\}$ means that $e = e_Y^i \in M_G^i$.

triangle added to W in the tripod graph G_W. □

Regarding the triangles in the Klee-graphs, one can easily observe the following:

Lemma 3.2 *Let G be a Klee-graph on at least 6 vertices. Then, G has at least two triangles and all triangles are vertex-disjoint.*

Proof: We prove it by induction on the number of vertices of G. The unique 6-vertex Klee-graph has two vertex-disjoint triangles, see Figure 2.

Now we focus on the Klee-graph construction step, i.e., we replace a vertex x in a graph G by a triangle abc obtaining a graph G'. By the induction hypothesis, in G there exists at most one triangle containing x. In G' we create a new triangle abc and destroy at most one triangle from G' — the one containing x. Notice that there is no triangle with any of the vertices a, b, c in G' except for abc, so the new triangle is vertex-disjoint from the other ones in G'. Therefore the construction step cannot decrease the number of triangles in the graph. □

The next easy lemma ensures that if we collapse a triangle into a vertex, we are still in the class of Klee-graphs.

Lemma 3.3 *Let G be a Klee-graph on at least 6 vertices and let abc be a triangle in G. Let G' be the graph obtained from G by contracting vertices a, b, c into a vertex x and removing the loops. Then, G' is a Klee-graph too.*

Proof: We prove it by induction on the number of vertices of G. If we contract one triangle in the unique 6-vertex Klee-graph we obtain K_4. So assume G has at least 8 vertices. By the definition of the Klee-graphs, G can be obtained from Klee-graph G_0 by replacing a vertex $y \in V(G_0)$ by a triangle pqr. If $pqr = abc$, we have $G' = G_0$ (taking $x = y$) and we are done. Otherwise, by Lemma 3.2, $\{a, b, c\} \cap \{p, q, r\} = \emptyset$ and the triangle abc exists in G_0. Let G_0' be the graph constructed by contracting the triangle abc in G_0 into one vertex x. Then, by induction hypothesis, G_0' is a Klee-graph and G' is obtained from G_0' by replacing y by a triangle pqr, so it is a Klee-graph too. □

Lemma 3.4 *Let G be a Klee-graph and let abc be a triangle in G. Then there exists a construction tree T for G such that the children of the root of T are $\{a\}$, $\{b\}$, $\{c\}$ and $V(G) \setminus \{a, b, c\}$. In other words, there exists a construction tree such that G is a tripod-graph for the tripod $V(G) \setminus \{a, b, c\}$.*

Proof: We prove by induction on $|V(G)|$. For $G = K_4$, the claim is obvious. Take G with $|V(G)| \geq 6$. By Lemma 3.2 there exists a triangle pqr disjoint from abc. By Lemma 3.3, the graph G' constructed from G by contracting the triangle pqr into a vertex x is also a Klee-graph. By induction hypothesis, there exists a construction tree T' for G' such that the children of the root of T' are $\{a\}$, $\{b\}$, $\{c\}$ and $V(G') \setminus \{a, b, c\}$. We replace the leaf $\{x\}$ by the node $\{p, q, r\}$ and its children $\{p\}$, $\{q\}$, $\{r\}$, as described in the definition of a construction tree. This way we obtain the desired tree T. $\square$

4 Counting perfect matchings

Let G be a Klee-graph with some fixed construction tree T and let W be a tripod in G. Since $|W|$ is odd, for any perfect matching M in G precisely one or all three legs of W are in M. Denote by $P_i(W)$ the number of perfect matchings in the tripod graph G_W which use only leg e^i_W ($i = 1, 2, 3$), and denote by $\bar{P}(W)$ the number of perfect matchings in G_W which use all three legs. Let us define $\mathcal{P}(W)$. A motivation for this product will be explained later.

$$\mathcal{P}(W) = P_1(W) \cdot P_2(W) \cdot P_3(W)$$

Note that the following hold:

- $P_1(W) = P_2(W) = P_3(W) = 1$ and $\bar{P}(W) = 0$ for $|W| = 1$;
- $P_1(W) = P_2(W) = P_3(W) = 1$ and $\bar{P}(W) = 1$ for $|W| = 3$;
- $\{P_1(W), P_2(W), P_3(W)\} = \{2, 1, 1\}$ and $\bar{P}(W) = 1$ for $|W| = 5$.

Lemma 4.1 *Let W be a tripod in a Klee-graph G, and G' a graph constructed by extending one vertex $x \in W$ into a triangle abc. Let W' be the tripod in G' that corresponds to W, i.e., $W' = W \setminus \{x\} \cup \{a, b, c\}$. Then $P_i(W) \leq P_i(W')$ for $i = 1, 2, 3$ and $\bar{P}(W) \leq \bar{P}(W')$. Moreover, $P_i(W) \geq 1$ for $i = 1, 2, 3$.*

Proof: Assume that $N_G(\{x\}) = \{x_a, x_b, x_c\}$ and each vertex $v \in \{a, b, c\}$ is connected to x_v in G'. Then any perfect matching M in G_W can be extended to a perfect matching M' in $G_{W'}$ by replacing the edge xx_v ($v \in \{a, b, c\}$) by the edge vx_v and by the edge with both endvertices in $\{a, b, c\} \setminus \{v\}$. Finally note that the extensions of distinct matchings are distinct.

To see that $P_i(W) \geq 1$ for $i = 1, 2, 3$ note that the part of the perfect matching M^i_G that is contained in $G[W]$ contributes to $P_i(W)$. $\square$

Lemma 4.2 *Let G be a Klee-graph with a tripod W of size greater than one. Let W_1, W_2 and W_3 be the children of W in the construction tree. Then the following formulas hold:*

$$\begin{array}{rcl} P_1(W) &=& P_1(W_1)\,P_1(W_2)\,P_1(W_3) \;+\; \bar{P}(W_1)\,P_3(W_2)\,P_2(W_3) \\ P_2(W) &=& P_2(W_1)\,P_2(W_2)\,P_2(W_3) \;+\; P_3(W_1)\,\bar{P}(W_2)\,P_1(W_3) \\ P_3(W) &=& P_3(W_1)\,P_3(W_2)\,P_3(W_3) \;+\; P_2(W_1)\,P_1(W_2)\,\bar{P}(W_3) \\ \bar{P}(W) &=& P_1(W_1)\,P_2(W_2)\,P_3(W_3) \;+\; \bar{P}(W_1)\,\bar{P}(W_2)\,\bar{P}(W_3) \end{array}$$

In particular, $\mathcal{P}(W) \geq \mathcal{P}(W_j)$ for $j = 1, 2, 3$.

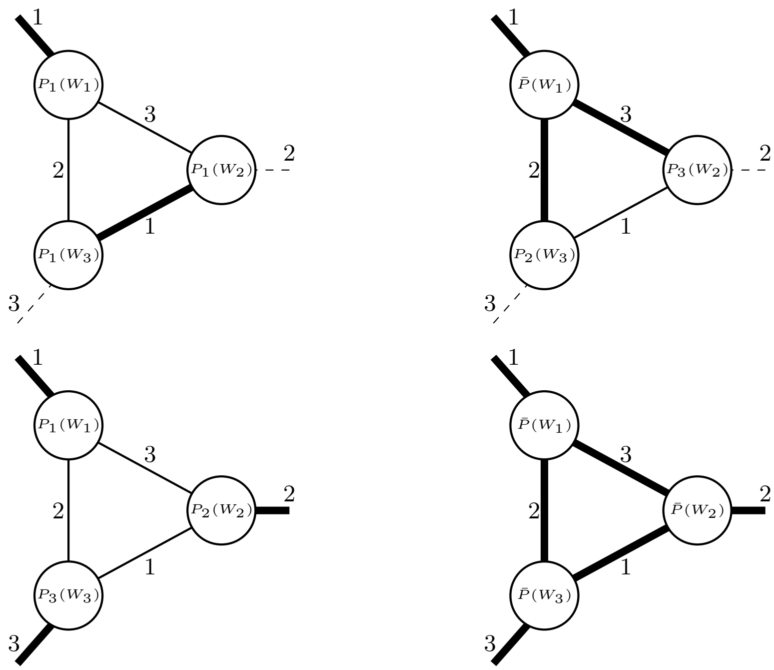

Fig. 6: Counting $P_1(W)$ and $\bar{P}(W)$.

Proof: In each equation, we consider two cases which are illustrated in Figure 6. By symmetry, it suffices to prove the formulas for $P_1(W)$ and $\bar{P}(W)$. Let us start with $P_1(W)$. The leg $e^1_{W_1}$ must be used and legs $e^2_{W_2}$ and $e^3_{W_3}$ must not. Recall that from every tripod we may use one or all three legs in any perfect matching. Therefore, if we use the leg $e^1_{W_1}$ then we may use all legs of W_1 — in this case we use legs $e^3_{W_2}$ and $e^2_{W_3}$ — or we may use only leg $e^1_{W_1}$ of W_1 and in this case we need to use the edge $e^1_{W_2} = e^1_{W_3}$ between tripods W_2 and W_3. To obtain the formula for $\bar{P}(W)$, note that we may either use all of the edges between tripods W_1, W_2, W_3 or none of them, which proves the desired equations.

Finally, since $P_i(W_j) \geq 1$ for $i, j \in \{1, 2, 3\}$, we have $P_1(W) \geq P_1(W_1)$, $P_2(W) \geq P_2(W_1)$, $P_3(W) \geq P_3(W_1)$ and therefore $\mathcal{P}(W) \geq \mathcal{P}(W_1)$. Similarly $\mathcal{P}(W) \geq \mathcal{P}(W_j)$ for $j = 2, 3$. $\square$

Before we proceed to the main result, we need to do some calculations by hand to provide the basis for the inductive proof of the main theorem.

Lemma 4.3 *Let W be a tripod of size at least 5. Then, either W is one of the tripods depicted in Figure 7 or $\mathcal{P}(W) \geq 4$.*

Proof: Note first that the left tripod in Figure 7 is the unique tripod on 5 vertices, so we may assume $|W| \geq 7$.

Let W_1, W_2 and W_3 be the children of W in the construction tree. Without loss of generality assume

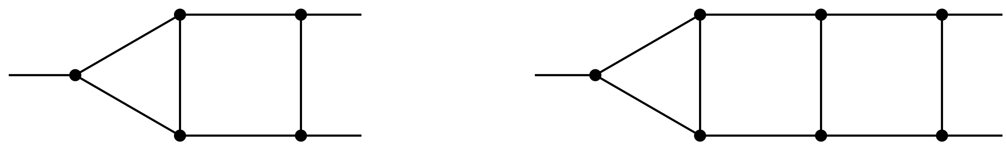

Fig. 7: The only tripods on at least 5 vertices satisfying $\mathcal{P}(W) < 4$.

that $|W_1| \geq 3$. If $|W_2| \geq 3$, then $\bar{P}(W_1), \bar{P}(W_2) \geq 1$ and, by Lemma 4.2:

$$\begin{aligned} P_1(W) &\geq P_1(W_1) + \bar{P}(W_1) \geq 2 \\ P_2(W) &\geq P_2(W_2) + \bar{P}(W_2) \geq 2, \end{aligned}$$

and so $\mathcal{P}(W) \geq 4$. Similarly the claim is proven for $|W_3| \geq 3$.

Suppose now that $|W_2| = |W_3| = 1$. Then $|W_1| \geq 5$. Let W_{11}, W_{12} and W_{13} be the children of W_1 in the construction tree. If $|W_{12}| \geq 3$, then $\bar{P}(W_{12}) \geq 1$ and, by Lemma 4.2:

$$\begin{aligned} P_1(W) &\geq P_1(W_1) + \bar{P}(W_1) \geq 1 + 1 = 2 \\ P_2(W) &\geq P_2(W_1) \geq P_2(W_{11})\, P_2(W_{12})\, P_2(W_{13}) + \bar{P}(W_{12}) \geq 2, \end{aligned}$$

and so $\mathcal{P}(W) \geq 4$. Similarly the claim is proven for $|W_{13}| \geq 3$.

The last case is when $|W_{12}| = |W_{13}| = 1$. If $|W_{11}| = 3$ then W is the right tripod depicted in Figure 7. Otherwise $|W_{11}| \geq 5$ and, by Lemma 4.1, $\mathcal{P}(W_{11}) \geq 2$. Note that

$$\begin{aligned} P_1(W) &\geq P_1(W_1) + \bar{P}(W_1) \\ &\geq P_1(W_{11}) + \bar{P}(W_{11}) + P_1(W_{11}) \\ &\geq 2P_1(W_{11}). \end{aligned}$$

Therefore

$$\mathcal{P}(W) = P_1(W)P_2(W)P_3(W) \geq 2P_1(W_{11})P_2(W_{11})P_3(W_{11}) = 2\mathcal{P}(W_{11}) \geq 2 \cdot 2 = 4,$$

and the claim is proven. □

Let us note that, unfortunately, it is not obvious how to use the sum $P_1(W) + P_2(W) + P_3(W)$ in the following inductive proof since it is possible that in the equations from Lemma 4.2 big values will be multiplied by small values resulting in something too small to preserve the bound. Therefore we make use of the product $\mathcal{P}(W) = P_1(W) \cdot P_2(W) \cdot P_3(W)$ in the main theorem of this section.

Theorem 4.4 *Let G be a Klee-graph with a tripod W satisfying $|W| \geq 5$. Then*

$$\mathcal{P}(W) \geq 2^{(|W|+15)/20}.$$

Proof: We use induction on the size of the tripod W. As an induction basis, we verify tripods of size $5 \leq |W| \leq 25$. First, let us focus on tripods depicted in Figure 7. The left one is the unique tripod of size 5 that satisfies

$$\{P_1(W), P_2(W), P_3(W)\} = \{1, 1, 2\} \quad \text{and} \quad \mathcal{P}(W) = 2 = 2^{(5+15)/20}.$$

The right one has 7 vertices and satisfies

$$\{P_1(W), P_2(W), P_3(W)\} = \{1, 1, 3\} \quad \text{and} \quad \mathcal{P}(W) = 3 > 2^{(7+15)/20}.$$

By Lemma 4.3, every other tripod W of size at least 5 satisfies $\mathcal{P}(W) \geq 4 = 2^{(25+15)/20}$. This finishes the proof of the induction basis.

Next let us proceed to the induction step. We may assume now that $|W| \geq 27$ and perform the following procedure:

I. Take $W^{tmp} := W$.

II. Let W_1^{tmp}, W_2^{tmp}, W_3^{tmp} be the children of W^{tmp} in the construction tree.

III. If at least two of W_i^{tmp} have size at least 5, stop.

IV. If none of W_i^{tmp} is a single vertex, stop.

V. If we are here for the fourth time, stop.

VI. Assign to W^{tmp} the one among W_i^{tmp} that has at least 5 vertices and go to Step II.

Note that if we do not stop at any of the Steps III and IV, it means that we have among $W_1^{tmp}, W_2^{tmp}, W_3^{tmp}$ one single vertex and either another single vertex or a triangle (three-vertex tripod), so at Step VI one of the W_i^{tmp} has at least $|W^{tmp}| - 4$ vertices. Therefore, if we do not stop at Step V, then at Step VI one of W_i^{tmp} has at least $|W| - 3 \cdot 4 \geq 15$ vertices.

We now do case analysis to bound $\mathcal{P}(W)$ regarding Steps III, IV or V, where the procedure stopped.

Procedure stopped at Step III: Since we do not use the relative order of $W_1^{tmp}, W_2^{tmp}, W_3^{tmp}$ we may assume that $|W_1^{tmp}| \geq 5$ and $|W_2^{tmp}| \geq 5$. As noted before, after one loop through Steps II-VI, the tripod W^{tmp} looses at most 4 vertices; therefore $|W^{tmp}| \geq |W| - 3 \cdot 4$. If $|W_3^{tmp}| \geq 5$, then by induction hypothesis and Lemma 4.2, we infer:

$$\begin{aligned}
\mathcal{P}(W) &\geq \mathcal{P}(W^{tmp}) \\
&= (P_1(W_1^{tmp})\, P_1(W_2^{tmp})\, P_1(W_3^{tmp}) + \bar{P}(W_1^{tmp})\, P_3(W_2^{tmp})\, P_2(W_3^{tmp}))\cdot \\
&\quad\ (P_2(W_1^{tmp})\, P_2(W_2^{tmp})\, P_2(W_3^{tmp}) + P_3(W_1^{tmp})\, \bar{P}(W_2^{tmp})\, P_1(W_3^{tmp}))\cdot \\
&\quad\ (P_3(W_1^{tmp})\, P_3(W_2^{tmp})\, P_3(W_3^{tmp}) + P_2(W_1^{tmp})\, P_1(W_2^{tmp})\, \bar{P}(W_3^{tmp})) \\
&\geq P_1(W_1^{tmp})\, P_1(W_2^{tmp})\, P_1(W_3^{tmp})\, P_2(W_1^{tmp})\, P_2(W_2^{tmp})\, P_2(W_3^{tmp})\cdot \\
&\quad\ P_3(W_1^{tmp})\, P_3(W_2^{tmp})\, P_3(W_3^{tmp}) \\
&= \mathcal{P}(W_1^{tmp})\, \mathcal{P}(W_2^{tmp})\, \mathcal{P}(W_3^{tmp}) \\
&\geq 2^{(|W_1^{tmp}|+15)/20} \cdot 2^{(|W_2^{tmp}|+15)/20} \cdot 2^{(|W_3^{tmp}|+15)/20} \\
&= 2^{(|W^{tmp}|+45)/20} \geq 2^{(|W|+33)/20}.
\end{aligned}$$

Otherwise, if $|W_3^{tmp}| \leq 3$ we have

$$P_1(W_3^{tmp}) = P_2(W_3^{tmp}) = P_3(W_3^{tmp}) = 1 \quad \text{and} \quad |W_1^{tmp}| + |W_2^{tmp}| \geq |W^{tmp}| - 3,$$

and hence:

$$\begin{aligned}
\mathcal{P}(W) &\geq \mathcal{P}(W^{tmp}) \\
&\geq P_1(W_1^{tmp})\,P_1(W_2^{tmp})\,P_1(W_3^{tmp})\,P_2(W_1^{tmp})\,P_2(W_2^{tmp})\,P_2(W_3^{tmp})\cdot \\
&\quad P_3(W_1^{tmp})\,P_3(W_2^{tmp})\,P_3(W_3^{tmp}) \\
&= \mathcal{P}(W_1^{tmp})\,\mathcal{P}(W_2^{tmp}) \\
&\geq 2^{(|W_1^{tmp}|+15)/20}\cdot 2^{(|W_2^{tmp}|+15)/20} \\
&\geq 2^{(|W^{tmp}|+27)/20} \geq 2^{(|W|+15)/20}.
\end{aligned}$$

Procedure stopped at Step IV: Since we did not stop at Step III, without loss of generality we may assume that $|W_1^{tmp}|\geq 5$ and $|W_2^{tmp}|=|W_3^{tmp}|=3$. Recall that $\bar{P}(W_j)=P_i(W_j)=1$ for $i=1,2,3$ and $j=2,3$ and note that $|W_1^{tmp}|=|W^{tmp}|-6\geq|W|-18$. So, we obtain

$$\begin{aligned}
\mathcal{P}(W) &\geq \mathcal{P}(W^{tmp}) \\
&= (P_1(W_1^{tmp})\,P_1(W_2^{tmp})\,P_1(W_3^{tmp})+\bar{P}(W_1^{tmp})\,P_3(W_2^{tmp})\,P_2(W_3^{tmp}))\cdot \\
&\quad (P_2(W_1^{tmp})\,P_2(W_2^{tmp})\,P_2(W_3^{tmp})+P_3(W_1^{tmp})\,\bar{P}(W_2^{tmp})\,P_1(W_3^{tmp}))\cdot \\
&\quad (P_3(W_1^{tmp})\,P_3(W_2^{tmp})\,P_3(W_3^{tmp})+P_2(W_1^{tmp})\,P_1(W_2^{tmp})\,\bar{P}(W_3^{tmp})) \\
&\geq P_1(W_1^{tmp})\cdot(P_2(W_1^{tmp})+P_3(W_1^{tmp}))\cdot(P_2(W_1^{tmp})+P_3(W_1^{tmp})) \\
&\geq 4P_1(W_1^{tmp})\,P_2(W_1^{tmp})\,P_3(W_1^{tmp})=4\mathcal{P}(W_1^{tmp}) \\
&\geq 2^{(|W_1^{tmp}|+55)/20}\geq 2^{(|W|+37)/20}.
\end{aligned}$$

Procedure stopped at Step V. In this case we need to use all four iterations of Step II and also to do further case analysis. Let $W^{(0)}=W$ and let $W^{(i+1)}$ be the biggest child of $W^{(i)}$ in the construction tree. Note that for $i=0,1,2$ the tripod $W^{(i+1)}$ is assigned to W^{tmp} at Step VI, but this definition makes sense also for $i=3$. Moreover, for $i=0,1,2,3$ children of $W^{(i)}$ in the construction tree different than $W^{(i+1)}$ have total size at most 4, since we did not break at Steps III and IV. Therefore

$$|W^{(i)}|\geq|W^{(i-1)}|-4\geq|W|-4i\quad\text{and}\quad|W^{(4)}|\geq|W|-16\geq 11.$$

Before we start, note the following inequalities implied by Lemma 4.2 for any tripod T and its children T_1,T_2,T_3 in the construction tree:

$$\begin{aligned}
P_j(T) &\geq P_j(T_k) && \text{for } j,k=1,2,3, \\
P_j(T) &\geq P_j(T_j)+\bar{P}(T_j) && \text{for } j=1,2,3, \\
\bar{P}(T) &\geq P_j(T_j) && \text{for } j=1,2,3.
\end{aligned}$$

In the next few paragraphs we analyze cases whether $W^{(i+1)}$ is $W_1^{(i)}$, $W_2^{(i)}$ or $W_3^{(i)}$, for $i=0,1,2,3$. Fortunately, for every i at most one case needs further investigation.

Without loss of generality we may assume that $W^{(1)} = W_1^{(0)} = W_1$. Therefore:

$$\begin{aligned} P_1(W^{(0)}) &\geq P_1(W^{(1)}) + \bar{P}(W^{(1)}) \\ P_2(W^{(0)}) &\geq P_2(W^{(1)}) \\ P_3(W^{(0)}) &\geq P_3(W^{(1)}) \\ \bar{P}(W^{(0)}) &\geq P_1(W^{(1)}). \end{aligned}$$

Now let us focus on positioning $W^{(2)}$ inside $W^{(1)}$.
Case 1: $W^{(2)} = W_1^{(1)}$**.** Then

$$\begin{aligned} P_1(W^{(0)}) &\geq P_1(W^{(1)}) + \bar{P}(W^{(1)}) \geq 2P_1(W^{(2)}) + \bar{P}(W^{(2)}) \geq 2P_1(W^{(2)}) \\ P_2(W^{(0)}) &\geq P_2(W^{(1)}) \geq P_2(W^{(2)}) \\ P_3(W^{(0)}) &\geq P_3(W^{(1)}) \geq P_3(W^{(2)}). \end{aligned}$$

So

$$\mathcal{P}(W) = \mathcal{P}(W^{(0)}) \geq 2\mathcal{P}(W^{(2)}) \geq 2^{(|W^{(2)}|+35)/20} \geq 2^{(|W|+27)/20}.$$

Case 2: $W^{(2)} = W_2^{(1)}$ **or** $W^{(2)} = W_3^{(1)}$**.** Note that these cases are symmetrical, so let us assume that $W^{(2)} = W_2^{(1)}$. Then

$$\begin{aligned} P_1(W^{(0)}) &\geq P_1(W^{(1)}) + \bar{P}(W^{(1)}) \geq P_1(W^{(2)}) + P_2(W^{(2)}) \\ P_2(W^{(0)}) &\geq P_2(W^{(1)}) \geq P_2(W^{(2)}) + \bar{P}(W^{(2)}) \\ P_3(W^{(0)}) &\geq P_3(W^{(1)}) \geq P_3(W^{(2)}). \end{aligned}$$

Now let us position $W^{(3)}$ inside $W^{(2)}$ by considering further subcases.
Case 2.1: $W^{(3)} = W_1^{(2)}$**.** Then

$$\begin{aligned} P_1(W^{(0)}) &\geq P_1(W^{(2)}) + P_2(W^{(2)}) \geq P_1(W^{(3)}) + P_2(W^{(3)}) \\ P_2(W^{(0)}) &\geq P_2(W^{(2)}) + \bar{P}(W^{(2)}) \geq P_2(W^{(3)}) + P_1(W^{(3)}) \\ P_3(W^{(0)}) &\geq P_3(W^{(2)}) \geq P_3(W^{(3)}). \end{aligned}$$

Thus,

$$\mathcal{P}(W) \geq (P_1(W^{(3)}) + P_2(W^{(3)}))^2 P_3(W^{(3)}) \geq 4\mathcal{P}(W^{(3)}) \geq 2^{(|W^{(3)}|+55)/20} \geq 2^{(|W|+43)/20}.$$

Case 2.2: $W^{(3)} = W_2^{(2)}$**.** Then

$$\begin{aligned} P_1(W^{(0)}) &\geq P_1(W^{(2)}) + P_2(W^{(2)}) \geq P_1(W^{(3)}) + P_2(W^{(3)}) \geq P_1(W^{(3)}) \\ P_2(W^{(0)}) &\geq P_2(W^{(2)}) + \bar{P}(W^{(2)}) \geq 2P_2(W^{(3)}) \\ P_3(W^{(0)}) &\geq P_3(W^{(2)}) \geq P_3(W^{(3)}). \end{aligned}$$

In this case

$$\mathcal{P}(W) \geq 2\mathcal{P}(W^{(3)}) \geq 2^{(|W^{(3)}|+35)/20} \geq 2^{(|W|+23)/20}.$$

Case 2.3: $W^{(3)} = W_3^{(2)}$**.** Here

$$\begin{aligned} P_1(W^{(0)}) &\geq P_1(W^{(2)}) + P_2(W^{(2)}) \geq P_1(W^{(3)}) + P_2(W^{(3)}) \\ P_2(W^{(0)}) &\geq P_2(W^{(2)}) + \bar{P}(W^{(2)}) \geq P_2(W^{(3)}) + P_3(W^{(3)}) \\ P_3(W^{(0)}) &\geq P_3(W^{(2)}) \geq P_3(W^{(3)}) + \bar{P}(W^{(3)}). \end{aligned}$$

Now we need to look at children of $W^{(3)}$ in the construction tree T once more, i.e., position $W^{(4)}$ inside $W^{(3)}$. Consider the following subcases:

Case 2.3.1: $W^{(4)} = W_1^{(3)}$**.** Then

$$\begin{aligned} P_1(W^{(0)}) &\geq P_1(W^{(3)}) + P_2(W^{(3)}) \geq P_1(W^{(4)}) + P_2(W^{(4)}) \\ P_2(W^{(0)}) &\geq P_2(W^{(3)}) + P_3(W^{(3)}) \geq P_2(W^{(4)}) + P_3(W^{(4)}) \\ P_3(W^{(0)}) &\geq P_3(W^{(3)}) + \bar{P}(W^{(3)}) \geq P_3(W^{(4)}) + P_1(W^{(4)}), \end{aligned}$$

and hence

$$\begin{aligned} \mathcal{P}(W) &\geq (P_1(W^{(4)}) + P_2(W^{(4)}))(P_2(W^{(4)}) + P_3(W^{(4)}))(P_3(W^{(4)}) + P_1(W^{(4)})) \\ &\geq 8\mathcal{P}(W^{(4)}) \geq 2^{(|W^{(4)}|+75)/20} \geq 2^{(|W|+59)/20}. \end{aligned}$$

Case 2.3.2: $W^{(4)} = W_2^{(3)}$**.** Then

$$\begin{aligned} P_1(W^{(0)}) &\geq P_1(W^{(3)}) + P_2(W^{(3)}) \geq P_1(W^{(4)}) + P_2(W^{(4)}) \\ P_2(W^{(0)}) &\geq P_2(W^{(3)}) + P_3(W^{(3)}) \geq P_2(W^{(4)}) + P_3(W^{(4)}) \\ P_3(W^{(0)}) &\geq P_3(W^{(3)}) + \bar{P}(W^{(3)}) \geq P_3(W^{(4)}) + P_2(W^{(4)}), \end{aligned}$$

and so

$$\mathcal{P}(W) \geq P_1(W^{(4)})(P_2(W^{(4)}) + P_3(W^{(4)}))^2 \geq 4\mathcal{P}(W^{(4)}) \geq 2^{(|W^{(4)}|+55)/20} \geq 2^{(|W|+39)/20}.$$

Case 2.3.3: $W^{(4)} = W_3^{(3)}$**.** Here we derive,

$$\begin{aligned} P_1(W^{(0)}) &\geq P_1(W^{(3)}) + P_2(W^{(3)}) \geq P_1(W^{(4)}) + P_2(W^{(4)}) \\ P_2(W^{(0)}) &\geq P_2(W^{(3)}) + P_3(W^{(3)}) \geq P_2(W^{(4)}) + P_3(W^{(4)}) \\ P_3(W^{(0)}) &\geq P_3(W^{(3)}) + \bar{P}(W^{(3)}) \geq 2P_3(W^{(4)}). \end{aligned}$$

So

$$\mathcal{P}(W) \geq 2\mathcal{P}(W^{(4)}) \geq 2^{(|W^{(4)}|+35)/20} \geq 2^{(|W|+19)/20}.$$

Since we have exhausted all cases, the theorem is proven. □

The goal of this section is now an easy corollary from Theorem 4.4.

Theorem 4.5 *Every Klee-graph G with at least 8 vertices has at least $3 \cdot 2^{(|V(G)|+12)/60}$ perfect matchings.*

Proof: By Lemma 3.4, G is a tripod graph for some triangle abc and tripod $W = V(G) \setminus \{a,b,c\}$. The number of perfect matchings in G is $P_1(W) + P_2(W) + P_3(W) + \bar{P}(W)$. By Theorem 4.4, $\mathcal{P}(W) = P_1(W)P_2(W)P_3(W) \geq 2^{(|W|+15)/20} = 2^{(|V(G)|+12)/20}$ and therefore

$$P_1(W) + P_2(W) + P_3(W) + \bar{P}(W) \geq 3\sqrt[3]{P_1(W)P_2(W)P_3(W)} \geq 3 \cdot 2^{(|V(G)+12)/60}.$$

□

5 Klee-graphs with few perfect matchings

In this section we give an infinite family of Klee-graphs that have a 'small' number of perfect matchings. Let F_n be the Fibonnacci sequence, i.e., $F_0 = 1$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$. Precisely, we show the following:

Theorem 5.1 *For every positive integer k there exists a Klee-graph G with $12k + 4$ vertices that has $3(k+1)F_{k+1}$ perfect matchings.*

Let $\phi = \frac{1+\sqrt{5}}{2}$. It is well known that $F_n < c_F \phi^n$ for some constant c_F. Therefore, as a corollary from Theorem 5.1, we obtain that there exists a constant c_0 such that for every positive integer n there exists a Klee-graph with at least n vertices and at most $c_0 n \phi^{n/12}$ perfect matchings. Since $\phi^{1/12} < 2^{1/17.285}$, for sufficiently large constant c we have that $c_0 n \phi^{n/12} < c\, 2^{n/17.285}$ and we obtain the concluding corollary of this section:

Corollary 5.2 *There exists a constant c such that for every positive integer n there exists a Klee-graph with at least n vertices and at most $c\, 2^{n/17.285}$ perfect matchings.*

Now let us prove Theorem 5.1. A *ladder graph* of level k is the tripod graph G_{L^k} of the following tripod L^k, defined recursively. L^1 is the unique 5-vertex tripod with construction tree $T(L^1)$ such that $e^2_{L^1}$ has an endvertex in the inner triangle. In the construction tree $T(L^{k+1})$ of the tripod L^{k+1}, the children L^{k+1}_1, L^{k+1}_2 and L^{k+1}_3 of the root node are defined as follows:

- the first child L^{k+1}_1 is $T(L^k)$,
- $|L^{k+1}_2| = 3$, i.e., L^{k+1}_2 is a triangle with three legs, and
- $|L^{k+1}_3| = 1$, i.e., L^{k+1}_3 is a single vertex.

See Figure 8 for details. Note that a ladder tripod has $4k + 1$ vertices.

Lemma 5.3 *For a ladder graph G_{L^k} the following holds: $P_1(L^k) = F_k$, $P_2(L^k) = k + 1$, $P_3(L^k) = 1$ and $\bar{P}(L^k) = F_{k-1}$.*

Proof: We prove by induction on k. For L^1 we have $P_1(L^1) = P_3(L^1) = \bar{P}(L^1) = 1$ and $P_2(L^1) = 2$.

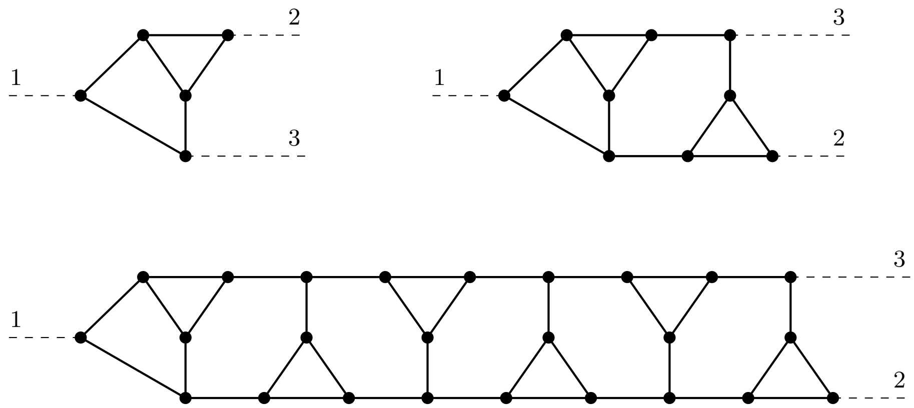

Fig. 8: Ladder tripods L^1, L^2 and L^6.

Let us look at the children of L^{k+1} in the construction tree $T(L^{k+1})$. We have $L_1^{k+1} = L^k$, $|L_2^{k+1}| = 3$ and $|L_3^{k+1}| = 1$. Therefore, by Lemma 4.2, it holds

$$\begin{aligned} P_1(L^{k+1}) &= P_1(L^k) + \bar{P}(L^k) = F_k + F_{k-1} = F_{k+1} \\ P_2(L^{k+1}) &= P_2(L^k) + P_3(L^k) = k + 1 + 1 = k + 2 \\ P_3(L^{k+1}) &= P_3(L^k) = 1 \\ \bar{P}(L^{k+1}) &= P_1(L^k) = F_k. \end{aligned}$$

This completes the proof. □

For an integer i, by $L^k_{(i)}$ we denote the tripod L^k with the labels of the edges cyclically shifted by i; that is, $M_j^{L^k_{(i)}} = M_{i+j}^{L^k}$, where the operation $i + j$ is taken modulo 3. A 3-*ladder graph* of level k is a tripod graph G_{S^k} of the tripod S^k, defined as follows: the children of S^k in the construction tree $T(S^k)$ are $S_i^k = T(L^k_{(i-1)})$ for $i = 1, 2, 3$ (see Figure 9). Note that a 3-ladder tripod has $12k + 3$ vertices.

Lemma 5.4 *The* 3*-ladder graph* G_{S^k} *satisfies the following for* $i = 1, 2, 3$*:*

$$P_i(S^k) = (k + 1)F_{k+1}.$$

Proof: By symmetry, we may consider only $P_1(S^k)$. By Lemma 4.2,

$$\begin{aligned} P_1(S^k) &= P_1(L^k_{(0)})\, P_1(L^k_{(1)})\, P_1(L^k_{(2)}) + \bar{P}(L^k_{(0)})\, P_3(L^k_{(1)})\, P_2(L^k_{(2)}) \\ &= P_1(L^k)\, P_2(L^k)\, P_3(L^k) + \bar{P}(L^k)\, P_2(L^k)\, P_3(L^k) \\ &= (k + 1)F_k + (k + 1)F_{k-1} \\ &= (k + 1)F_{k+1}. \end{aligned}$$

□

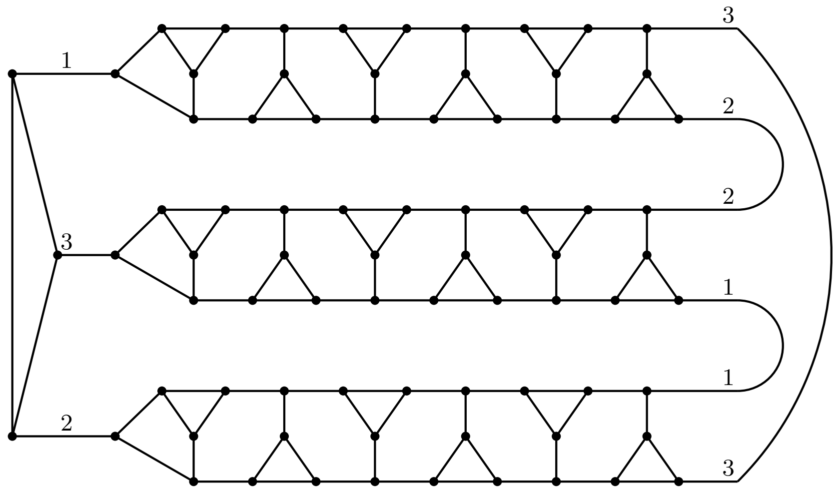

Fig. 9: 3-ladder graph of level 6.

We now complete the proof of Theorem 5.1. Let G_k be a graph constructed from G_{S^k} by contracting the triangle $V(G_{S^k}) \setminus S^k$ into a single vertex. Then G_k is a Klee-graph by Lemma 3.3 with $12k+4$ vertices and $P_1(S^k) + P_2(S^k) + P_3(S^k) = 3(k+1)F_{k+1}$ perfect matchings.

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