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*Strong parity vertex coloring of plane graphs**

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A strong parity vertex coloring of a 2-connected plane graph is a coloring of the vertices such that every face is incident with zero or an odd number of vertices of each color. We prove that every 2-connected loopless plane graph has a strong parity vertex coloring with 97 colors. Moreover the coloring we construct is proper. This proves a conjecture of Czap and Jendrol' [Discuss. Math. Graph Theory 29 (2009), pp. 521–543.]. We also provide examples showing that eight colors may be necessary (ten when restricted to proper colorings).

Keywords: graph, strong parity vertex coloring, strong parity chromatic number, proper coloring, face, discharging

1 Introduction

The notions of strong parity vertex coloring and the strong parity chromatic number were defined by Czap and Jendrol' [3]. Let us recall their definition in an equivalent form. Let G be a nontrivial connected plane graph, and let f be one of its faces. (Throughout the paper, graphs are allowed to have parallel edges but no loops.) Consider a (possibly improper) vertex coloring of G. The face f satisfies the *strong parity vertex coloring condition* (*spv-condition* for short) with respect to the coloring if for each color c of the coloring, there is zero or an odd number of occurrences of vertices colored with c on a closed facial walk of f. The coloring is a *strong parity vertex coloring* (*spv-coloring* for short) if the spv-condition holds for every face of G. Assume now that G is 2-connected. Then the minimum number of colors in an spv-coloring of G is called the *strong parity chromatic number* of G and is denoted by $\chi_s(G)$. The restriction to 2-connected graphs in the definition of χ_s is essential, since there are plane graphs of connectivity one that do not admit any spv-coloring (an example of Czap and Jendrol' [3] consists of two triangles sharing

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one vertex). Similarly, the existence of an spv-coloring cannot be guaranteed for graphs with loops (which is why we exclude them in our definition).

Czap and Jendroľ [3] conjectured that there is a constant bound K on χ_s in the class of 2-connected plane graphs. Furthermore, they suggested that the best possible bound equals 6, providing an infinite family of graphs with $\chi_s = 6$. The main result of our paper confirms the conjecture for the class of 2-connected plane graphs with an added restriction to proper colorings:

Theorem 1.1 *Every* 2*-connected plane (loopless) graph has a proper spv-coloring with at most* 97 *colors.*

The proof is given in Section 2. (During the preparation of this paper, another proof—giving a slightly worse constant—was independently found by Czap, Jendroľ, and Voigt [4].) In Section 3, we present examples showing that the best possible value of K in the above conjecture is at least 8, or at least 10 with the restriction to proper colorings.

It should be noted that prior to the introduction of parity vertex colorings, a related type of edge coloring was considered in [1, 2]. An edge variant of parity coloring (called facial parity edge coloring) was recently studied, e.g., in [5].

In the remainder of this section, we establish the basic notation used throughout the paper; the notions not mentioned here are standard in graph theory [6]. As mentioned above, graphs are assumed to be loopless, but parallel edges are allowed. A graph is called *trivial* if it is empty or consists of a single vertex. Let G be a plane graph; let v be a vertex, e_1 and e_2 edges, and f a face of G. Then $F(v)$ or $F(e_1)$ denotes the set of faces incident with v and e_1 respectively. The *boundary vertices* and *boundary edges* of f are all the vertices and edges of G, respectively, incident with f. The sets of these vertices and edges are denoted by $V(f)$ and $E(f)$ respectively. We refer to $|V(f)|$ as the *length* of f.

The degree of v in G, i.e., the number of edges of G incident with v, is denoted by $d(v)$. If $d(v) = 2$, the vertex v is a 2*-vertex*; when $d(v) > 2$, we call v a *high-degree* vertex or a vertex of *high* degree. The edges e_1 and e_2 are *parallel* if they are not loops and share their end-vertices. When e_1 and e_2 are parallel and constitute the boundary of f, the face f is called a *digon*. A path P is *trivial* if it comprises a single vertex, that is, the length of P equals 0. Every vertex of P other than its end-vertices is called an *internal vertex* of P. Finally, we remark that for all the notation defined, the relevant graph may be referred to by a subscript whenever necessary. For example, we write $F_G(v)$ or $d_G(v)$ if this graph is G.

2 Upper bound

This section is devoted to the proof of the main result, Theorem 1.1. In Section 2.1, we prove certain structural properties of a minimal counterexample. These are used in an application of the discharging method in Section 2.2.

We now introduce a graph operation to be used in the proof of Lemma 2.4. Let G be a plane graph and $v \in V(G)$ a vertex of degree $d \geq 2$, and let the edges incident with v be enumerated in a clockwise order as $e_i = vv_i$, $i \in \mathbb{Z}_d$ (we write $\mathbb{Z}_d$ for the set $\{0, \dots, d-1\}$ with addition modulo d). Suppose that the vertices v_i are pairwise different (that is, v is not incident with a pair of parallel edges). The *annihilation* of v is the construction of a plane graph G' from G defined as follows:

(1) add edges $e'_i = v_i v_{i+1}$, $i \in \mathbb{Z}_d$, embedded in the plane so that for each i, the edges e_i, e_{i+1}, and e'_i, in this order, constitute a facial walk;

(2) delete v together with all the edges e_i.

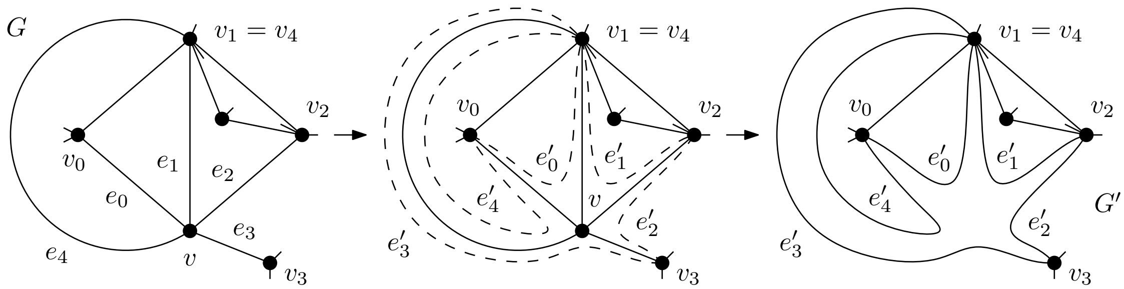


Fig. 2.1. The annihilation of a vertex v. The original graph G is on the left, the resulting graph G' on the right.

Intuitively, one may achieve the desired embeddings of the edges e'_i by drawing each e'_i 'close enough' to the curve consisting of the embeddings of e_i and e_{i+1}; see Figure 2.1 for an example of a properly conducted annihilation.

Regarding the faces of G and G', it is obvious that the following holds:

Observation 2.1 *Let G' be obtained from G by the annihilation of a vertex $v \in V(G)$. Then*

(1) *every face of G not in $F_G(v)$ is also a face of G';*

(2) *each face $g \in F_G(v)$ has its counterpart g' in G' such that a facial walk of g' may be obtained from a facial walk of g by replacing each of its subsequences of the form $e_i v e_{i+1}$ with e'_i, and hence $V(g') = V(g) - \{v\}$;*

(3) *there is precisely one more face in G', having the sequence $v_0 e'_{d-1} v_{d-1} e'_{d-2} \dots v_1 e'_0 v_0$ as its facial walk.*

The assumption that v is not incident with a pair of parallel edges is essential: without it, the annihilation of v may produce a loop as well as a cutvertex. On the other hand, the following lemma shows that parallel edges incident with v are the only reason for such a result:

Lemma 2.2 *Let v be a vertex of a 2-connected (loopless) plane graph G, $|V(G)| \geq 4$, such that v is incident with no pair of parallel edges. Then the graph G' obtained from G by annihilating v is 2-connected (and loopless).*

Proof: We use the well-known fact that a connected loopless plane graph G on at least three vertices is 2-connected if and only if the facial walk of each of its faces is a cycle. (The 'only if' direction is Proposition 4.2.6 in [6]. Conversely, for any cutvertex v of G there is a face whose boundary contains two neighbors of v in different components of $G - v$. Since each cycle is contained within some block, the boundary of this face cannot be a cycle.)

We use this criterion for both G and G' in the following. (Note that G' is connected.) Take an arbitrary face f' of G', and let W be a facial walk of f'. We may assume that f' is not a face of G, otherwise there is nothing to show. Thus by Observation 2.1, W is either the walk $v_0 e'_{d-1} v_{d-1} e'_{d-2} \dots v_1 e'_0 v_0$ (up to the choice of the end-vertex), or it arises from a facial walk of a face of G by replacing each of its subsequences of the form $e_i v e_{i+1}$ with e'_i. In both cases, it follows from the assumptions that W is a cycle. □

Finally, we include a technical lemma that will greatly simplify the case analysis in the proof of Claim 1 in Section 2.2.

Lemma 2.3 *Let (l_i), (l'_i), $i = 0, \dots, k$, be tuples of positive integers such that $l_j \leq l'_j$ for every $j \neq k$, and $l'_k \geq l'_j$ for every $j \neq k$ with $l_j < l'_j$. Then $\sum_{i=0}^k l'_i \leq \sum_{i=0}^k l_i$ or $\sum_{i=0}^k 1/l'_i \leq \sum_{i=0}^k 1/l_i$.*

Proof: We first prove that if $\sum_{i=0}^k l'_i = \sum_{i=0}^k l_i$ and the tuples are distinct, then $\sum_{i=0}^k 1/l'_i < \sum_{i=0}^k 1/l_i$. We proceed by induction on the size of the set $J := \{j : j \neq k, l_j < l'_j\}$. From the assumption that (l'_i) and (l_i) are distinct and have the same sum (and $l_j \leq l'_j$ for $j \neq k$), it follows that J is nonempty. Now, fix j_0 as some index in J, and let $d := l'_{j_0} - l_{j_0}$. Consider a tuple (l''_i) such that $l''_{j_0} = l'_{j_0} = l_{j_0} + d$, $l''_k = l_k - d$, and $l''_j = l_j$ for each remaining index j. Clearly, $\sum_{i=0}^k l''_i = \sum_{i=0}^k l'_i$, $l''_k \geq l'_k$, and the number of $j \neq k$ such that $l''_j \neq l'_j$ equals $|J| - 1$. We have

$$\sum_{i=0}^{k} \frac{1}{l''_i} = \sum_{i=0}^{k} \frac{1}{l_i} + \left(\frac{1}{l''_{j_0}} - \frac{1}{l_{j_0}}\right) + \left(\frac{1}{l''_k} - \frac{1}{l_k}\right) = \sum_{i=0}^{k} \frac{1}{l_i} - d\left(\frac{1}{l_{j_0}(l_{j_0}+d)} - \frac{1}{l''_k(l''_k+d)}\right),$$

and since

$$l''_k \geq l'_k \geq l'_{j_0} > l_{j_0}$$

by the assumptions and the choice of j_0, it follows immediately that

$$\sum_{i=0}^{k} \frac{1}{l''_i} < \sum_{i=0}^{k} \frac{1}{l_i}. \tag{2.1}$$

When $|J| = 1$ (the base case of the induction), the tuple (l''_i) equals (l'_i) and there is nothing more to prove. Otherwise, we may apply the induction hypothesis to (l''_i) and (l'_i), in this order, obtaining $\sum_{i=0}^k 1/l'_i < \sum_{i=0}^k 1/l''_i$; this together with (2.1) gives the desired conclusion.

Now we prove the lemma. We may suppose that $\sum_{i=0}^k l'_i > \sum_{i=0}^k l_i$. If $l'_k \geq l_k$, then trivially $\sum_{i=0}^k 1/l'_i < \sum_{i=0}^k 1/l_i$. Otherwise, we claim that there exists a tuple (l''_i) such that $l_j \leq l''_j \leq l'_j$ for every $j \neq k$, $l''_k = l'_k$, and $\sum_{i=0}^k l''_i = \sum_{i=0}^k l_i$. Indeed, (l''_i) may be obtained from (l'_i) by replacing some of the values l'_j $(j < k)$ with smaller ones, using the fact that $\sum_{i=0}^{k-1}(l'_i - l_i) > l_k - l'_k$. Note that (l''_i) is distinct from both (l_i) and (l'_i). It follows from the above that $\sum_{i=0}^k 1/l''_i < \sum_{i=0}^k 1/l_i$. Furthermore, since $l''_i \leq l'_i$ for all $i \leq k$ and the tuples are not equal, $\sum_{i=0}^k 1/l'_i < \sum_{i=0}^k 1/l''_i$. The proof is now complete. □

2.1 Reducibility

Let G be a counterexample to Theorem 1.1 with the minimum number of vertices, and subject to this condition, with the minimum number of edges.

In Lemma 2.4 below, we infer several constraints applying to G. Based on these constraints, we derive bounds for the (reduced) face degree of a vertex in G in Lemma 2.5. As per standard terminology, a graph contradicting Lemma 2.4 is said to be *reducible*.

Before stating the lemma, we introduce some terminology. Let v be a vertex and f a face of G. The *face-vertex neighborhood* of v in G, denoted by $N^F(v)$, is defined as $\left(\bigcup_{g \in F(v)} V(g)\right) - \{v\}$. Similarly, the *$f$-reduced face-vertex neighborhood* of v, referred to as $N^F(v,f)$, is the set $\left(\bigcup_{g\in F(v),\, g\neq f} V(g)\right) - \{v\}$. We call the sizes of these sets the *face degree* of v and *f-reduced face degree* of v respectively, writing $d^F(v)$ and $d^F(v,f)$ respectively. As with the other notation, the graph G is included as a subscript if necessary. For instance, we may write $N^F_G(v)$ or $d^F_G(v,f)$.

Lemma 2.4 *The graph G has the following properties:*

(1) $|V(G)| > 97$*;*

(2) *G does not contain parallel edges; in particular, G is without digons;*

(3) *no facial walk of a face of G contains four consecutive* 2*-vertices;*

(4) *for every vertex v of G, it holds* $d^F(v) \geq 97$*;*

(5) *for every two vertices u and v of G such that* $F(u)\cap F(v) = \{f\}$*, it holds* $d^F(u,f)+d^F(v,f) \geq 96$.

Proof: By assumption, G is a 2-connected graph. We prove each of the assertions by contradiction. To see (1), consider an assignment of a different color to each vertex of G.

We proceed to show assertion (2). Let e_1 and e_2 denote two parallel edges in G. We distinguish two cases. If e_1 together with e_2 delimit a digon f, we simply delete one of the two edges, say e_1, obtaining a (2-connected) graph G'. By the minimality of G, the graph G' has a proper spv-coloring c with at most 97 colors. As all faces of G except f have their counterparts in G' with the same sets of boundary vertices, and c is proper, c is also a proper spv-coloring of G.

When, on the other hand, the curve $\mathcal{C}$ comprising the embeddings of e_1 and e_2 is not the boundary of a digon, we produce two graphs G_1 and G_2 by deleting the interior and exterior of $\mathcal{C}$ from G, respectively. Both these graphs are clearly 2-connected (in particular, each has at least three vertices), and smaller than G with respect to the given ordering; thus each has a proper spv-colorings with at most 97 colors. The colorings can be chosen so that they coincide at the common vertices, i.e., the end-vertices of e_1, and the number of colors in their union c is minimal. Then c is a proper spv-coloring of G using at most 97 colors.

Next, suppose $x_1x_2x_3x_4$ is a path contradicting statement (3). Let v_1 be the neighbor of x_1 in G other than x_2, and let v_4 be the neighbor of x_4 other than x_3. The vertices v_1 and v_4 are distinct and different from all x_i, $i = 1,\dots,4$, otherwise the facial walk would contain just four or five vertices, and by the 2-connectedness of G these would be the only vertices of G; a contradiction to assertion (1). We construct a graph G' by contracting the path $x_1\dots x_4v_4$ into a single vertex v_4'. It remains 2-connected due to statement (1), and hence by assumption, G' has a proper spv-coloring c with at most 97 colors. As v_1 and v_4' are adjacent in G', we obtain $c(v_1) \neq c(v_4')$. Now, we use the coloring for the corresponding vertices of G and assign the color $c(v_1)$ to x_2, x_4, and the color $c(v_4')$ to x_1, x_3, v_4. This way the occurrence of the colors $c(v_1)$ and $c(v_4')$ preserves the parity on the corresponding facial walks, and we obtain a proper spv-coloring of G with no more than 97 colors.

Now we focus on assertion (4). Suppose it does not hold. We perform the annihilation of v, obtaining a graph G'. By parts (1) and (2), G has obviously more than 3 vertices and has no parallel edges incident with v, and since G is 2-connected, Lemma 2.2 assures that G' is 2-connected, as well. By the minimality of G, G' has a proper spv-coloring c with at most 97 colors. Using c for G and assigning to v a color not used by c on any vertex in $N^F_G(v)$, but if possible present in c, we obtain a proper coloring of G of cardinality less than or equal to 97.

By Observation 2.1, the only faces of G whose sets of boundary vertices differ from those of their counterparts in G' are the elements of $F_G(v)$, but by the choice of the color of v, the spv-condition is maintained for them. Hence, the coloring of G is also an spv-coloring; a contradiction.

Finally we deal with statement (5). Suppose it does not hold. We construct a graph G'' by annihilating u. As above, G'' is 2-connected, and hence we may annihilate v in G'' to obtain the graph G'. Since u and v have precisely one common incident face in G, they are not adjacent; therefore, the annihilation of u does not create any new edges at v. This means, by part (2), that there is no pair of parallel edges incident with v in G'', and thus, considering statement (1) again, Lemma 2.2 can be applied to the annihilation of v. We conclude that G' is 2-connected. As it is also smaller than G with respect to our order, there is a proper spv-coloring c' of G' using at most 97 colors.

We extend the coloring to G as follows. If there exists a color used by c' on a vertex in $V_G(f) - N_G^F(u,f) - N_G^F(v,f)$ and on no vertex in $N_G^F(u,f) \cup N_G^F(v,f)$, we assign this color to both u and v. Otherwise we color each of u and v with a different color not used by c' on any vertex in $N_G^F(u,f) \cup N_G^F(v,f)$, but if possible appearing in c'. Either case yields a coloring c of G with no more than 97 colors.

For the desired contradiction, it remains to show that c is a proper spv-coloring of G. As the neighbors of u and v belong to $N_G^F(u,f) \cup N_G^F(v,f)$, the coloring is indeed proper. Next, by Observation 2.1 and the assumption about $F(u) \cap F(v)$, we see that each face g of G has its counterpart g' in G' with $V_G(g)$ equal to $V_{G'}(g') \cup \{u,v\}$ if $f = g$, $V_{G'}(g') \cup \{u\}$ if $g \in F_G(u) - \{f\}$, $V_{G'}(g') \cup \{v\}$ if $g \in F_G(v) - \{f\}$, and $V_{G'}(g')$ otherwise. Considering the particular choice of the colors of u and v in either case, it is straightforward that the spv-condition holds for every face of G. □

Let f be a face of the graph G. The number of high-degree vertices (i.e., vertices of degree at least 3) on the boundary of f is called the *weight* of f and is denoted by $w(f)$. The face f is a *pseudodigon* if $w(f) = 2$ and f is not a digon. We say that f is *small* if $w(f) < 20$, and *large* if $w(f) \geq 20$. The *configuration* of a vertex v of G is the tuple obtained by ordering the elements from the multiset $\{w(g) : g \in F(v)\}$ in a nondecreasing manner.

Lemma 2.5 *Let v be a vertex of G of degree $k \geq 2$. Let the faces incident with v be denoted by $f_0, \ldots, f_{k-1}$ in a clockwise order around v. Let $l_i = w(f_i)$ for every such face f_i. The following bounds hold for the face degree of v:*

(1) *if $k \geq 3$, then $d^F(v) \leq 4\sum_{i=0}^{k-1} l_i - 5k - 3\sigma$, where σ is the number of all $i = 0, \ldots, k-1$ with the property that $l_i + l_{i+1} \leq 25$ (indices modulo k);*

(2) *if $k = 2$, then $d^F(v) \leq 4(l_0 + l_1) - 6$;*

(3) *If $k = 3$, then the configuration of v is different from $(2, 8, 19)$ and $(3, 8, 19)$.*

For the f_0-reduced face degree, we have the following:

(4) *if $k \geq 3$, then $d^F(v, f_0) \leq 4\sum_{i=1}^{k-1} l_i - 5k + 9$;*

(5) *if $k = 3$ and $l_1 + l_2 \leq 25$, then $d^F(v, f_0) \leq 4(l_1 + l_2) - 9$.*

Proof: Let f_j be a face of G incident with the vertex v. Traversing the boundary cycle of f_j clockwise from v, let $P_v(f_j)$ be the subpath of the boundary cycle starting with the successor of v and ending with the predecessor of the last high-degree vertex before v. (See Figure 2.2 for an illustration.) Let $p_v(f_j)$

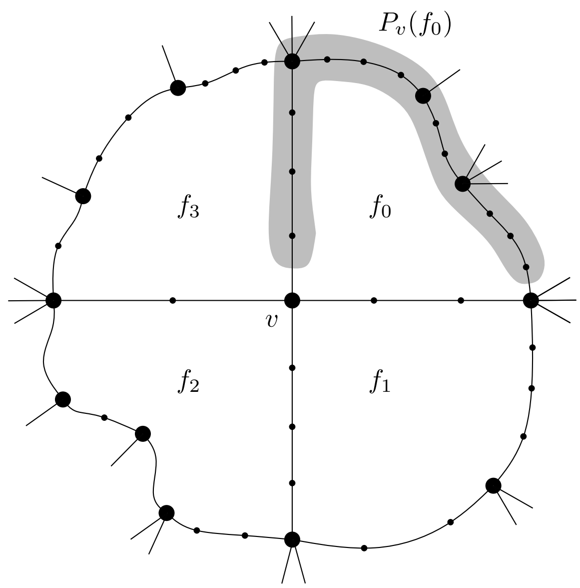


Fig. 2.2. The definition of the path $P_v(f_i)$ for $i = 0$ and a vertex v of degree 4. High-degree vertices are shown as larger dots.

be the number of vertices of $P_v(f_j)$. Since each vertex in $N^F(v)$ is included in at least one of the paths $P_v(f_i)$ ($i = 0, \ldots, k-1$), we have

$$d^F(v) \le \sum_{i=0}^{k-1} p_v(f_i). \tag{2.2}$$

We now bound $p_v(f_i)$. Assume first that $k \ge 3$. Recall that every two consecutive high-degree vertices on the boundary of f_i are separated by at most three vertices of degree 2 (Lemma 2.4 (3)). Since exactly two out of the l_i high-degree vertices are excluded from $P_v(f_i)$, the path $P_v(f_i)$ decomposes into $l_i - 2$ segments, each consisting of at most three vertices of degree 2 (in G) followed by a high-degree vertex, and one final segment consisting of up to three vertices of degree 2 and no high-degree vertex. The final segment may be empty. We find:

$$p_v(f_i) \le 4(l_i - 2) + 3 = 4l_i - 5. \tag{2.3}$$

We also observe an improvement in the following special case. Let us call the path $P_v(f_i)$ *deficient* if it starts or ends with a high-degree vertex (of G), or if it contains two consecutive high-degree vertices. In that case, one of the above defined segments contains zero instead of three vertices of degree 2, and we obtain:

$$p_v(f_i) \le 4l_i - 8 \quad \text{if } P_v(f_i) \text{ is deficient.} \tag{2.4}$$

If $k = 2$, analogous reasoning yields

$$p_v(f_i) \leq \begin{cases} 4l_i - 9 & \text{if } P_v(f_i) \text{ contains three consecutive high-degree vertices,} \\ 4l_i - 7 & \text{if } P_v(f_i) \text{ starts with two high-degree vertices,} \\ 4l_i - 5 & \text{if } P_v(f_i) \text{ ends at a high-degree vertex,} \\ 4l_i - 4 & \text{if } P_v(f_i) \text{ is deficient,} \\ 4l_i - 2 & \text{otherwise.} \end{cases} \tag{2.5}$$

We can now prove part (2) of the lemma. It may be assumed that $P_v(f_0)$ starts with a high-degree vertex, since adjacent 2-vertices have the same face degree. By (2.2) and (2.5),

$$d^F(v) \leq p_v(f_0) + p_v(f_1) \leq (4l_0 - 4) + (4l_1 - 2) = 4(l_0 + l_1) - 6.$$

Next, we derive part (1). From (2.2) and (2.3), it follows that

$$d^F(v) \leq 4\sum_{i=0}^{k-1} l_i - 5k. \tag{2.6}$$

We need to improve this estimate by 3σ, where σ is as defined in the lemma. Let i be such that $l_i + l_{i+1} \leq 25$ (indices modulo k) and let w be the first vertex of $P_v(f_{i+1})$. If the degree of w is 2, then by part (2) of the lemma, $d^F(w) \leq 94$, contradicting Lemma 2.4 (4). Thus, $P_v(f_{i+1})$ starts with a high-degree vertex and we can apply (2.4) in place of (2.3) to bound $p_v(f_{i+1})$. This results in an improvement to the upper bound in (2.6) by 3 for each i satisfying $l_i + l_{i+1} \leq 25$, and hence in an improvement by 3σ in total. Part (1) of the lemma follows.

Let us proceed to part (3). First, note that $\{l_0, l_1, l_2\} \neq \{2, 8, 19\}$ by part (1): indeed, we would have

$$d^F(v) \leq 4 \cdot (l_0 + l_1 + l_2) - 5 \cdot 3 - 3 \cdot 2 = 95,$$

contradicting Lemma 2.4 (4).

Suppose then that $l_0 = 3$, $l_1 = 8$ and $l_2 = 19$. No vertex of degree 2 in G is incident with both f_0 and f_2, since part (2) of the lemma would imply that its face degree is bounded by $4(3 + 19) - 6 = 82$, contradicting Lemma 2.4 (4). Thus, $P_v(f_0)$ starts with a high-degree vertex, and the same can be proved for $P_v(f_1)$ by an identical argument.

We claim that no vertex of degree 2 in G is incident with both f_1 and f_2. Suppose to the contrary that there is such a vertex, and let z be the last 2-vertex encountered on the clockwise boundary cycle of f_1 before v. The fact that $P_v(f_0)$ starts with a high-degree vertex implies that $P_z(f_2)$ ends at a high-degree vertex. Furthermore, since $P_v(f_1)$ starts with a high-degree vertex, $P_z(f_1)$ either starts with two consecutive high-degree vertices (if z is a neighbor of v) or contains three consecutive high-degree vertices. By (2.5),

$$d^F(z) \leq (4l_1 - 7) + (4l_2 - 5) = 96,$$

a contradiction with Lemma 2.4 (4). It follows that besides $P_v(f_0)$ and $P_v(f_1)$, the path $P_v(f_2)$ is also deficient. By (2.4),

$$d^F(v) \leq 4 \cdot (l_0 + l_1 + l_2) - 3 \cdot 8 = 96,$$

which again contradicts Lemma 2.4 (4).

We turn to part (4) of the lemma. Unlike the situation in the proof of part (1), $N^F(v, f_0)$ is not entirely covered by the sets $V(P_v(f_i))$, where $i = 1, \dots, k-1$. On the other hand, only a few vertices of $N^F(v, f_0)$ are missing in the union of these sets, namely the vertices on the boundary of f_{k-1} not contained in $P_v(f_{k-1})$. If we let u denote the vertex that follows $P_v(f_{k-1})$ on the boundary of f_{k-1}, then the missing vertices are u and the (up to three) vertices of degree 2 following u. Consequently,

$$d^F(v, f_0) \leq \sum_{i=1}^{k-1} p_v(f_i) + 4, \tag{2.7}$$

and by (2.3),

$$d^F(v, f_0) \leq 4\sum_{i=1}^{k-1} l_i - 5(k-1) + 4 = 4\sum_{i=1}^{k-1} l_i - 5k + 9.$$

Part (5) follows easily from (2.7) using the estimate (2.3) for the face f_1 and the estimate (2.4) for f_2 (note that $l_1 + l_2 \leq 25$ implies that $P_v(f_2)$ is deficient, just as in the proof of (1)). The proof is thus complete. □

2.2 Discharging

Having explored the properties of the graph G, we are ready to use the discharging method to arrive at a contradiction.

We assign an initial charge to the vertices and faces of G as follows:

- each vertex v receives $d(v) - 6$ units of charge;
- each face f receives $2|V(f)| - 6$ units of charge.

The following observation is a well-known consequence of Euler's formula.

Observation 2.6 *The sum of the charges defined above is* -12.

In the first phase, we redistribute the charges according to Rules 1–2:

Rule 1 *Every face that is not a pseudodigon sends two units of charge to each incident* 2*-vertex. Each pseudodigon does the same, except that one of the respective* 2*-vertices receives no charge.*

Observe that after the application of Rule 1, the charge of each face is nonnegative. In addition, the charge of every large face is at least $2 \cdot 20 - 6 = 34$.

Rule 2 *Every small face distributes its remaining charge evenly to all incident high-degree vertices (i.e., vertices of degree at least* 3*). Each large face (i.e., a face of weight at least* 20*) behaves in the same way, except that it retains a charge of* 4*.*

After applying the above rules the first phase is completed. In the second phase, Rule 3 is applied to the vertices that ended up with negative charge after the first phase.

Rule 3 *If a vertex has a negative charge of c and is incident with a large face f, then it receives the charge of $-c$ from f.*

We will show that the final charge of every vertex and face in G is nonnegative, contradicting Observation 2.6.

Recall from Subsection 2.1 that the *configuration* of a vertex v of G is obtained by ordering the multiset $\{w(g)\colon g \in F(v)\}$ in a nondecreasing way. If we remove (one copy of) the element $w(f)$ from this ordered multiset, we obtain the *f-reduced configuration* of v.

It will be convenient to alter the definition of the weight of a face f and the configuration of a vertex v as follows. The *modified weight* $w'(f)$ of f is defined as 3 if $w(f) = 2$, and $w(f)$ otherwise. Replacing the weight of each face by its modified weight in the definition of the configuration of v, we obtain the *modified configuration* of v. The *modified f-reduced configuration* of v is obtained from the f-reduced configuration of v in an analogous way.

First we analyze how much charge a vertex v of high degree d receives by Rule 2. Denote the faces incident with v by f_i, $i = 0, \dots, d-1$, and let n_i be the number of 2-vertices incident with f_i. After applying Rule 1, each f_i has charge $2|V(f_i)| - 6 - 2n_i$ if f_i is not a pseudodigon, and $2|V(f_i)| - 6 - 2(n_i - 1)$ otherwise. By Lemma 2.4 (2), f_i is not a digon so in both cases the charge can be written as $2w'(f_i) - 6$. Hence, when f_i is a small face, it sends v the charge of

$$\frac{2w'(f_i) - 6}{w(f_i)} = 2 - \frac{6}{w'(f_i)}.$$

(The equality is true as $w(f_i) = w'(f_i)$ if f_i is not a pseudodigon, and $2w'(f_i) - 6 = 0$ otherwise.) On the other hand, when f_i is a large face, it sends v

$$\frac{2w'(f_i) - 6 - 4}{w(f_i)} = 2 - \frac{10}{w'(f_i)}$$

units of charge. Note that in both cases the charge received by v from f_i is nonnegative. In total, the vertex v obtains the nonnegative charge of

$$\begin{aligned} &\sum_{\substack{i \\ w'(f_i)<20}} \left(2 - \frac{6}{w'(f_i)}\right) + \sum_{\substack{i \\ w'(f_i)\geq 20}} \left(2 - \frac{10}{w'(f_i)}\right) \\ &= 2d - 6 \sum_{\substack{i \\ w'(f_i)<20}} \frac{1}{w'(f_i)} - 10 \sum_{\substack{i \\ w'(f_i)\geq 20}} \frac{1}{w'(f_i)}. \end{aligned} \tag{2.8}$$

Next, we establish the following two essential claims. For convenience, we refer to the vertices with a negative charge after the first phase as *special* vertices. Since the initial charge of a vertex v is $d(v) - 6$ and each vertex receives a nonnegative charge during the application of Rule 2, every special vertex has degree at most 5.

Claim 1 *Every special vertex is incident with a large face.*

Proof: We proceed by contradiction, assuming that v is a special vertex not incident with any large face. First suppose that $d(v) = 2$; let f_1 and f_2 be the two faces incident with v. As the initial charge of v is -4, at least one of these faces, say f_1, is a pseudodigon by Rule 1. By assumption, $w(f_2) \leq 19$. Hence $d^F(v) \leq 78$ by Lemma 2.5 (2), a contradiction to Lemma 2.4 (4).

Therefore, let v be a special vertex of high degree d. Summing its initial charge and the charge (2.8) received by Rule 2 gives

$$3d - 6 - 6\sum_i \frac{1}{w'(f_i)} < 0,$$

or equivalently,

$$\sum_i \frac{1}{w'(f_i)} > \frac{d}{2} - 1. \tag{2.9}$$

We proceed by case analysis; let (l'_i) denote the modified configuration of v. (We write (l'_i) instead of (l_i) as a reminder that the configuration is a modified one.) Assume first that v is of degree 3. Then $l'_0 \le 5$, otherwise (2.9) fails since its left hand side is at most $3 \cdot (1/6)$ which equals its right hand side $1/2$. Since v is not incident with any large face, we have $l'_1, l'_2 \le 19$. We aim to use Lemma 2.5 to bound $d^F(v)$. Although it is formulated for ordinary (non-modified) configurations, the monotonicity of the upper bounds ensures that the lemma remains valid if the configuration is a modified one. By Lemma 2.5 (1), $d^F(v) \le 4\sum_i l'_i - 21$. Consequently, Lemma 2.4 (4) implies that

$$\sum_i l'_i \ge 30. \tag{2.10}$$

If $l'_0 = 3$ and $l'_1 \le 9$, then by (2.10) (l'_i) is one of the three tuples $(3, 8, 19)$, $(3, 9, 18)$, and $(3, 9, 19)$. The first of these is excluded by Lemma 2.5 (3), and the remaining two contradict (2.9). If $l'_0 = 3$ and $l'_1 > 9$, then by Lemma 2.3 applied to the tuples $(3, 10, 15)$ and (l'_i), in this order, either $\sum_i l'_i \le 28$ or $\sum_i 1/l'_i \le 1/2$. However, that contradicts (2.10) or (2.9), respectively.

Hence $l'_0 \ge 4$. If l'_0, $l'_1 = 4$, then $\sum_i l'_i \le 27$, which is impossible by (2.10). Otherwise we may use Lemma 2.3 for the tuples $(4, 5, 20)$ and (l'_i), and obtain a contradiction to (2.10) or (2.9) again.

Thus $d \ge 4$; as we have remarked above, $d \le 5$. Lemmas 2.5 (1) and 2.4 (4) imply (2.10) again. In particular, (l'_i) cannot be of the form $(3, 3, 3, x)$. If $d = 4$, then by Lemma 2.3 applied to the tuple $(3, 3, 4, 12)$, $\sum_i l'_i \le 22$ or $\sum_i 1/l'_i \le 1$, contradicting (2.10) or (2.9). For $d = 5$, we obtain a similar contradiction by applying Lemma 2.3 to $(3, 3, 3, 3, 6)$, which yields $\sum_i l'_i \le 18$ or $\sum_i 1/l'_i \le 3/2$. $\square$

Claim 2 *Every large face has a nonnegative final charge.*

Proof:

Let f be an arbitrary large face of G. We start by listing the possible f-reduced or modified f-reduced configurations of special vertices incident with f, and for each case we note a lower bound on the charge of these vertices. Take such a special vertex v.

If v is a 2-vertex, then by Rule 1 its f-reduced configuration is (2) and the charge equals -2.

Now suppose that v is of high degree d. Let (l'_i), $i = 1, \ldots, d-1$, be its modified f-reduced configuration, and let d' denote the number of large faces incident with v. As noted earlier, $d \le 5$. By considering the initial charge of v and (2.8), we see that after applying Rule 2, v has charge

$$3d - 6 - 6\sum_{\substack{i \\ w'(f_i)<20}} \frac{1}{w'(f_i)} - 10\sum_{\substack{i \\ w'(f_i)\ge 20}} \frac{1}{w'(f_i)}.$$

(modified) f-reduced configuration	*charge*
(2)	-2
$(3,x)$, $x \leq 11$	$\geq 1/2 - 6/x \geq -3/2$
$(4,x)$, $x \leq 5$	$\geq 1 - 6/x \geq -1/2$
$(3,3,3)$	$\geq -1/2$

Tab. 1. The proof of Claim 2: the list of possible f-reduced (the first line) or modified f-reduced (the other lines) configurations of special vertices incident with the face f, together with the charge of these vertices.

As the charge is negative by assumption,

$$3d - 6 - 6 \sum_{\substack{i \\ w'(f_i)<20}} \frac{1}{w'(f_i)} - \frac{d'}{2} < 0 \tag{2.11}$$

by the definition of large face. Furthermore, $w'(f_i) \geq 3$ always, and hence

$$3d - 6 - 2(d - d') - d'/2 < 0.$$

Since the left hand side equals $d - 6 + 3d'/2$, we deduce that $d' = 1$ (i.e., f is the only large face incident with v), and that $d \leq 4$.

Assume first that $d = 3$. Then (2.11) reduces to

$$\frac{5}{2} - 6\left(\frac{1}{l_1'} + \frac{1}{l_2'}\right) < 0.$$

We infer that either $l_1' = 3$ and $l_2' \leq 11$, or $l_1' = 4$ and $l_2' \leq 5$. The charge of v is at least $1/2 - 6/l_2'$ in the former, and at least $1 - 6/l_2'$ in the latter case.

Now let $d = 4$. By (2.11),

$$\frac{11}{2} - 6\sum_i \frac{1}{l_i'} < 0.$$

If some l_i' were greater than or equal to 4, this inequality would not hold. Hence $(l_i') = (3,3,3)$ and the charge of v is at least $-1/2$. We summarize the results in Table 1.

Let S denote the set of all special vertices incident with f and R their total charge after the completion of the first phase. We observe the following:

Any two vertices $u, v \in S$ have at least two common incident faces. (2.12)

Suppose the contrary. In view of the possible f-reduced or modified f-reduced configurations of u and v listed in Table 1, Lemma 2.5 (4) and (5) implies that both $d^F(u,f)$ and $d^F(v,f)$ are at most 47. On the other hand, Lemma 2.4 (5) and the assumption that f is the only face incident with both u and v imply that $d^F(u,f) + d^F(v,f) > 95$, a contradiction.

We proceed to prove Claim 2 by contradiction. Suppose that f has a negative charge after the application of Rule 3. Since the charge of f after the first phase is 4 units (by Rule 2 for large faces), this is equivalent to the condition

$$R < -4. \tag{2.13}$$

Considering the lower bounds for charges in Table 1, we see that there are at least three special vertices.

Let $v \in S$ be a 2-vertex; the other face f' incident with v is a pseudodigon. By Rule 1 and (2.12), every other vertex v' in S is one of the two high-degree vertices v_1, v_2 incident with f'. Therefore $S = \{v, v_1, v_2\}$; it follows that v_1 and v_2 are both of degree 3 by assumption (2.13). This means that $F(v_1) = F(v_2)$, and hence the configuration of both vertices is the same. Considering (2.13) again, the modified f-reduced configuration of both v_1 and v_2 is $(3, 3)$.

At this point, we digress by making an auxiliary observation:

Let g be a face of G with $w(g) \leq 3$. The intersection of the boundaries of f and g consists of pairwise disjoint paths, of which at most one is nontrivial; the internal vertices of all these paths are of degree 2 in G. Furthermore, any two vertices of degree 3 incident with both f and g are precisely the end-vertices of such a nontrivial path, and hence, there are at most two such vertices. (2.14)

To prove this, let us denote the intersection of the boundaries of f and g by H. The assertion about the structure of H is obvious by considering the 2-connectedness of G and the weights of both f and g. Let $\mathcal{P}$ denote the set of the respective paths. If v is a vertex of degree 3 incident with f and g, then H must contain an edge incident with v, i.e., v lies on—and hence is an end-vertex of—a nontrivial path in $\mathcal{P}$. The second assertion easily follows.

Applying (2.14) to the face incident with v_1 different from f and f', and recalling that f' is a pseudodigon, we infer that $w(f) = 2$; a contradiction.

Thus all vertices in S are of high degree. Suppose that $S = \{v_1, v_2, v_3\}$. Then by assumption (2.13), the modified f-reduced configuration of v_1, v_2, and v_3 is $(3, 3)$. Hence by (2.14), no face other than f is incident with the three vertices. By (2.12), G contains three different faces f_{ij}, $1 \leq i < j \leq 3$, incident with both v_i and v_j and different from f. By (2.14), the boundary of f is precisely $\bigcup \mathcal{P}$; thus $w(f) = 3$, a contradiction to the assumption that f is large.

Therefore $|S| \geq 4$. We claim that G contains a face $f' \neq f$ incident with all the vertices in S. To prove this, consider three vertices u, x, v in S consecutively encountered on a facial walk of f. By (2.12), there is a curve $\mathcal{C}_{uv}$ connecting u and v through a face $f_{uv} \neq f$ of G. Consider any vertex $y \in S - \{u, x, v\}$. There is a curve $\mathcal{C}_{xy}$ connecting x and y through a face $f_{xy} \neq f$. If we let $\mathcal{C}_{vu}$ be a curve connecting v to u through f, then $\mathcal{C}_{uv} \cup \mathcal{C}_{vu}$ is a closed curve separating x from y. Consequently, $\mathcal{C}_{uv}$ intersects $\mathcal{C}_{xy}$, and therefore y is incident with the face f_{uv}. The assertion easily follows.

Now let $k := |S|$. Then $w(f') \geq k$, and consequently the f-reduced configuration of each v in S contains a number greater than or equal to k. From Table 1, we see that

$$R \geq k\left(\frac{1}{2} - \frac{6}{k}\right) = \frac{k}{2} - 6,$$

the right side of which is at least -4 by the condition on k. This contradicts assumption (2.13). □

With the help of the two preceding claims, we can easily finish the proof. By Claim 1 and Rule 3, every special vertex—and hence every vertex—ends up with a nonnegative charge. The final charge of every face is nonnegative as well; Rule 2 and Claim 2 guarantee this for small and large faces respectively. However, as already mentioned, this contradicts Observation 2.6.

3 Lower bound

In this section, we provide examples showing that the best possible constant bound on χ_s for the class of 2-connected plane simple graphs is at least 8, and the corresponding bound for proper spv-colorings is at least 10. Note that for the class of 2-connected plane (multi)graphs, the latter example implies a lower bound of 10 for general spv-colorings. Indeed, if we replace each of its edges by a digon bounding a face, then every spv-coloring of the resulting graph is necessarily a proper spv-coloring of the original simple graph.

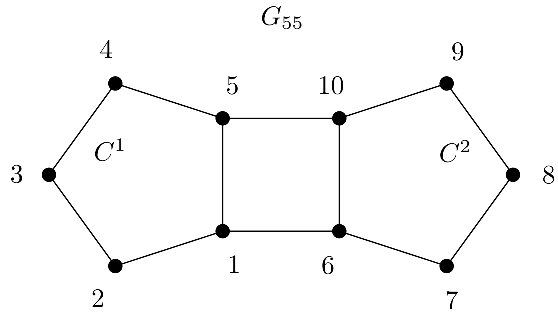
First, we focus on the bound for proper spv-colorings. We construct a graph G_{55} on ten vertices by linking two disjoint cycles C^1, C^2 on five vertices with two additional edges whose endvertices in each cycle are adjacent. See Figure 3.1 (a).

By the spv-conditions for the two faces of G_{55} of length 5, every proper spv-coloring c must assign each vertex of C^1 a different color; the same holds for C^2. The spv-condition for the face of G_{55} of length 10 then implies that c uses each color precisely once.

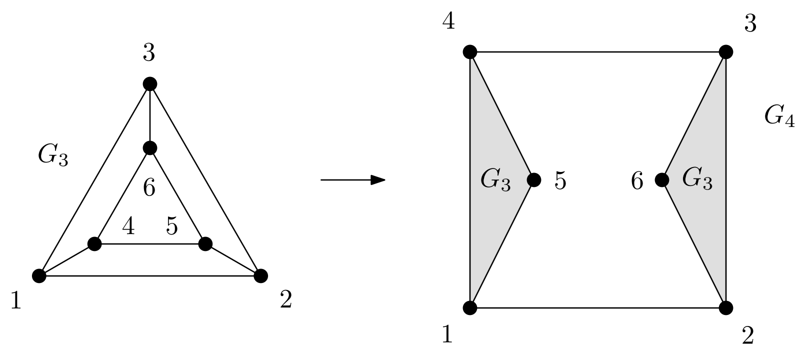
Second, we consider the bound on χ_s (where the coloring is allowed to be improper). Take a three-sided prism G_3 embedded in the plane so that one of its triangular faces is the outer face f_3. As observed by Czap and Jendroľ [3, proof of Lemma 5.1], every coloring c of G_3 such that every face of G_3 distinct from f_3 satisfies the spv-condition colors each boundary vertex of f_3 with a different color.

Now construct a graph G_4 from a cycle on four vertices by replacing every other edge with a copy of G_3 in such a way that the outer face f_4 of G_4 is of length 4; see Figure 3.1 (b). Let c' be a coloring of G_4 satisfying the spv-condition for each face of G_4 other than f_4. When restricted to the vertices of any of the copies of G_3, c' has the property of the coloring c discussed above. This and the spv-condition for the face of G_4 of length 6 imply that c' assigns a different color to each boundary vertex of f_4.

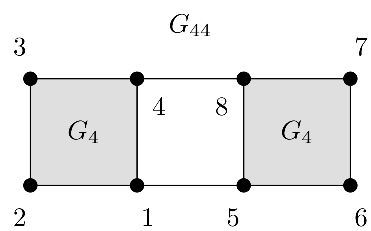
Finally, we reproduce the construction of G_{55} with copies of G_4 in place of the cycles on five vertices. Thereby we obtain a graph G_{44} with the outer face f_{44} of length 8, such that every spv-coloring of G_{44} is injective on $V(f_{44})$. The graph G_{44} is shown in Figure 3.1 (c). □



(a) The graph G_{55}.



(b) The graphs G_3 (left) and G_4 (right).



(c) The graph G_{44}.

Fig. 3.1. Illustrations for Section 3. The labeled gray areas represent the respective subgraphs not depicted in detail. For each graph, the relevant coloring is unique up to symmetry; it is indicated by numerical labels.

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