Discrete Mathematics and Theoretical Computer Science DMTCS vol. **15:2**, 2013, 33–48

On the enumeration of d-minimal permutations

Mathilde Bouvel[1†] and Luca Ferrari[2‡]

[1] *LaBRI UMR 5800, Université de Bordeaux and CNRS, Bordeaux, France*
[2] *Dipartimento di Matematica e Informatica "U. Dini", Università degli Studi di Firenze, Firenze, Italy*

received 23rd May 2011, revised 27th February 2013, 5th March 2013, accepted 21st March 2013.

We suggest an approach for the enumeration of minimal permutations having d descents which uses skew Young tableaux. We succeed in finding a general expression for the number of such permutations in terms of (several) sums of determinants. We then generalize the class of skew Young tableaux under consideration; this allows in particular to recover a formula for Eulerian numbers which is a direct consequence of a result of MacMahon.

Keywords: permutation, descent, pattern containment order, duplication-loss model, skew Young tableau

1 Introduction

This article deals with minimal permutations with d descents (also called d-minimal permutations here). This family of permutations has been introduced in [BoRo] in the study of the whole genome duplication-random loss model of genome rearrangement. In this context, genomes are represented by permutations, and minimal permutations with $d = 2^p$ descents are the basis of excluded patterns that describes the class of permutations that can be obtained from the identity with cost at most p.

In order to describe properties of this class of permutations, its basis has been studied, and the first natural question to address is to count how many excluded patterns it contains. In [BP] some partial results on the enumeration of minimal permutations with d descents have been obtained: namely, minimal permutations with d descents and of size n have been enumerated by closed formulas, for $n = d+1, d+2$ and $2d$ ($d+1$ and $2d$ being lower and upper bounds for the size of a minimal permutation with d descents – see [BP]). In [MY], further results on the enumeration of minimal permutations with d descents have been obtained using multivariate generating functions, allowing in particular to derive a closed formula enumerating those of size $2d-1$ as well as some asymptotic results.

In this work we offer an alternative approach for the enumeration of minimal permutations with d descents, making extensive use of a bijection between these permutations and a family of skew Young

†Email: `bouvel@labri.fr`. Mathilde Bouvel conveys special acknowledgements to the *Dipartimento di Sistemi e Informatica* for the kind hospitality during her visit in May 2008 in which this research started.
‡Email: `luca.ferrari@unifi.it`

tableaux. This gives a general formula for the number $p_{d+k,d}$ of minimal permutations with d descents and of size $d+k$, as a sum of determinants of matrices (Theorem 4.2). Its specialization for $k=1,2$ and d allows to recover the expressions of $p_{d+1,d}$, $p_{d+2,d}$ and $p_{2d,d}$ given in [BP]. When specializing it for $k=3$, we also obtain a closed formula for $p_{d+3,d}$ (Theorem 4.4). Moreover, an analysis of the general form of the formula for $p_{d+k,d}$ allows us to find an explicit expression also in the case $k=4$. Finally, the family of skew Young tableaux under consideration has a natural generalization which is investigated in Section 5.

In closing this introduction, we remark that the interest in d-minimal permutations is very lively nowadays, as witnessed by the contemporary research paper [CGM], in which the authors consider the problem of enumerating d-minimal permutations in connection with skew Young tableaux. We wish to point out that our work partially overlaps with [CGM], where the authors independently prove one of the main theorem of our work.

2 Preliminary definitions and results

For any integer n, S_n denotes the set of permutations of $[n]=\{1,2,\ldots,n\}$. A permutation $\sigma\in S_n$ will be represented either by the word $\sigma(1)\ldots\sigma(n)$ or by the $n\times n$ grid, where a cell contains a dot if and only if it is at coordinates $(i,\sigma(i))$ for some $i\in[n]$.

The *pattern containment order* on permutations [P] is defined as follows. A permutation $\pi\in S_k$ is contained in (or is a pattern of) $\sigma\in S_n$ when there exist integers $1\le i_1<\ldots<i_k\le n$ such that π and $\sigma(i_1)\ldots\sigma(i_k)$ are order-isomorphic sequences, *i.e.* they are such that $\pi(\ell)<\pi(m)\Leftrightarrow\sigma(i_\ell)<\sigma(i_m)$ for all $\ell,m\in[k]$.

A *descent* in a permutation $\sigma\in S_n$ is an integer $i\in[n-1]$ such that $\sigma(i)>\sigma(i+1)$. Similarly, an *ascent* is an integer $i\in[n-1]$ such that $\sigma(i)<\sigma(i+1)$.

A *minimal permutation with d descents*, or *d-minimal permutation*, of length n is a permutation of S_n that is minimal in the sense of the pattern-containment relation for the property of having d descents. In other words, it is a permutation with d descents such that, when removing any of its entries and suitably renaming the remaining elements, the resulting permutation of S_{n-1} has $d-1$ descents. For instance (see Figure 1), the permutation $\sigma = 14\,12\,9\,3\,13\,5\,15\,10\,6\,2\,1\,11\,8\,7\,4$ is minimal with 11 descents, as it has exactly 11 descents and every permutation it contains as a pattern has at most 10 descents.

In [BP], minimal permutations with d descents have been characterized as follows:

Theorem 2.1 *A permutation σ is minimal with d descents if and only if it has exactly d descents and its ascents i satisfy the "diamond property",* i.e. *are such that $2\le i\le n-2$ and $\sigma(i-1)\sigma(i)\sigma(i+1)\sigma(i+2)$ forms an occurrence of either the pattern 2143 or the pattern 3142.*

As explained in [BP], this characterization allows to represent d-minimal permutations by means of certain labelled posets. These posets, labelled with the integers from 1 to n, are made of chains somehow linked by diamond-shaped structures (corresponding to the ascents of the permutation). Figure 1 shows an example of this one-to-one correspondence. Notice that each of these labelled posets represents a unique d-minimal permutation, whereas the underlying unlabelled poset can be seen as representing a set of d-minimal permutations (those that are in correspondence with a legal labelling of the poset).

Posets and labelled posets in these families are in one-to-one correspondence with skew Ferrers diagrams and skew Young tableaux having special properties. These combinatorial objects have been widely studied in the literature, in particular from an enumerative point of view (see, for instance, the recent

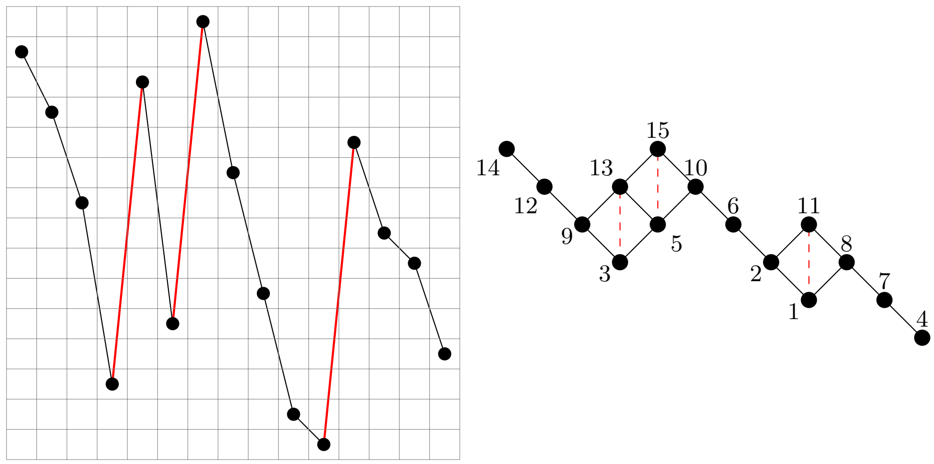


Fig. 1: The 11-minimal permutation $\sigma = 14\,12\,9\,3\,13\,5\,15\,10\,6\,2\,1\,11\,8\,7\,4$, and the corresponding poset.

paper [BaRo]). In the following sections, we explicitly describe the correspondence between unlabelled (resp. labelled) posets and skew Ferrers diagrams (resp. skew Young tableaux), as well as some enumerative results on these objects, and how they can be used for our purposes.

3 Connection with skew Young tableaux

In order to explain how the poset representation of d-minimal permutations described in the previous section can be conveniently interpreted by using skew Young tableaux, we first need to recall some definitions.

A *partition* of an integer n is a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_i \geq \lambda_{i+1}$ for $1 \leq i \leq k-1$ and $\sum_{i=1}^{k} \lambda_i = n$. The integer n is called the *size* of the partition, and we write $n = |\lambda|$. The number of parts k will be denoted by $k = \ell(\lambda)$ (this is also called the *length* of λ).

A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ can be represented by its *Ferrers diagram*, which is obtained by drawing k rows of contiguous unit cells, from top to bottom, such that row i contains λ_i cells, and with the first cells of these k rows vertically aligned. An example is shown in Figure 2. We will also denote by λ the Ferrers diagram associated with the partition λ. The size $|\lambda|$ obviously corresponds to the number of cells of the Ferrers diagram, and the number of rows is given by $\ell(\lambda)$.

For our purposes, a *Young tableau* is a filling of a Ferrers diagram λ using distinct positive integers from 1 to $n = |\lambda|$, with the property that the values are (strictly) decreasing along each row and each column of the Ferrers shape. This constitutes a slight deviation from the classical definition, which requires the word "increasing" instead of the word "decreasing". However, it is clear that all the properties and results on (classical) Young tableaux can be translated into our setting by simply replacing the total order "$\leq$" by the total order "$\geq$" on $\mathbf{N}$. In Figure 2, a Young tableau of shape $\lambda = (8, 6, 3, 3, 2, 1)$ is shown. As for Ferrers diagrams, the size of a Young tableau is given by the number of its cells.

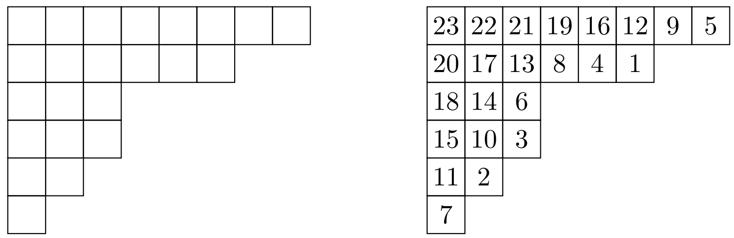


Fig. 2: The Ferrers diagram associated with the partition $\lambda = (8, 6, 3, 3, 2, 1)$, and a Young tableau on this shape.

The main definition we need in our work is that of a *skew Young tableau*. The definition can be given exactly as for a Young tableau, with the only difference that the underlying shape consists of a Ferrers diagram λ with a Ferrers diagram μ removed (starting from the top-left corner). Such a skew shape is usually denoted λ/μ. We refer the reader to [St] for the formal definition and some important facts concerning the enumeration of skew Young tableaux. In Figure 3 a skew Young tableau of skew shape $(8, 6, 3, 3, 2, 1)/(3, 2, 2, 1)$ is depicted. As before, the size of a skew Young tableau denotes its number of cells.

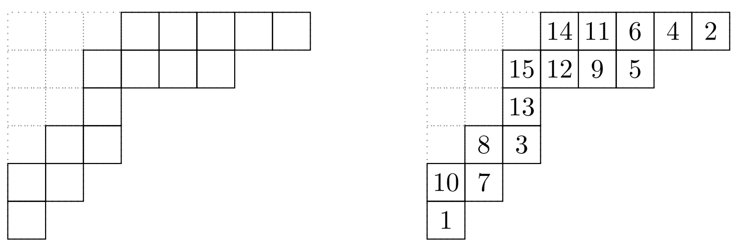


Fig. 3: The skew shape $(8, 6, 3, 3, 2, 1)/(3, 2, 2, 1)$, and a skew Young tableau on this shape.

As announced at the beginning of the present section, we can translate the poset representation of a d-minimal permutation into a suitable skew Young tableau.

Proposition 3.1 *The set of d-minimal permutations of length $d + k$ is in bijection with the set of skew Young tableaux whose skew shapes λ/μ satisfy $|\lambda/\mu| = d + k$ and have k rows such that two consecutive rows have precisely two columns in common.*

Proof: A d-minimal permutation of length $d+k$ consists of k descending runs and, denoting with a, b, c, d four consecutive elements such that a, b and c, d belong to different descending runs, then necessarily $a > b$, $c > d$, $a < c$ and $b < d$ (see Theorem 2.1). Then, starting from a d-minimal permutation π of length $d + k$, one can construct a skew Young tableau as follows: starting from the bottom, the i-th row of the tableau consists of the elements of the i-th descending run of π; moreover two consecutive rows are required to have exactly two columns in common. The resulting tableau is a skew Young due to the above recalled diamond property of d-minimal permutations. $\square$

In Figure 4 the skew Young tableau determined by the permutation whose poset representation is given in Figure 1 is shown.

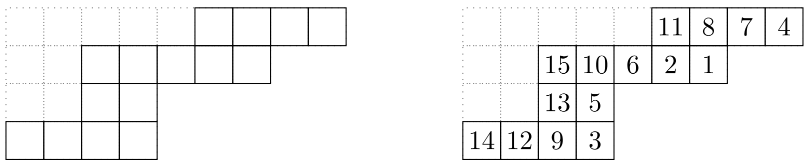
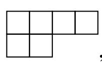
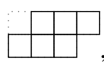
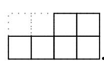


Fig. 4: The skew shape corresponding to the underlying unlabelled poset of Figure 1, and the skew Young tableau corresponding to the permutation σ of Figure 1.

Example. For $k = 2$ and $d = 4$, the set of 4-minimal permutations of length 6 is in bijection with the set consisting of all skew Young tableaux of one of the following skew shapes:

- $(4, 2)/\emptyset$, *i.e.* ,
- $(4, 3)/(1)$, *i.e.* ,
- $(4, 4)/(2)$, *i.e.* .

There are 9, 14 and 9 skew Young tableaux of these shapes, respectively, giving a total of 32 4-minimal permutations of length 6.

Remarks. Let π be a d-minimal permutation and suppose it has k descending runs. Obviously this means that π has $k - 1$ ascents and that $|\pi| = d + k$. Moreover, suppose that λ/μ is the skew shape associated with π. Recall that $\ell(\lambda)$ and $\ell(\mu)$ denote the number of rows of λ and μ respectively. Finally, let ℓ_i be the length of the i-th descending run of π. Some straightforward consequences of the above bijection are the following:

1. The skew shape associated with π is connected.
2. $\ell(\lambda) = k$.
3. Set $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_k)$. Then $\lambda_i = \sum_{j=1}^{k-i+1} \ell_j - 2(k - i)$.
4. $\ell(\mu) < \ell(\lambda)$ and, more precisely, $\ell(\mu) = (\ell(\lambda) - 1) - \#(\text{starting descending runs of length 2 in } \pi)$.
5. Set $\mu = (\mu_1, \mu_2, \ldots \mu_k)$. Then $\mu_i = \lambda_{i+1} - 2$.

4 Some enumerative results

The main goal of the present section is to enumerate d-minimal permutations of length n, with $d+1 \leq n \leq 2d$. A general result in this direction can be obtained by considering the above described bijection with skew Young tableaux. In particular, a classical result due to Aitken is our starting point.

Theorem 4.1 ([A]) *Let λ/μ be a skew shape, with $|\lambda/\mu| = N$ and $\ell(\lambda) = n$. Then, the number $f^{\lambda/\mu}$ of skew Young tableaux of shape λ/μ is*

$$f^{\lambda/\mu} = N!\,\det\left(\frac{1}{(\lambda_i - \mu_j - i + j)!}\right)_{i,j=1..n}. \tag{1}$$

Formula (1) can be deduced from the well known Jacobi-Trudi identity, as shown, for instance, in [St2]. Moreover, in such a formula the entry (i, j) of the considered matrix is intended to be 0 if the expression $\lambda_i - \mu_j - i + j$ is negative.

The main result of this section is essentially a corollary of Theorem 4.1, in the case in which the skew shape λ/μ has the properties of Proposition 3.1.

Theorem 4.2 *Denote by $p_{d+k,d}$ the number of d-minimal permutations of length $d+k$ (so that $1 \leq k \leq d$). Then*

$$p_{d+k,d} = \sum_{\substack{a_1,a_2,\dots,a_k \geq 2 \\ a_1+a_2+\cdots+a_k = d+k}} (d+k)! \cdot \det(A(a_1,\dots,a_k)),$$

where $A(a_1,\dots,a_k)$ is the following matrix:

$$\begin{pmatrix}
\frac{1}{a_1!} & \frac{1}{(a_1+a_2-1)!} & \frac{1}{(a_1+a_2+a_3-2)!} & \cdots & \cdots & \cdots & \frac{1}{(a_1+\cdots+a_k-k+1)!} \\
1 & \frac{1}{a_2!} & \frac{1}{(a_2+a_3-1)!} & \cdots & \cdots & \cdots & \frac{1}{(a_2+\cdots+a_k-k+2)!} \\
\chi_{a_2=2} & 1 & \frac{1}{a_3!} & \ddots & & & \vdots \\
0 & \chi_{a_3=2} & 1 & \ddots & & & \vdots \\
0 & 0 & \chi_{a_4=2} & \ddots & & & \vdots \\
\vdots & \ddots & 0 & \ddots & & & \vdots \\
\vdots & & & \ddots & & & \vdots \\
\vdots & & & & & & \frac{1}{(a_{k-2}+a_{k-1}+a_k-2)!} \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \frac{1}{(a_{k-1}+a_k-1)!} \\
0 & \cdots & \cdots & 0 & \chi_{a_{k-1}=2} & 1 & \frac{1}{a_k!}
\end{pmatrix}.$$

Here χ_P denotes the characteristic function of the property P (i.e., $\chi_P = 1$ when P is true and $\chi_P = 0$ otherwise). In other words, $A(a_1,\dots,a_k)$ is the $k \times k$ matrix whose entries $a_{i,j}$ obey the following *equalities:*

$$\begin{aligned} a_{i,j} &= \frac{1}{(a_i + \cdots + a_j + i - j)!}, \qquad \text{when } i \leq j, \\ a_{i,i-1} &= 1, \\ a_{i,i-2} &= \chi_{a_{i-1}=2}, \\ a_{i,j} &= 0, \qquad \text{when } i > j+2. \end{aligned}$$

Proof: Theorem 4.1 ensures that

$$p_{d+k,d} = \sum_{\lambda/\mu} (d+k)! \det\left(\frac{1}{(\lambda_i - \mu_j - i + j)!}\right)$$

where the sum is over all skew shapes λ/μ of size $d+k$ having k rows and such that two consecutive rows have exactly two columns in common.

For such a skew shape λ/μ, let us define the sequence $a = (a_1, a_2, \ldots, a_k)$ by $a_i = \lambda_i - \mu_i$, $1 \leq i \leq k$. The sequence a is such that $a_1, a_2, \ldots, a_k \geq 2$ and $a_1 + a_2 + \cdots + a_k = d + k$. From the remark at the end of the previous section, we additionally have that $\mu_k = 0$ (point 4.) and that $\mu_i = \lambda_{i+1} - 2$ for all $1 \leq i \leq k-1$ (point 5.). It is now a trivial matter to check that the sequence $a = (a_1, a_2, \ldots, a_k)$ completely and uniquely determines λ/μ. Hence, the sum in the above formula can be taken over sequences $a = (a_1, a_2, \ldots, a_k)$ such that $a_1, a_2, \ldots, a_k \geq 2$ and $a_1 + a_2 + \cdots + a_k = d + k$.

In what follows, we give expressions for the entries $a_{i,j} = \frac{1}{(\lambda_i - \mu_j - i + j)!}$ in terms of $(a_1, a_2, \ldots, a_k)$.

If $i \leq j$, then, by Theorem 4.1, $a_{i,j} = \frac{1}{(\lambda_i - \mu_j - i + j)!}$. Due to the remarks stated at the end of the previous section, we have:

$$a_i + a_{i+1} + \ldots + a_j = (\lambda_i - \mu_i) + (\lambda_{i+1} - \mu_{i+1}) + \ldots (\lambda_j - \mu_j) = \lambda_i - \mu_j + 2(j - i).$$

This yields for the denominator of the above fraction the following expression (leaving aside the factorial):

$$\lambda_i - \mu_j - i + j = a_i + a_{i+1} + \ldots + a_j + i - j,$$

as desired.

If $j = i - 1$, then we have immediately:

$$a_{i,i-1} = \frac{1}{(\lambda_i - \mu_{i-1} - 1)!} = \frac{1}{(2-1)!} = 1.$$

Concerning the case $j = i - 2$, since $\mu_{i-1} = \lambda_i - 2$ and $\mu_{i-2} \geq \mu_{i-1}$, we observe that $\lambda_i - \mu_{i-2} \leq 2$, and that the equality holds precisely when $\mu_{i-1} = \mu_{i-2}$, *i.e.* when $a_i = \lambda_{i-1} - \mu_{i-1} = 2$. Thus we get:

$$a_{i,i-2} = \frac{1}{(\lambda_i - \mu_{i-2} - i + i - 2)!} = \chi_{a_{i-1}=2}.$$

Finally, if $i > j + 2$, then the denominator of $a_{i,j}$ is easily seen to be negative, hence $a_{i,j} = 0$. □

From a theoretical point of view, Theorem 4.2 completely solves the problem of the enumeration of d-minimal permutations with respect to their length, giving a formula for $p_{d+k,d}$. Unfortunately, it is clear that such a formula is very difficult to use in concrete cases, due to its intrinsic complexity. However, using our result we are able to recover some known cases (namely $k = 1, 2, d$), as well as to get a formula for the case $k = 3$ (that is, d-minimal permutations of length $d + 3$), which was first discovered in [MY] with different methods in terms of generating functions. Moreover, essentially by inspection of the formula found in Theorem 4.2, we also describe the general form of a formula for $p_{d+k,d}$ and specialize it to a closed form when $k = 4$.

We start by collecting in a single theorem the known cases $k = 1, 2, d$, showing how they can be derived from Theorem 4.2.

Theorem 4.3 *([BP]) The following equalities hold:*

$$\begin{aligned} p_{d+1,d} &= 1, \\ p_{d+2,d} &= 2^{d+2} - (d+1)(d+2) - 2, \\ p_{2d,d} &= C_d, \end{aligned}$$

where we denote by $C_d = \frac{1}{d+1}\binom{2d}{d}$ the d-th Catalan number.

Proof: When $k = 1$, the formula of Theorem 4.2 becomes completely trivial:

$$p_{d+1,d} = (d+1)! \cdot \left| \frac{1}{(d+1)!} \right| = 1.$$

In the case $k = 2$, we have a single sum where a 2×2 determinant appears:

$$\begin{aligned} p_{d+2,d} &= \sum_{\substack{a_1,a_2 \geq 2 \\ a_1+a_2=d+2}} (d+2)! \cdot \begin{vmatrix} \frac{1}{a_1!} & \frac{1}{(d+1)!} \\ 1 & \frac{1}{a_2!} \end{vmatrix} = \sum_{a=2}^{d} \left(\binom{d+2}{a} - (d+2) \right) \\ &= 2^{d+2} - 2(d+3) - (d-1)(d+2) = 2^{d+2} - 2 - (d+1)(d+2). \end{aligned}$$

If $k = d$, the formula of Theorem 4.2 gives a determinantal expression of $p_{2d,d}$:

$$p_{2d,d} = (2d)! \cdot \begin{vmatrix} \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} & \frac{1}{6!} & \cdots & \frac{1}{(d+1)!} \\ 1 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} & \cdots & \frac{1}{d!} \\ 1 & 1 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \cdots & \frac{1}{(d-1)!} \\ 0 & 1 & 1 & \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(d-2)!} \\ 0 & 0 & 1 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(d-3)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{2!} \end{vmatrix}.$$

From Theorem 4.1, it can be seen that this determinant counts Young tableaux of rectangular shape having d rows and 2 columns, which are counted by Catalan numbers (see [St2]). $\square$

Notice that there are several ways to recover the equality $p_{2d,d} = C_d$. Indeed, the fact that $p_{2d,d}$ counts Young tableaux of rectangular shape having d rows and 2 columns is also clear from Proposition 3.1, and another combinatorial proof of $p_{2d,d} = C_d$ is given in [BP]. Also, as pointed out to the authors by Krattenthaler, the equality between the determinant in the proof of Theorem 4.3 and the d-th Catalan number can be obtained by letting q tend to 1, A tend to ∞ and $L_i = 2 - i$ in formula (3.12) of [K].

We next proceed with the evaluation of $p_{d+3,d}$.

Theorem 4.4 *The following equality holds:*

$$p_{d+3,d} = 3^{d+3} - (d^2+4d+7)\cdot 2^{d+2} + \frac{1}{2}d^4 + \frac{5}{2}d^3 + 5d^2 + 6d + 1.$$

Proof: Applying Theorem 4.2 in the case $k = 3$, we obtain:

$$p_{d+3,d} = \sum_{\substack{a,b,c\geq 2\\ a+b+c=d+3}} (d+3)!\cdot \begin{vmatrix} \frac{1}{a!} & \frac{1}{(a+b-1)!} & \frac{1}{(d+1)!} \\ 1 & \frac{1}{b!} & \frac{1}{(b+c-1)!} \\ \chi_{b=2} & 1 & \frac{1}{c!} \end{vmatrix}.$$

Expanding the determinant and grouping the terms where $\chi_{b=2}$ appears, we then obtain:

$$\begin{aligned} p_{d+3,d} = &\sum_{\substack{a,b,c\geq 2\\ a+b+c=d+3}} \left(\frac{(d+3)!}{a!b!c!} + (d+2)(d+3) - \frac{(d+3)!}{a!(b+c-1)!} - \frac{(d+3)!}{c!(a+b-1)!}\right) \\ &+ \sum_{\substack{a,c\geq 2\\ a+c=d+1}} \left(\frac{(d+3)!}{(a+1)!(c+1)!} - \frac{(d+2)(d+3)}{2}\right), \end{aligned}$$

and we compute a closed form for each of the six sums appearing above.
It is readily checked that

$$\sum_{\substack{a,b,c\geq 2\\ a+b+c=d+3}} (d+2)(d+3) = \frac{(d+2)(d+3)(d-1)(d-2)}{2}$$

$$\text{and} \quad \sum_{\substack{a,c\geq 2\\ a+c=d+1}} \frac{(d+2)(d+3)}{2} = \frac{(d+2)(d+3)(d-2)}{2}.$$

For the terms $\sum_{\substack{a,b,c\geq 2\\ a+b+c=d+3}} \frac{(d+3)!}{a!(b+c-1)!} = \sum_{\substack{a,b,c\geq 2\\ a+b+c=d+3}} \frac{(d+3)!}{c!(a+b-1)!}$, we may write:

$$\begin{aligned} &\sum_{\substack{a,b,c\geq 2\\ a+b+c=d+3}} \frac{(d+3)!}{a!(b+c-1)!} = \sum_{a=2}^{d-1}\sum_{b=2}^{d+1-a} \frac{(d+3)!}{a!(d+2-a)!} = \sum_{a=2}^{d-1}(d-a)\frac{(d+3)!}{a!(d+2-a)!} \\ &= d(d+3)\sum_{a=2}^{d-1}\frac{(d+2)!}{a!(d+2-a)!} - (d+2)(d+3)\sum_{a=2}^{d-1}\frac{(d+1)!}{(a-1)!(d+1-(a-1))!} \\ &= d(d+3)\left(2^{d+2} - 2 - 2(d+2) - \frac{(d+2)(d+1)}{2}\right) \\ &\qquad - (d+2)(d+3)\left(2^{d+1} - 2 - (d+1) - \frac{d(d+1)}{2}\right) \\ &= (d+3)\left(2^{d+1}(d-2) - d(d+1) + 6\right). \end{aligned}$$

The two remaining sums are handled using the fact that binomial (resp. trinomial) coefficients count set partitions of a set into 2 (resp. 3) disjoint subsets. Namely, denoting the disjoint union by $\uplus$, we have:

$$\begin{aligned}
\sum_{\substack{a,c\geq 2\\ a+c=d+1}} \frac{(d+3)!}{(a+1)!(c+1)!} &= \sum_{\substack{a,c\geq 3\\ a+c=d+3}} \frac{(d+3)!}{a!c!} \\
&= \sum_{I\uplus J=[d+3]} \chi_{|I|\geq 3}\cdot\chi_{|J|\geq 3} \\
&= \sum_{I\uplus J=[d+3]} (1-\chi_{|I|<3})\cdot(1-\chi_{|J|<3}) \\
&= \sum_{I\uplus J=[d+3]} 1 \quad - \quad 2\sum_{I\uplus J=[d+3]} \chi_{|I|<3} \quad + \sum_{I\uplus J=[d+3]} \chi_{|I|<3}\cdot\chi_{|J|<3} \\
&= 2^{d+3} - 2\left(1+(d+3)+\frac{(d+3)(d+2)}{2}\right) \qquad \text{as soon as } d\geq 1
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\substack{a,b,c\geq 2\\ a+b+c=d+3}} \frac{(d+3)!}{a!b!c!} &= \sum_{I\uplus J\uplus K=[d+3]} (1-\chi_{|I|<2})\cdot(1-\chi_{|J|<2})\cdot(1-\chi_{|K|<2}) \\
&= \sum_{I\uplus J\uplus K=[d+3]} 1 \quad - \quad 3\sum_{I\uplus J\uplus K=[d+3]} \chi_{|I|<2} \\
&\qquad + \quad 3\sum_{I\uplus J\uplus K=[d+3]} \chi_{|I|<2}\cdot\chi_{|J|<2} \quad - \sum_{I\uplus J\uplus K=[d+3]} \chi_{|I|<2}\cdot\chi_{|J|<2}\cdot\chi_{|K|<2} \\
&= 3^{d+3} \quad - \quad 3\left(\sum_{J\uplus K=[d+3]} 1 + \quad (d+3)\cdot\sum_{J\uplus K=[d+2]} 1\right) \\
&\qquad + \quad 3\big(1+2(d+3)+(d+3)(d+2)\big) \qquad \text{since } d\geq 0 \\
&= 3^{d+3} - 3\left(2^{d+3}+(d+3)2^{d+2}\right) + 3\big(1+(d+3)(d+4)\big).
\end{aligned}$$

The closed formula for $p_{d+3,d}$ is then simply obtained by summing up the contributions of these 6 terms. $\square$

Table 1 shows the first few terms of the sequence $(p_{d+3,d})_d$.

d	3	4	5	6	7	8	9
$p_{d+3,d}$	5	84	686	3936	18387	75372	283052

Tab. 1: The first few terms of the sequence $(p_{d+3,d})_d$

We close the section with the evaluation of $p_{d+4,d}$.

We start by observing that the formula in Theorem 4.2 suggests that $p_{d+k,d}$ is a linear combination of suitable powers of i, for $i = 1, \dots k$, with polynomial coefficients. Specifically, we have an expression of the form

$$p_{d+k,d} = k^{d+k} + P_1^{(k)}(d) \cdot (k-1)^{d+k} + P_2^{(k)}(d) \cdot (k-2)^{d+k} + \cdots + P_{k-1}^{(k)}(d) \cdot 1^{d+k}, \qquad (2)$$

where the $P_i^{(k)}(d)$'s are polynomials of suitable degrees. In particular, when $k = 4$, we get

$$p_{d+4,d} = 4^{d+4} + P_1^{(4)}(d) \cdot 3^{d+4} + P_2^{(4)}(d) \cdot 2^{d+4} + P_3^{(4)}(d) \cdot 1^{d+4}.$$

In this case we can be more precise: we expand the generic determinant in Theorem 4.2 and find an upper bound on the contribution of each term to the degrees of the $P_i^{(4)}(d)$'s, namely $\deg P_1^{(4)}(d) \leq 2, \deg P_2^{(4)}(d) \leq 4, \deg P_3^{(4)}(d) \leq 6$. Thus, using our formula (and suitable software, such as Maple) to compute the first values of $p_{d+4,d}$, we can write a linear system whose indeterminates are the coefficients of the $P_i^{(4)}(d)$'s. Solving such a system (again using Maple), we finally obtain the following:

Theorem 4.5 *For all $d \geq 1$,*

$$\begin{aligned} p_{d+4,d} = {} & 4^{d+4} - (d^2 + 5d + 10)3^{d+3} + \left(\frac{1}{2}d^4 + \frac{7}{2}d^3 + 10d^2 + 16d + 4\right)2^{d+2} \\ & - \left(\frac{1}{6}d^6 + d^5 + \frac{4}{3}d^4 + \frac{1}{3}d^3 - \frac{1}{2}d^2 - \frac{19}{3}d + 4\right). \end{aligned}$$

Our analysis of the case $k = 4$ should clearly show that an analogous derivation of the term $p_{d+5,d}$, and of all the subsequent ones, could be carried on, at least in principle. The problem of suitably organizing the many expressions arising actually prevents from doing it in practice.

5 A generalization

The main motivation of the present paper is the study of d-minimal permutations and, in particular, their enumeration. Our approach is based on a bijection between the set of d-minimal permutations and a special class of skew Young tableaux, namely those in which every pair of consecutive rows has precisely two columns in common. Denote by $SkYT_2(n, k)$ this set of tableaux, n being the number of cells and k the number of rows. We can generalize this setting in a very natural way, by defining the set $SkYT_h(n, k)$ of skew Young tableaux having n cells and k rows such that any two consecutive rows have precisely $h \geq 1$ columns in common. In this final section we wish to relate these tableaux with some families of permutations, as well as to describe some enumerative results for low values of h.

Our first result is a generalization of Theorem 3.1.

Theorem 5.1 *The set $SkYT_h(d + k, k)$ is in bijection with the set of permutations π of length $d + k$ having exactly d descents and satisfying the following property (call it $DES_h(\pi)$):*

$$DES_h(\pi) \begin{cases} \textit{for every } i \leq h-1, \textit{ by deleting } i \textit{ elements of } \pi \\ \textit{and renaming the remaining ones in the usual way,} \\ \textit{the resulting permutation } \tau \textit{ has precisely } d-i \textit{ descents.} \end{cases}$$

Proof: Denote by $S^{(h)}_{d+k}(d)$ the set of permutations π of length $d+k$ having precisely d descents and satisfying $DES_h(\pi)$. Define a map $f : S^{(h)}_{d+k}(d) \to SkYT_h(d+k,k)$ by suitably generalizing the one given in Proposition 3.1: starting from the bottom, the i-th row of the tableau consists of the elements of the i-th descending run of π, and two consecutive rows are required to have exactly h columns in common. We claim that this map is well-defined. Indeed suppose, by way of contradiction, that in $f(\pi)$ there is a column in which a is above b and $a < b$. Without loss of generality, we can assume that a is in the cell immediately above b. Then, removing the $h-1$ entries of π preceding a, following b and belonging to the columns common to the rows of a and b, we obtain a permutation τ having $d-h+2$ descents, as shown in Figure 5. Hence, π does not satisfy $DES_h(\pi)$. The fact that f is injective and surjective is trivial, and follows directly from its definition. $\square$

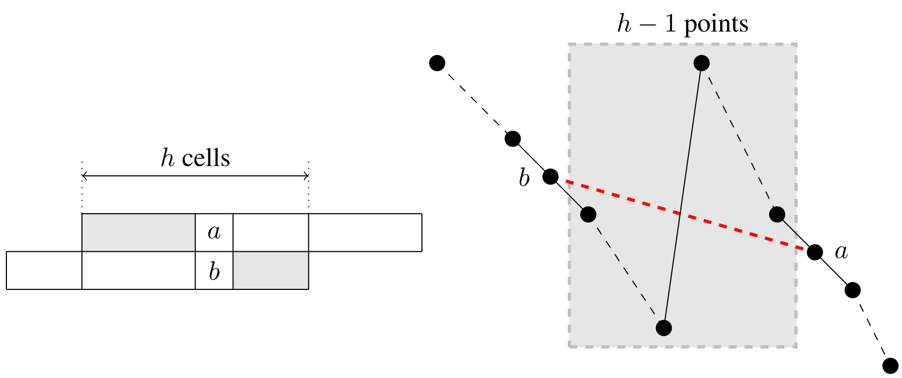


Fig. 5: Proof of Theorem 5.1. The gray area corresponds to the $h-1$ points removed in the proof.

It is clear that, when $h = 2$, we get precisely Theorem 3.1, since the resulting class of permutations is that of d-minimal permutations.

We also have a characterization of the above classes of permutations in terms of patterns, which follows quite easily from the above theorem, and so will be stated without proof.

Theorem 5.2 *A permutation σ belongs to $S^{(h)}_{d+k}(d)$, for some k, if and only if it has exactly d descents and its ascents occur in the middle of a consecutive pattern of the form $\pi = \pi_1\pi_2$, where π_1 and π_2 are words of the same length h, both decreasing and $\pi_1 < \pi_2$ componentwise.*

Having introduced this generalized setting, it is natural to ask what happens when $h < 2$.

If $h = 1$, what we obtain is the class of permutations having exactly d descents. It is well known that the number of permutations of length n having d descents is given by the Eulerian number $A_{n,d}$ (sequence A008292 in [Sl]). Using our approach, we find the following determinant expression for Eulerian numbers:

$$A_{d+k,d} = \sum_{\substack{a_1,a_2,\ldots,a_k \geq 1 \\ a_1+a_2+\cdots+a_k = d+k}} (d+k)! \cdot \det(B(a_1,\ldots,a_k)),$$

where $B(a_1, \ldots, a_k)$ is the following matrix:

$$\begin{pmatrix} \frac{1}{a_1!} & \frac{1}{(a_1+a_2)!} & \frac{1}{(a_1+a_2+a_3)!} & \cdots & \frac{1}{(a_1+\cdots+a_{k-1})!} & \frac{1}{(a_1+\cdots+a_k)!} \\ 1 & \frac{1}{a_2!} & \frac{1}{(a_2+a_3)!} & \cdots & \frac{1}{(a_2+\cdots+a_{k-1})!} & \frac{1}{(a_2+\cdots+a_k)!} \\ 0 & 1 & \frac{1}{a_3!} & \cdots & \frac{1}{(a_3+\cdots+a_{k-1})!} & \frac{1}{(a_3+\cdots+a_k)!} \\ \vdots & \vdots & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & \frac{1}{a_{k-1}!} & \frac{1}{(a_{k-1}+a_k)!} \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{a_k!} \end{pmatrix}.$$

In other words, $B(a_1, \ldots, a_k)$ is the $k \times k$ matrix whose entries $b_{i,j}$ obey the following equalities:

$$\begin{aligned} b_{i,j} &= \frac{1}{(a_i + \cdots + a_j)!}, \qquad \text{when } i \leq j, \\ b_{i,i-1} &= 1, \\ b_{i,j} &= 0, \qquad \text{when } i \geq j+2. \end{aligned}$$

The above formula for Eulerian numbers comes as no surprise. Indeed, it can be seen as a consequence of Example 2.2.4 in [St1], where it is shown that each of the above summands counts permutations of length $d + k$ having specified descent set. This result is attributed to MacMahon by Stanley.

We point out that the determinant of the matrix $B(a_1, \ldots, a_k)$ has a very neat recursive expression, from which a closed formula can be deduced.

Proposition 5.1 *Set* $D(a_1, \ldots, a_k) = \det(B(a_1, \ldots, a_k))$*. Then*

$$D(a_1, a_2, \ldots, a_k) = \frac{1}{a_1!} \cdot D(a_2, a_3, \ldots, a_k) - D(a_1 + a_2, a_3, \ldots, a_k).$$

Proof: Just expand $D(a_1, \ldots, a_k)$ with respect to its first column. □

Corollary 5.1 *The following formula holds:*

$$D(a_1, \ldots, a_k) = \sum_{i=1}^{k} (-1)^{k-i} \cdot \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_i) \\ \alpha \in \mathbf{PL}(a_1,\cdots,a_k)}} \frac{1}{|\alpha_1|! \cdot \ldots \cdot |\alpha_i|!}, \tag{3}$$

where $\mathbf{PL}(a_1, \cdots, a_k)$ *denotes the set of linear partitions of the totally ordered set* $\{a_1, \ldots, a_k\}$ *and* $|\alpha_i|$ *is the sum of the elements of the block* α_i*.*

Proof: We start by observing that, when $k = 1$, the outer sum of the r.h.s of (3) reduces to a single summand (for $i = 1$), as well as the inner sum, which has the unique summand $\frac{1}{a_1!}$. Moreover, when $k = 2$, the r.h.s. of (3) consists of two summands, which are $(-1) \cdot \frac{1}{(a_1+a_2)!}$ (for $i = 1$) and $\frac{1}{a_1!} \cdot \frac{1}{a_2!}$ (for $i = 2$), and this coincides with the expression of $D(a_1, a_2)$.

We can now conclude our proof using an inductive argument. The set $\mathbf{PL}(a_1, \cdots, a_k)$ can be partitioned into two subsets, namely the linear partitions in which a_1 occurs as a singleton (call this subset X)

and the linear partitions in which a_1 occurs in a block of cardinality at least 2 (call this subset Y). Using this partition of $\mathbf{PL}(a_1, \cdots, a_k)$ we can split the sum in the r.h.s of (3) into two sums, the first taking into account the contribution of X and the second taking into account the contribution of Y. We thus obtain the following equalities:

$$\sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_i)\\ \alpha\in\mathbf{PL}(a_1,\cdots,a_k)}} \frac{1}{|\alpha_1|!\cdot\ldots\cdot|\alpha_i|!} = \frac{1}{a_1!}\cdot \sum_{\substack{\beta=(\beta_1,\ldots,\beta_j)\\ \beta\in\mathbf{PL}(a_2,\cdots,a_k)}} \frac{1}{|\beta_1|!\cdot\ldots\cdot|\beta_j|!} + \sum_{\substack{\gamma=(\gamma_1,\ldots,\gamma_t)\\ \gamma\in\mathbf{PL}(\{a_1,a_2\},a_3,\cdots,a_k)}} \frac{1}{|\gamma_1|!\cdot\ldots\cdot|\gamma_t|!},$$

whence, using the induction hypothesis and the above proposition:

$$\sum_{i=1}^{k}(-1)^{k-i}\cdot \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_i)\\ \alpha\in\mathbf{PL}(a_1,\cdots,a_k)}} \frac{1}{|\alpha_1|!\cdot\ldots\cdot|\alpha_i|!} = \frac{1}{a_1!}\cdot D(a_2,\ldots,a_k) - D(a_1+a_2,a_3,\ldots,a_k) = D(a_1,\ldots,a_k),$$

as desired. □

Remark. An alternative approach to the case $h = 1$ could be done via the notion of Hessenberg matrix. An (upper) *Hessenberg matrix* is a square matrix having zero entries below the first subdiagonal. Hessenberg matrices prove their usefulness especially in numerical analysis and computer programming, being a sort of normal form to which any square matrix can be reduced in a finite number of steps. There are also some papers in the literature concerning the evaluation of the determinant of certain Hessenberg matrices having special form (see for instance [BS] and [LCT]). In [T], the determinant of Hessenberg matrices having all the elements of the first subdiagonal equal to 1 is considered (this is precisely the kind of matrices we meet in the determinant formula for Eulerian numbers).

Theorem 5.1 does not have any meaning when $h = 0$. The corresponding set $SkYT_0(n,k)$ consists of all skew Young tableaux having n cells and k rows such that any two consecutive rows only have the corners of two cells in common. In this case, it is immediate to see that $SkYT_0(n,k)$ is in bijection with all surjective functions from an n-set to a k-set: just interpret the elements of a tableau as balls and the rows of a tableau as boxes. Thus we get immediately that $|SkYT_0(n,k)| = k!\cdot S(n,k)$, where the $S(n,k)$'s are the Stirling numbers of the second kind. We can also use Theorem 4.1 to get an analog of Theorem 4.2; indeed, we can derive the following formula:

$$|SkYT_0(d+k,k)| = \sum_{\substack{a_1,a_2,\ldots,a_k\geq 1\\ a_1+a_2+\cdots+a_k=d+k}} (d+k)!\cdot\det(C(a_1,\ldots,a_k)),$$

where $C(a_1, \ldots, a_k)$ is the following triangular matrix:

$$\begin{pmatrix} \frac{1}{a_1!} & \frac{1}{(a_1+a_2+1)!} & \frac{1}{(a_1+a_2+a_3+2)!} & \cdots & \frac{1}{(a_1+\cdots+a_{k-1}+k-2)!} & \frac{1}{(a_1+\cdots+a_k+k-1)!} \\ 0 & \frac{1}{a_2!} & \frac{1}{(a_2+a_3+1)!} & \cdots & \frac{1}{(a_2+\cdots+a_{k-1}+k-3)!} & \frac{1}{(a_2+\cdots+a_k+k-2)!} \\ 0 & 0 & \frac{1}{a_3!} & \cdots & \frac{1}{(a_3+\cdots+a_{k-1}+k-4)!} & \frac{1}{(a_3+\cdots+a_k+k-3)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{a_{k-1}!} & \frac{1}{(a_{k-1}+a_k+1)!} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{a_k!} \end{pmatrix}.$$

From here it is then immediate to obtain $|SkYT_0(d+k,k)| = \sum_{\substack{a_1,a_2,\ldots,a_k \geq 1 \\ a_1+a_2+\cdots+a_k=d+k}} \binom{d+k}{a_1,\ldots a_k}$ which is known to be the number of surjective functions from an $(d+k)$-set to a k-set, as already shown a few lines above.

6 Further work

Even if our approach to the enumeration of d-minimal permutations allows us to completely solve the problem from a purely theoretical point of view, it is doubtless that its application to concrete cases leads to some technical difficulties. This is of course due to the intrinsic complexity of the sums of determinants appearing in Theorem 4.2. However, it seems plausible that at least a few more cases than those we deal with in the present paper can be managed by means of our technique. A major difficulty is to find (an upper bound on) the degrees of the polynomials of formula (2). In the case $k = 4$ we proved our bounds with a case by case analysis, but we have not been able to find similar bounds for general k.

Another interesting problem that remains untouched concerns the study of the structure of the poset determined by a minimal permutation with d descents, defined in [BP] and recalled here in Section 2. For instance, one can observe that a d-minimal permutation corresponds to a linear extension of the associated poset. Moreover, an interesting (and classical) line of research could be the investigation of the properties of the distributive lattice of the sup-irreducibles of these posets.

Acknowledgments

We would like to thank Christian Krattenthaler for helpful suggestions on how to handle our determinant that have improved our results.

References

[A] A. C. Aitken, *The monomial expansion of determinantal symmetric functions* Proc. Royal Soc. Edinburgh (A) 61 (1943) 300–310.

[BaRo] Y. Baryshnikov, D. Romik, *Enumeration formulas for Young tableaux in a diagonal strip* Israel J. Math. 178 (2010) 157–186.

[BS] A. T. Benjamin, M. A. Shattuck, *Recounting determinants for a class of Hessenberg matrices* Integers 7 (2007) # A55 (7 pp.).

[BP] M. Bouvel, E. Pergola, *Posets and permutations in the duplication-loss model: minimal permutations with d descents* Theoret. Comput. Sci. 411 (2010) 2487–2501.

[BoRo] M. Bouvel, D. Rossin, *A variant of the tandem duplication - random loss model of genome rearrangement* Theoret. Comput. Sci. 410 (2009) 847–858.

[CGM] W. Y. C. Chen, C. C. Y. Gu, K. J. Ma, *Minimal permutations and 2-regular skew tableaux* Adv. in Appl. Math. 47 (2011) 795–812.

[K] C. Krattenthaler, *Advanced determinant calculus* Sém. Lothar. Combin. 42 (1999) Article B42q (67 pp.).

[LCT] H.-C. Li, Y.-M. Chen, E.-T. Tan, *Counting determinants of Fibonacci-Hessenberg matrices using LU factorizations* Integers 9 (2009) #A37 (23 pp.).

[MY] T. Mansour, S. H. F. Yan, *Minimal permutations with d descents* European J. Combin. 31 (2010) 1445–1460.

[P] V. R. Pratt, *Computing permutations with double-ended queues, parallel stacks and parallel queues* Proceedings of the fifth annual ACM symposium on Theory of Computing, 1973, 268–277.

[Sl] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences* at http://oeis.org.

[St] R. P. Stanley, *On the enumeration of skew Young tableaux* Adv. in Appl. Math. 30 (2003) 283–294.

[St1] R. P. Stanley, *Enumerative Combinatorics, Vol. 1* Cambridge University Press, Cambridge, 1997.

[St2] R. P. Stanley, *Enumerative Combinatorics, Vol. 2* Cambridge University Press, Cambridge, 1999.

[T] U. Tamm, *The determinant of a Hessenberg matrix and some applications in discrete mathematics* preprint.