On the number of maximal independent sets in a graph

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Miller and Muller (1960) and independently Moon and Moser (1965) determined the maximum number of maximal independent sets in an n-vertex graph. We give a new and simple proof of this result.

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Let G be a (simple, undirected, finite) graph. A set $S \subseteq V(G)$ is *independent* if no edge of G has both its endpoints in S. An independent set S is *maximal* if no independent set of G properly contains S. Let $\mathrm{MIS}(G)$ be the set of all maximal independent sets in G. Miller and Muller (1960) and Moon and Moser (1965) independently proved that the maximum, taken over all n-vertex graphs G, of $|\,\mathrm{MIS}(G)|$ equals

$$g(n) := \begin{cases} 3^{n/3} & \text{if } n \equiv 0 \pmod 3 \\ 4 \cdot 3^{(n-4)/3} & \text{if } n \equiv 1 \pmod 3 \\ 2 \cdot 3^{(n-2)/3} & \text{if } n \equiv 2 \pmod 3 \ . \end{cases}$$

This result is important for various reasons. For example, $g(n)$ bounds the time complexity of various algorithms that output all maximal independent sets (Bron and Kerbosch, 1973; Lawler et al., 1980; Tsukiyama et al., 1977; Tomita et al., 2006; Johnson et al., 1988; Eppstein, 2003; Eppstein et al., 2010). Here we give a new and simple proof of this upper bound on $|\,\mathrm{MIS}(G)|$.

Theorem 1 ((Miller and Muller, 1960; Moon and Moser, 1965)) *For every n-vertex graph G,*

$$|\,\mathrm{MIS}(G)| \leq g(n) \ .$$

Proof: We proceed by induction on n. The base case with $n \leq 2$ is easily verified. Now assume that $n \geq 3$. Let G be a graph with n vertices. Let d be the minimum degree of G. Let v be a vertex of degree

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d in G. Let $N[v]$ be the closed neighbourhood of v. If $I \in \mathrm{MIS}(G)$ then $I \cap N[v] \neq \emptyset$, otherwise $I \cup \{v\}$ would be an independent set. Moreover, if $w \in I \cap N[v]$ then $I \setminus \{w\} \in \mathrm{MIS}(G - N[w])$. Thus

$$|\mathrm{MIS}(G)| \leq \sum_{w \in N_G[v]} |\mathrm{MIS}(G - N_G[w])| \ ,$$

Since $\deg(w) \geq d$ and g is non-decreasing, by induction,

$$|\mathrm{MIS}(G)| \leq (d+1) \cdot g(n-d-1) \ .$$

Note that

$$4 \cdot 3^{(n-4)/3} \leq g(n) \leq 3^{n/3} \ .$$

If $d \geq 3$ then

$$|\mathrm{MIS}(G)| \leq (d+1) \cdot 3^{(n-d-1)/3} \leq 4 \cdot 3^{(n-4)/3} \leq g(n) \ .$$

If $d = 2$ then

$$|\mathrm{MIS}(G)| \leq 3 \cdot g(n-3) = g(n) \ .$$

If $d = 1$ and $n \equiv 1 \pmod 3$ then since $n - 2 \equiv 2 \pmod 3$,

$$\mathrm{MIS}(G) \leq 2 \cdot g(n-2) \leq 2 \cdot 2 \cdot 3^{(n-2-2)/3} = 4 \cdot 3^{(n-4)/3} = g(n) \ .$$

If $d = 1$ and $n \equiv 0 \pmod 3$ then since $n - 2 \equiv 1 \pmod 3$,

$$\mathrm{MIS}(G) \leq 2 \cdot g(n-2) \leq 2 \cdot 4 \cdot 3^{(n-2-4)/3} < 3^{n/3} = g(n) \ .$$

If $d = 1$ and $n \equiv 2 \pmod 3$ then since $n - 2 \equiv 0 \pmod 3$,

$$\mathrm{MIS}(G) \leq 2 \cdot g(n-2) \leq 2 \cdot 3^{(n-2)/3} = g(n) \ .$$

This proves that $|\mathrm{MIS}(G)| \leq g(n)$, as desired. □

For completeness we describe the example by Miller and Muller (1960) and Moon and Moser (1965) that proves that Theorem 1 is best possible. If $n \equiv 0 \pmod 3$ then let M_n be the disjoint union of $\frac{n}{3}$ copies of K_3. If $n \equiv 1 \pmod 3$ then let M_n be the disjoint union of K_4 and $\frac{n-4}{3}$ copies of K_3. If $n \equiv 2 \pmod 3$ then let M_n be the disjoint union of K_2 and $\frac{n-2}{3}$ copies of K_3. Observe that $|\mathrm{MIS}(M_n)| = g(n)$.

Note that Vatter (2011) gave another proof of Theorem 1, and also described a connection between this result and the question, "What is the largest integer that is the product of positive integers with sum n?" Also note that Dieter Kratsch proved that $|\mathrm{MIS}(G)| \leq 3^{n/3}$ using a similar proof to that presented here; see Gaspers (2010, page 177). Thanks to the authors of (Vatter, 2011; Gaspers, 2010) for pointing out these references.

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